

## A THREE-FACTOR CONVERGENCE MODEL OF INTEREST RATES\*

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**Abstract.** A convergence model of interest rates explains the evolution of the domestic short rate in connection with the European rate. The first model of this kind was proposed by Corzo and Schwartz in 2000 and its generalizations were studied later. In all these models, the European rates are modelled by a one-factor model. This, however, does not provide a satisfactory fit to the market data. A better fit can be obtained using the model, where the short rate is a sum of two unobservable factors. Therefore, we build the convergence model for the domestic rates based on this evolution of the European market. We study the prices of the domestic bonds in this model, which are given by the solution of the partial differential equations. In general, it does not have an explicit solution. Hence we suggest an analytical approximative formula and derive order of its accuracy in a particular case.

**Key words.** interest rate, convergence model, bond price, partial differential equation, closed-form solution, analytic approximation, order of accuracy

**AMS subject classifications.** 91B28, 35K15

**1. Introduction.** Interest rate is a rate charged for the use of the money. As an example we show Euribor (European Interbank Offered Rates) interest rates on the interbank market. Figure 1.1 displays the interest rates with different maturities (so called term structures) at a given day and the evolution of the selected interest rate during a given time period.

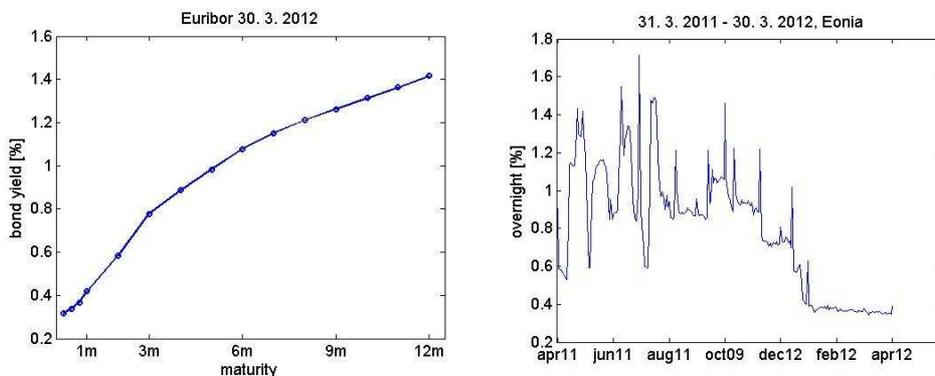


FIG. 1.1. *Example of the term structure of the interest rates (left) and the evolution of interest rate in time (right). Source: <http://www.euribor-ebf.eu>.*

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Interest rates in the countries are not independent. This is particularly true if the country plans to adopt the euro currency. In this case, the domestic interest rates are closely related to the rates in the monetary union, especially shortly before the time of accession. This is documented in Figure 1.2 which presents the Bribor (Bratislava Interbank Offered Rates) together with the Euribor rates in the last quarter before Slovakia adopted euro.

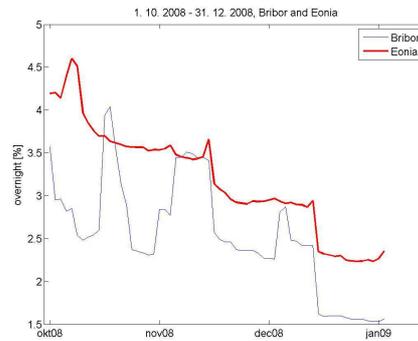


FIG. 1.2. *Bribor and Eonia short rate in the last quarter of 2008.*

The first model describing this phenomenon has been proposed by Corzo and Schwarz in [4]. They model the instantaneous European rate  $r_e$  by a mean-reverting stochastic process  $dr_e = \kappa_e(\theta_r - r_e)dt + \sigma_e dw_e$  with constant volatility  $\sigma_e$  (known as Vasicek process in the context of interest rates modelling, see the model in [11]). The domestic instantaneous rate is then reverting not to a constant value, but to the current level of the European rate (with a possibility of a minor divergence, see [4] for details). However, Vasicek model is not necessarily the most suitable model for the European rates, or the short term interest rates in general. To eliminate the possibility of negative interest rates in Vasicek model, the model with volatility proportional to the square root of the interest rate has been proposed by Cox, Ingersoll and Ross in [5] (CIR model hereafter). The discussion of the proper form of volatility started with an influential paper [2] by Chan, Karolyi, Longstaff and Sanders (CKLS model hereafter) who considered more general nonlinear volatility, proportional to the power of the interest rate, i.e.,  $dr_e = \kappa_e(\theta_e - r_e)dt + \sigma_e r_e^{\gamma_e} dw_e$ . They have shown by analysing the market data that Vasicek and CIR models are rejected when tested as restrictions of a more general model.

If the short rate is modelled by a stochastic differential equation, the other interest rates can be computed from the bond prices, which are solutions to the parabolic partial differential equation. Among the models described above, only the Vasicek and CIR models have the solution known in a closed form. For a general CIR model, although found superior in modelling the evolution of the short rate, the bond prices are not so easily computed. Recently, analytical approximation formulae have been suggested in [3], [9], [10].

Generalization of the convergence model by Corzo and Schwarz has been studied in the thesis [8] and the paper [12]. The European short rate is assumed to follow CIR and general CKLS models respectively. It is shown in [8] that in the case of uncorrelated Wiener processes governing the evolution of the European and domestic short rate, the pricing of a domestic bond can be reduced to solving a system of

ordinary differential equations. For the general case, an analytical approximation formula has been suggested in [12]. This model was then fit to real Euro area and Slovak data in the last quarter before Slovakia joined the monetary union. The resulting fit is, however, not satisfying. This is true for the modelling European rates by the CIR model in the first place. Hence the first question when building a convergence model is the suitable model for the European interest rates. This has been found in the thesis [6]. The instantaneous interest rate is modelled as a sum of two unobservable mean reverting factors. Their sum is also considered to be unobservable, instead of identifying it with an overnight rate, to prevent the possible effect of speculations on the market affecting the overnight. This model achieves a much better fit, see the comparison in the Figure 1.3.

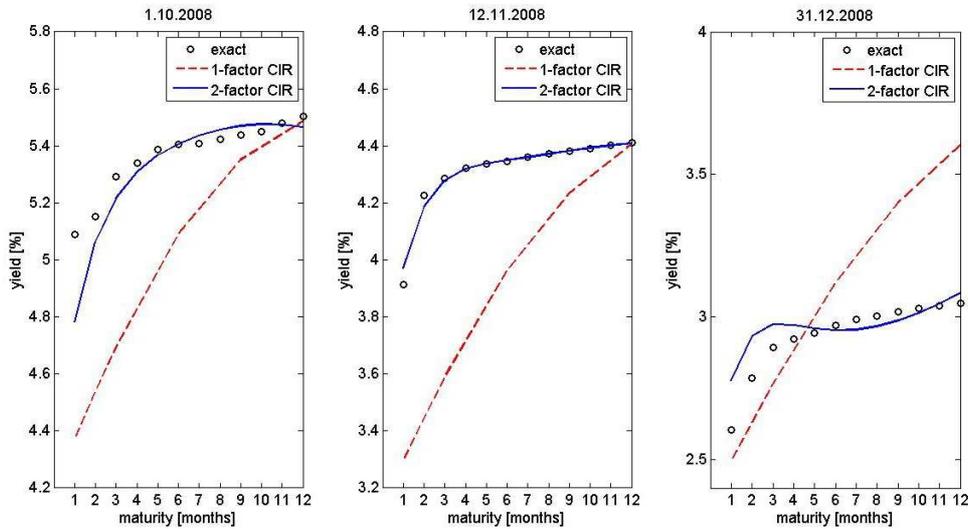


FIG. 1.3. *Fitting the European term structures from the last quarter of 2008 using the 1-factor CIR and 2-factor CIR models, selected days. Source: [12], [6].*

In this paper we propose the convergence model, where the European short rate is modelled as the sum of two factor of the CKLS type. Pricing European bonds is derived in the cited work [6]. Here we concentrate on pricing the domestic bonds - finding explicit solutions, proposing an analytical approximation for the general case and its preliminary analysis.

The paper is organized as follows: In section 2 we define the model in terms of a system of stochastic differential equations. Section 3 deals with bond pricing, which is firstly considered in the general case and then in the special cases which will be needed in the rest of the paper. In particular, we derive a closed form solution for the Vasicek-type of a model and a reduction to a system of ODEs for a special case of the CIR-type model. Based on the Vasicek closed form solution, we propose an analytical approximation formula for the general CKLS-type model. Using the ODE representation of the exact solution of the CIR model, we derive the order of accuracy of the approximation formula in this case. In section 4 we test the proposed approximation numerically. We end the paper with some concluding remarks in section 5.

**2. Formulation of the model.** We propose the following model for the joint dynamics of the European  $r_e$  and domestic  $r_d$  instantaneous interest rate. The European rate  $r_e = r_1 + r_2$  is modelled as the sum of the two mean-reverting factors  $r_1$  and  $r_2$ , while the domestic rate  $r_d$  reverts to the European rate. Volatilities of the processes are assumed to have a general CKLS form. Hence

$$\begin{aligned} dr_1 &= \kappa_1(\theta_1 - r_1)dt + \sigma_1 r_1^{\gamma_1} dw_1 \\ dr_2 &= \kappa_2(\theta_2 - r_2)dt + \sigma_2 r_2^{\gamma_2} dw_2 \\ dr_d &= \kappa_d((r_1 + r_2) - r_d)dt + \sigma_d r_d^{\gamma_d} dw_d \end{aligned}$$

with  $Cor(dw) = \mathcal{R}dt$ , where  $dw = (dw_1, dw_2, dw_d)^T$  is a vector of Wiener processes with correlation matrix  $\mathcal{R}$ , whose elements (i.e., correlations between  $r_i$  and  $r_j$ ) we denote by  $\rho_{ij}$ .

Figure 2.1 and Figure 2.2 show the evolution of the factors and the interest rates for the following set of parameters:  $\kappa_1 = 3, \theta_1 = 0.02, \sigma_1 = 0.05, \gamma_1 = 0.5, \kappa_2 = 10, \theta_2 = 0.01; \sigma_2 = 0.05, \gamma_2 = 0.5, \kappa_d = 1, \sigma_d = 0.02, \gamma_d = 0.5, \rho_{ij} = 0$  for all  $i, j$ .

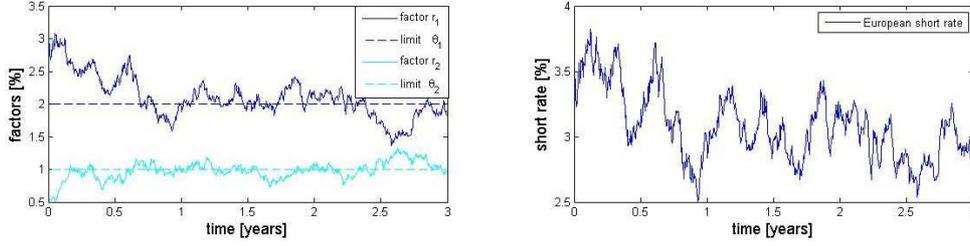


FIG. 2.1. Simulation of the factors  $r_1, r_2$  (left) and the European short rate  $r_e = r_1 + r_2$  (right).

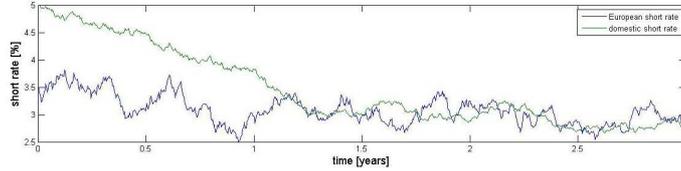


FIG. 2.2. Simulation of the European short rate  $r_e$  and the domestic short rate  $r_d$ .

**3. Bond prices.** To compute the bond prices, it is necessary to specify so called market prices of risk for each factor, in addition to the stochastic differential equations for the short rates. Denoting the market prices of risk as  $\lambda_1 = \lambda_1(t, r_1, r_2, r_d)$ ,  $\lambda_2 = \lambda_2(t, r_1, r_2, r_d)$ ,  $\lambda_d = \lambda_d(t, r_1, r_2, r_d)$  we obtain the following PDE for the price  $P = P(\tau, r_1, r_2, r_d)$  of the bond with time to maturity  $\tau = T - t$  (cf. [7]):

$$\begin{aligned} & -\frac{\partial P}{\partial \tau} + [\kappa_d((r_1 + r_2) - r_d) - \lambda_d \sigma_d r_d^{\gamma_d}] \frac{\partial P}{\partial r_d} + [\kappa_1(\theta_1 - r_1) - \lambda_1 \sigma_1 r_1^{\gamma_1}] \frac{\partial P}{\partial r_1} \\ & + [\kappa_2(\theta_2 - r_2) - \lambda_2 \sigma_2 r_2^{\gamma_2}] \frac{\partial P}{\partial r_2} + \frac{\sigma_d^2 r_d^{2\gamma_d}}{2} \frac{\partial^2 P}{\partial r_d^2} + \frac{\sigma_1^2 r_1^{2\gamma_1}}{2} \frac{\partial^2 P}{\partial r_1^2} + \frac{\sigma_2^2 r_2^{2\gamma_2}}{2} \frac{\partial^2 P}{\partial r_2^2} \\ & + \rho_{1d} \sigma_d r_d^{\gamma_d} \sigma_1 r_1^{\gamma_1} \frac{\partial^2 P}{\partial r_d \partial r_1} + \rho_{2d} \sigma_d r_d^{\gamma_d} \sigma_2 r_2^{\gamma_2} \frac{\partial^2 P}{\partial r_d \partial r_2} + \rho_{12} \sigma_1 r_1^{\gamma_1} \sigma_2 r_2^{\gamma_2} \frac{\partial^2 P}{\partial r_1 \partial r_2} - r_d P = 0. \end{aligned}$$

The PDE holds for all  $r_d, r_1, r_2 > 0$  and  $\tau \in [0, T]$  and it satisfies the initial condition  $P(0, r_d, r_1, r_2) = 1$  for all  $r_d, r_1, r_2 > 0$ .

**3.1. Vasicek and CIR type convergence models.** We define the Vasicek-type convergence model as the model, where the volatilities of the factors are all constant (i.e.,  $\gamma_1 = \gamma_2 = \gamma_d = 0$ ), as a generalization of the one-factor model [11]. Similarly as in this one-factor model, we consider constant market prices of risk, i.e.,  $\lambda_1(t, r_1, r_2, r_d) = \lambda_1, \lambda_2(t, r_1, r_2, r_d) = \lambda_2, \lambda_d(t, r_1, r_2, r_d) = \lambda_d$ , where  $\lambda_1, \lambda_2$  and  $\lambda_d$  are constants.

Similarly, as in one-factor and two-factor models proposed in [5] we define the CIR-type convergence model as the model with  $\gamma_1 = \gamma_2 = \gamma_d = 1/2$  and the market prices of risk proportional to the square roots of the corresponding factors, i.e.,  $\lambda_1(t, r_1, r_2, r_d) = \lambda_1\sqrt{r_1}, \lambda_2(t, r_1, r_2, r_d) = \lambda_2\sqrt{r_2}, \lambda_d(t, r_1, r_2, r_d) = \lambda_d\sqrt{r_d}$ , where  $\lambda_1, \lambda_2$  and  $\lambda_d$  are constants.

The PDE for the bond price then reads as

$$\begin{aligned} & -\frac{\partial P}{\partial \tau} + \mu_d \frac{\partial P}{\partial r_d} + \mu_2 \frac{\partial P}{\partial r_1} + \mu_3 \frac{\partial P}{\partial r_2} + \frac{\sigma_d^2 r_d^{2\gamma_d}}{2} \frac{\partial^2 P}{\partial r_d^2} + \frac{\sigma_1^2 r_1^{2\gamma_1}}{2} \frac{\partial^2 P}{\partial r_1^2} + \frac{\sigma_2^2 r_2^{2\gamma_2}}{2} \frac{\partial^2 P}{\partial r_2^2} \\ & + \rho_{1d} \sigma_d r_d^{\gamma_d} \sigma_1 r_1^{\gamma_1} \frac{\partial^2 P}{\partial r_d \partial r_1} + \rho_{2d} \sigma_d r_d^{\gamma_d} \sigma_2 r_2^{\gamma_2} \frac{\partial^2 P}{\partial r_d \partial r_2} + \rho_{12} \sigma_1 r_1^{\gamma_1} \sigma_2 r_2^{\gamma_2} \frac{\partial^2 P}{\partial r_1 \partial r_2} - r_d P = 0, \end{aligned}$$

where

$$\mu_d = a_1 + a_2 r_d + a_3 r_1 + a_4 r_2, \mu_2 = b_1 + b_2 r_1, \mu_3 = c_1 + c_2 r_2$$

(note that they are in fact the so called risk neutral drifts, cf. [1] for the relation between bond pricing and the risk neutral measure) with

- $a_1 = -\lambda_d \sigma_d, a_2 = -\kappa_d, a_3 = \kappa_d, a_4 = \kappa_d, b_1 = \kappa_1 \theta_1 - \lambda_1 \sigma_1, b_2 = -\kappa_1, c_1 = \kappa_2 \theta_2 - \lambda_2 \sigma_2, c_2 = -\kappa_2$  in the Vasicek-type model,
- $a_1 = 0, a_2 = -\kappa_d - \lambda_d \sigma_d, a_3 = \kappa_d, a_4 = \kappa_d, b_1 = \kappa_1 \theta_1, b_2 = -\kappa_1 - \lambda_1 \sigma_1, c_1 = \kappa_2 \theta_2, c_2 = -\kappa_2 - \lambda_2 \sigma_2$  in the CIR-type model.

We show that in the Vasicek case and the uncorrelated version (i.e., if the Wiener processes  $w_1, w_2, w_d$  are uncorrelated) of the CIR case, the solution of the PDE, can be written in a separable form

$$P(r_d, r_1, r_2, \tau) = e^{A(\tau)r_d + B(\tau)r_1 + C(\tau)r_2 + D(\tau)}. \quad (3.1)$$

Furthermore, in the Vasicek model the functions  $A, B, C, D$  can be written in the closed form. In the CIR model, they are solutions to the system of ordinary differential equations, which can be solved numerically much easier than the original PDE.

To prove the claim about the Vasicek model we insert the expected form of the solution (3.1) into the PDE with  $\gamma_i = 0$ . We obtain

$$\begin{aligned} & r_d(-\dot{A} + a_2 A - 1) + r_1(-\dot{B} + a_3 A + b_2 B) + r_2(-\dot{C} + a_4 A + c_2 C) \\ & + (-\dot{D} + a_1 A + b_1 B + c_1 C + \frac{\sigma_d^2}{2} A^2 + \frac{\sigma_1^2}{2} B^2 + \frac{\sigma_2^2}{2} C^2 \\ & + \rho_{1d} \sigma_d \sigma_1 AB + \rho_{2d} \sigma_d \sigma_2 AC + \rho_{12} \sigma_1 \sigma_2 BC) = 0, \end{aligned}$$

which implies the following system of ordinary differential equations:

$$\begin{aligned}
\dot{A} &= a_2A - 1, \\
\dot{B} &= a_3A + b_2B, \\
\dot{C} &= a_4A + c_2C, \\
\dot{D} &= a_1A + b_1B + c_1C + \frac{\sigma_d^2}{2}A^2 + \frac{\sigma_1^2}{2}B^2 + \frac{\sigma_2^2}{2}C^2 + \rho_{1d}\sigma_d\sigma_1AB \\
&\quad + \rho_{2d}\sigma_d\sigma_2AC + \rho_{12}\sigma_1\sigma_2BC,
\end{aligned} \tag{3.2}$$

with initial conditions  $A(0) = B(0) = C(0) = D(0) = 0$ . Functions  $A, B, C$  are easily found to be equal to (here and in the subsequent analysis we assume that  $a_2 \neq b_2$  and  $a_2 \neq c_2$ , and we omit the very special case when the coefficients are equal)

$$\begin{aligned}
A(\tau) &= \frac{1 - e^{a_2\tau}}{a_2}, \quad B(\tau) = \frac{a_3(b_2(1 - e^{a_2\tau}) - a_2(1 - e^{b_2\tau}))}{a_2b_2(a_2 - b_2)}, \\
C(\tau) &= \frac{a_4(c_2(1 - e^{a_2\tau}) - a_2(1 - e^{c_2\tau}))}{a_2c_2(a_2 - c_2)}.
\end{aligned}$$

The function  $D$  can be found by integration. For the sake of brevity we omit the details.

Now we consider the uncorrelated CIR case. Substituting  $\gamma_i = 1/2$  and zero correlations  $\rho_{ij} = 0$ ; and inserting the expected form of the solution (3.1) into the PDE we obtain

$$\begin{aligned}
r_d(-\dot{A} + a_2A + \frac{\sigma_d^2}{2}A^2 - 1) + r_1(-\dot{B} + a_3A + b_2B + \frac{\sigma_1^2}{2}B^2) \\
+ r_2(-\dot{C} + a_4A + c_2C + \frac{\sigma_2^2}{2}C^2) + (-\dot{D} + a_1A + b_1B + c_1C) = 0,
\end{aligned}$$

which implies the system of ordinary differential equations

$$\begin{aligned}
\dot{A} &= a_2A + \frac{\sigma_d^2}{2}A^2 - 1, \\
\dot{B} &= a_3A + b_2B + \frac{\sigma_1^2}{2}B^2, \\
\dot{C} &= a_4A + c_2C + \frac{\sigma_2^2}{2}C^2, \\
\dot{D} &= a_1A + b_1B + c_1C,
\end{aligned} \tag{3.3}$$

with initial conditions  $A(0) = B(0) = C(0) = D(0) = 0$ . Firstly, we find the function  $A$  by separation of variables. Then, we independently numerically solve the ODEs for  $B$  and  $C$ , and finally by numerical integration we obtain the function  $D$ .

Figure 3.1 shows the examples of term structures from the CIR-type model, where we have taken  $\lambda_d = \lambda_1 = \lambda_2 = 0$ . The remaining parameters are the same as in the section 2. Note the variety of the term structure shapes which can be obtained for the same values of both the domestic short rate  $r_d$  and the European short rate  $r_e$ , depending on the decomposition of  $r_e$  into the factors  $r_1$  and  $r_2$ .

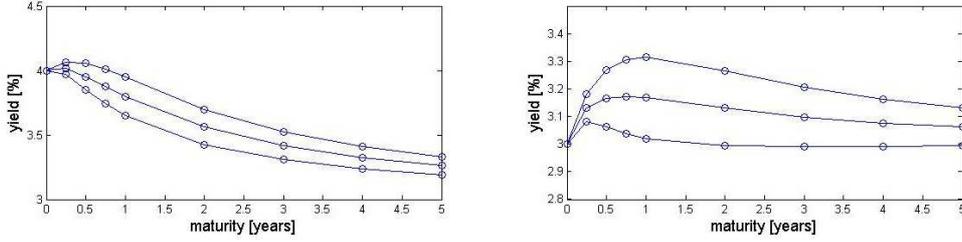


FIG. 3.1. Examples of term structures in the CIR-type convergence model. Domestic short rate  $r_d$  equals 4% (left) and 3% (right). European short rate  $r_e$  equals 5%, the term structures correspond to its different decompositions into factors:  $r_1 = 4\%, r_2 = 1\%$ ;  $r_1 = 2.5\%, r_2 = 2.5\%$ ;  $r_1 = 1\%, r_2 = 4\%$ .

**3.2. Analytical approximation formula for general convergence model.**

In the general case of the convergence model the assumption (3.1) does not lead to a solution. We use the idea of finding an approximative formula which has been successfully used in simpler models (one-factor models in [10], two-factor models in [6] and [12]). We consider the closed form solution from the model of the Vasicek type and replace its constant volatilities  $\sigma_1, \sigma_2, \sigma_d$  by instantaneous volatilities  $\sigma_1 r_1^{\gamma_1}, \sigma_2 r_2^{\gamma_2}, \sigma_d r_d^{\gamma_d}$ . In this way we obtain the approximation  $P^{ap} = P^{ap}(\tau, r_1, r_2, r_d)$ .

**3.3. Order of accuracy in the case of uncorrelated CIR model.**

Recall that we have the separated form of the solution (3.1) for the bond price in CIR model with zero correlations  $\rho_{ij}$  and the system of ODEs (3.3). The system (3.3) enables us to compute the derivatives of the functions  $A, B, C, D$  at  $\tau = 0$  (see Table 3.1) and consequently the Taylor series expansion of  $\ln P(\tau, r_1, r_2, r_d)$  around  $\tau = 0$ .

TABLE 3.1

Calculation of the derivatives of functions  $A, B, C, D$  from the CIR model with zero correlations

$i$	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
$A^i(0)$	0	-1	$-a_2$	$-a_2^2 + \sigma_d^2$	$-a_2^3 + 4a_2\sigma_d^2$
$B^i(0)$	0	0	$-a_3$	$-a_3a_2 - a_3b_2$	$-a_2^2a_3 + a_3\sigma_d^2 - a_2a_3b_2 - a_3b_2^2$
$C^i(0)$	0	0	$-a_4$	$-a_4a_2 - a_4c_2$	$-a_2^2a_4 + a_4\sigma_d^2 - a_2a_4c_2 - a_4c_2^2$
$D^i(0)$	0	0	$-a_1$	$-a_1a_2 - b_1a_3 - c_1a_4$	$-a_1a_2^2 + a_1\sigma_d^2 - a_2a_3b_1 - a_3b_1b_2 - a_2a_4c_1 - a_4c_1c_2$

The approximation formula  $P^{ap}$  is given in the closed form, hence the Taylor series can be computed also for  $\ln P^{ap}(\tau, r_1, r_2, r_d)$ . (Alternatively, we can use the system of ODEs similarly as in the case of the exact solution.) The derivatives needed in the expansion are shown in Table 3.2.

Comparing the expressions in Table (3.1) and Table (3.2) we obtain the order of the difference  $\ln P^{ap}(\tau, r_1, r_2, r_d) - \ln P(\tau, r_1, r_2, r_d)$ , which can be interpreted in terms of the relative error in bond prices and the absolute error in term structures, as stated in the following theorem and its corollary.

**THEOREM 3.1.** Let  $P^{CIR, \rho=0}$  be the bond price in the CIR-type convergence model with zero correlations and let  $P^{CIR, \rho=0, ap}$  be its approximation proposed in section 3.2. Then

$$\ln P^{CIR, \rho=0, ap} - \ln P^{CIR, \rho=0} = -\frac{1}{24} \sigma_d^2 (a_1 + a_2 r_d + a_3 r_1 + a_4 r_2) \tau^4 + o(\tau^4)$$

TABLE 3.2

Calculation of the derivatives of functions  $A, B, C, D$  from the approximation of the CIR model with zero correlations.

$i$	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
$A^i(0)$	0	-1	$-a_2$	$-a_2^2$	$-a_2^3$
$B^i(0)$	0	0	$-a_3$	$-a_3a_2 - a_3b_2$	$-a_2^2a_3 - a_2a_3b_2 - a_3b_2^2$
$C^i(0)$	0	0	$-a_4$	$-a_4a_2 - a_4c_2$	$-a_2^2a_4 - a_2a_4c_2 - a_4c_2^2$
$D^i(0)$	0	0	$-a_1$	$-a_1a_2 - b_1a_3 - c_1a_4 + \sigma_d^2r_d$	$-a_1a_2^2 - a_2a_3b_1 - a_3b_1b_2$ $-a_3b_1b_2 - a_2a_4c_1 - a_4c_1c_2$ $+3a_2\sigma_d^2r_d$

for  $\tau \rightarrow 0^+$ .

Note that the form of the leading term of the approximation error (i.e.,  $-\frac{1}{24}\sigma_d^2$  times the risk neutral domestic drift) is the same as in the two-factor convergence model [12], where the analogical strategy of forming the approximative formula has been used.

COROLLARY 3.2.

1. The relative error of the bond price satisfies

$$\frac{P^{CIR,\rho=0,ap} - P^{CIR,\rho=0}}{P^{CIR,\rho=0}} = -\frac{1}{24}\sigma_d^2(a_1 + a_2r_d + a_3r_1 + a_4r_2)\tau^4 + o(\tau^4)$$

for  $\tau \rightarrow 0^+$ .

2. The error in interest rates  $R$  can be expressed as

$$R^{CIR,\rho=0,ap} - R^{CIR,\rho=0} = \frac{1}{24}\sigma_d^2(a_1 + a_2r_d + a_3r_1 + a_4r_2)\tau^3 + o(\tau^3)$$

for  $\tau \rightarrow 0^+$ .

*Proof.* The first corollary is a consequence of the Taylor expansion of the exponential function  $e^x = 1 + x + o(x)$  for  $x \rightarrow 0^+$ . The second corollary follows from the formula  $R(\tau, r) = -\frac{\ln P(\tau, r)}{\tau}$  for calculating the interest rates  $R$  from the bond prices  $P$  (cf. [1], [7]).  $\square$

**4. Numerical experiment.** We consider the term structures presented in Figure 3.1 and compare them with the approximate values obtained by the proposed formula. The results are summarized in Table 4.1 and Table 4.2. The accuracy is very high (note that Euribor is quoted to three decimal places) even for higher maturities.

**5. Conclusions.** The paper deals with an approximation of the bond price in a three factor convergence model of the CKLS type on the basis of the closed form solution of the Vasicek model. We have numerically tested the proposed approximation on the CIR model with a zero correlations, for which the exact solution can be expressed in a simpler form and also analytically derived its accuracy in this case. The difference of logarithms of the exact solution and the proposed approximation is of the order  $O(\tau^4)$ .

Our next aim is to derive the order of accuracy in the general case. The special form of the solution in the case considered in this paper makes the analysis more direct, however, it is possible to study the accuracy of the approximation of the bond

TABLE 4.1

Exact interest rates and their approximations obtained by the proposed formula. The domestic short rate is 4%, the European short rate is 5%, the columns correspond to the different values of the factors:  $r_1 = 4\%$ ,  $r_2 = 1\%$  (left),  $r_1 = 2.5\%$ ,  $r_2 = 2.5\%$  (middle),  $r_1 = 1\%$ ,  $r_2 = 4\%$  (right).

maturity	exact	approx.	exact	approx.	exact	approx.
0	4.00000	4.00000	4.00000	4.00000	4.00000	4.00000
0.25	4.06607	4.06607	4.01638	4.01638	3.96668	3.96668
0.5	4.05591	4.05591	3.95219	3.95219	3.84847	3.84847
0.75	4.00932	4.00931	3.87493	3.87493	3.74055	3.74054
1	3.94734	3.94733	3.7995	3.79949	3.65166	3.65165
2	3.69802	3.69796	3.56221	3.56217	3.4264	3.42638
3	3.52184	3.52171	3.41487	3.41479	3.30791	3.30788
4	3.40688	3.40669	3.32208	3.32196	3.23728	3.23724
5	3.32995	3.32972	3.26077	3.26062	3.19158	3.19153

TABLE 4.2

Exact interest rates and their approximations obtained by the proposed formula. The domestic short rate is 3%, the European short rate is 5%, the columns correspond to the different values of the factors:  $r_1 = 4\%$ ,  $r_2 = 1\%$  (left),  $r_1 = 2.5\%$ ,  $r_2 = 2.5\%$  (middle),  $r_1 = 1\%$ ,  $r_2 = 4\%$  (right).

maturity	exact	approx.	exact	approx.	exact	approx.
0	3.00000	3.00000	3.00000	3.00000	3.00000	3.00000
0.25	3.18127	3.18127	3.13158	3.13158	3.08189	3.08189
0.5	3.26898	3.26898	3.16526	3.16526	3.06154	3.06154
0.75	3.30582	3.30583	3.17144	3.17144	3.03705	3.03705
1	3.31524	3.31524	3.1674	3.16741	3.01957	3.01957
2	3.26573	3.2657	3.12992	3.12991	2.99411	2.99412
3	3.20515	3.20508	3.09818	3.09816	2.99122	2.99124
4	3.1615	3.1614	3.0767	3.07667	2.9919	2.99194
5	3.13134	3.13121	3.06215	3.06211	2.99296	2.99301

prices also without this structure (see [12] for the analysis of a two-factor convergence model). Furthermore, we will look for a suitable calibration algorithm and calibrate the model to the real data to see, whether the increased complexity leads to a significant improvement in fitting the market data.

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