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# ERROR CONTROL BASED MODEL REDUCTION FOR MULTISCALE PROBLEMS

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**Abstract.** In this contribution we review a posteriori based discretization methods for variational multiscale problems and suggest a suitable conceptual approach for an efficient numerical treatment of parametrized variational multiscale problems where the parameters are either chosen from a low dimensional parameter space or consists of parameter functions from some compact low dimensional manifold that is embedded in some high dimensional or even infinite dimensional function space. The approach is based on combinations of ideas from established numerical multiscale methods and efficient model reduction approaches such as the reduced basis method.

**Key words.** multiscale methods, a posteriori error estimates, reduced basis method

**AMS subject classifications.** 15A15, 15A09, 15A23

**1. Introduction.** In this contribution we are interested in efficient approximation of parameterized multiscale problems in a very general parameterized variational setting. Let $U, V$ denote trial and test function spaces mapping from $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$ to $\mathbb{R}$. We look at solutions $u_\mu^\epsilon \in U$ of a parametrized variational problem of the form

$$R_\mu^\epsilon[u_\mu^\epsilon](v) = 0 \qquad \forall v \in V. \tag{1.1}$$

with an $\epsilon$ and $\mu$-dependent mapping $R_\mu^\epsilon : U \to V'$ where $\epsilon$ denotes a parameter that indicates the multiscale character of the problem, and $\mu : \Omega \to \mathbb{R}^p, p \in \mathbb{N}$ denotes a vector of parameter functions that do not depend on $\epsilon$.

In the most simple case we consider linear elliptic homogenization problems, where

$$U = V = H_0^1(\Omega), \qquad R_\mu^\epsilon[u] := \mathcal{A}_\mu^\epsilon[u] - F_\mu, \tag{1.2}$$

$$\mathcal{A}_\mu^\epsilon[u^\epsilon](v) = \int_\Omega \mu_1 A^\epsilon \nabla u \nabla v, \qquad F_\mu(v) = \int_\Omega \mu_2 f v \tag{1.3}$$

for some $\epsilon$ dependent diffusion tensor $A^\epsilon(x) = A(\frac{x}{\epsilon}) : \Omega \to \mathbb{R}^{d\times d}$ and a source function $f : \Omega \to \mathbb{R}$. Here $\mu_1, \mu_2$ denote two components of $\mu : \Omega \to \mathbb{R}^p, p = 2$.

Classical numerical multiscale methods such as multiscale finite elements [20, 13], variational multiscale methods [21], multiscale finite volume methods [2, 29] or the heterogeneous multiscale method [11] are designed to approximate variational problems of type (1.1) for fixed parameters $\mu$. On the other hand, classical model reduction approaches such as the reduced basis method [32, 15] are designed to treat parametrized systems for fixed, but moderate $\epsilon$. In this contribution we review numerical multiscale and reduced basis approaches with particular view to the a priori versus a posteriori character in the construction of approximation spaces and corresponding discretization schemes. Based on these observations we come up with a general model reduction approach for parametrized multiscale problems of type (1.1) and discuss specific realizations of this approach.

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In Section 2 we outline several numerical multiscale approaches. In the following Section 3 we discuss available adaptive schemes based on a posteriori error estimates. Section 4 then introduces the reduced basis approach and discusses its limitations in the multiscale scenario. Combing ideas from reduced basis methods with numerical multiscale approaches, we then introduce a general concept for model reduction of multiscale problems in Section 5 and discuss offline versus online computational complexities in Section 6

**2. General setting for numerical multiscale methods.** Numerical multiscale methods make use of a possible separation of scales in the underlying problem. The macroscopic scale is defined by a priori chosen macroscopic approximation spaces $U_H \subset U, V_H \subset V$, typically chosen as piecewise polynomial functions on a uniform coarse partition $\mathcal{T}_H$ of $\Omega$. The fine scale in the multiscale problem is usually defined by a priori chosen microscopic approximation spaces $U_h \subset U, V_h \subset V$, also typically chosen as piecewise polynomial functions on a uniform fine partition $\mathcal{T}_h$ of $\Omega$. For suitable choices of polynomial degrees and meshes the spaces should satisfy $U_H \subset U_h \subset U$, and $V_H \subset V_h \subset V$, respectively. In this setting, let us denote with $\pi_{U_H} : U \to U_H$, $\pi_{V_H} : V \to V_H$ projections into the coarse spaces. We then define fine parts of $U, U_h$, or $V, V_h$ through

$$U_f := \{u \in U : \pi_{U_H}(u) = 0\}, U_{f,h} := \{u_h \in U_h : \pi_{U_H}(u_h) = 0\},$$
$$V_f := \{v \in V : \pi_{V_H}(v) = 0\}, V_{f,h} := \{v_h \in V_h : \pi_{V_H}(v_h) = 0\}.$$

The solution $u_\mu^\epsilon \in U$ of (1.1) can now be decomposed into $u_c + u_f \in U_H \oplus U_f$, satisfying

$$R_\mu^\epsilon[u_c + u_f](v_H) = 0 \qquad \forall v_H \in V_H, \tag{2.1}$$
$$R_\mu^\epsilon[u_c + u_f](v_f) = 0 \qquad \forall v_f \in V_f. \tag{2.2}$$

This is a coupled system of a macroscopic and a fine scale variational problem for $(u_H, u_f)$ that is equivalent to the original formulation (1.1).

A discrete counterpart is immediately defined by replacing $U_f, V_f$ by its discrete counterparts $U_{f,h}, V_{f,h}$, i.e. $u_{\mu,h}^\epsilon \in U_h$ is defined through its decomposition $u_{\mu,h}^\epsilon = u_H + u_{f,h} \in U_H \oplus U_{f,h}$, satisfying

$$R_\mu^\epsilon[u_H + u_{f,h}](v_H) = 0 \qquad \forall v_H \in V_H, \tag{2.3}$$
$$R_\mu^\epsilon[u_H + u_{f,h}](v_{f,h}) = 0 \qquad \forall v_{f,h} \in V_{f,h}. \tag{2.4}$$

Depending on the choice of trail and test functions, and by further localization of the fine scale equation (2.4) a variety of numerical multiscale methods can be recovered. For a detailed derivation of the multiscale finite element method, the variational multiscale method, and the heterogeneous multiscale method in such a framework we refer to the expositions in [26] and [18].

To exemplify this approach, let us derive the heterogeneous multiscale method in the case of the simple elliptic multiscale problem (1.2), (1.3). Therefore, we first introduce a partition $\mathcal{T}_H$ of $\Omega$ and a macroscopic conforming discrete function spaces $U_H = V_H \subset H_0^1(\Omega)$, e.g. by choosing globally continuous, piecewise polynomial finite element spaces. Furthermore, we choose quadrature rules $(\omega_{T,q}, x_{T,q})_{q=1}^Q$ for $T \in \mathcal{T}_H$. With each quadrature point $x_{T,q}$, we associate a local function space

$$U_{f,x_{T,q}}^\delta := \{u_{f,x_{T,q}} = u_{f,h}|_{Y^\delta(x_{T,q})} : u_{f,h} \in U_{f,h})\}$$

where $Y^\delta(x_{T,q})$ is an appropriate discrete $\delta$-environment of $x_{T,q}$ that can be decomposed with elements from the fine mesh $\mathcal{T}_h$. We define local corrector operators $\mathcal{Q}_{x_{T,q}} : U_H \to U^\delta_{f,x_{T,q}}$ through

$$R^\epsilon_\mu[u_H + \mathcal{Q}_{x_{T,q}}(u_H)](v_{f,x_{T,q}}) = 0 \qquad \forall v_{f,x_{T,q}} \in U^\delta_{f,x_{T,q}}. \tag{2.5}$$

The definition of the correctors is a localized form of the fine scale equation (2.4). A corresponding local reconstruction operator $\mathcal{R}_{x_{T,q}}$ is then given as

$$\mathcal{R}_{x_{T,q}}(u_H) = u_H + \mathcal{Q}_{x_{T,q}}(u_H) \tag{2.6}$$

and we obtain the heterogeneous multiscale solution $u_H \in U_H$ by using numerical quadrature and replacing $u_H + u_{f,h}$ by the localized reconstruction $\mathcal{R}_{x_{T,q}}(u_H)$ in the coarse scale equation (2.3). We thus obtain, e.g. in the simple elliptic multiscale problem (1.2), (1.3) that $u_H$ satisfies

$$\sum_{T\in\mathcal{T}_H}\sum_{q=1}^{Q}\omega_{T,q}\fint_{Y^\epsilon(x_{T,q})}\mu_1 A^\epsilon\nabla\mathcal{R}_{x_{T,q}}(u_H)\cdot\nabla\Phi_H = \int_\Omega \mu_2 f\Phi_H, \qquad \forall\Phi_H\in U_H \tag{2.7}$$

where we suppose $\epsilon < \delta$, i.e. $Y^\epsilon(x_{T,q}) \subset Y^\delta(x_{T,q})$, and again $Y^\epsilon(x_{T,q})$ is an appropriate discrete $\delta$-environment of $x_{T,q}$ that can be decomposed with elements from the fine mesh $\mathcal{T}_h$.

**3. A priori versus a posteriori based construction of approximation spaces.** Discretizations of partial differential equations are based on approximations in finite dimensional spaces $U_h, V_h$ that are supposed to approximate $U, V$. If the finite dimensional spaces are chosen as subspaces, a discrete formulation can be obtained e.g. by Galerkin projection of the original problem onto the discrete spaces. The approximate model for (1.1) then reads: Find $u^\epsilon_{\mu,h} \in U_h$ such that

$$R^\epsilon_\mu[u^\epsilon_{\mu,h}](v_h) = 0 \qquad \forall v_h \in V_h. \tag{3.1}$$

Concerning the construction of the approximation spaces, we distinguish between a priori and a posteriori approaches. In an a priori approach, the construction reflects a priori error estimates that are typically of the form

$$||u^\epsilon_\mu - u^\epsilon_{\mu,h}|| \le C \inf_{u_h\in U_h} ||u^\epsilon_\mu - u_h|| \tag{3.2}$$

In this case, the true error is estimated by the best-approximation error of an element $u^\epsilon_\mu \in U$ in the discrete approximation space $U_h$. Appropriate discretization spaces in this setting should thus be able to have uniform approximation quality with respect to arbitrary elements from $U$. In grid based methods, such spaces are typically constructed as piecewise polynomial spaces on uniformly refined grids. For multiscale problems, however, the usage of piecewise polynomial spaces would lead to very fine meshes, as fast data oscillations would have to be resolved, although the global behavior of the solutions could be well represented with polynomials on coarse meshes. A way out of this discrepancy between data approximation and representation of the macroscopic behavior of the solution is the construction of locally adapted basis functions that incorporate local responses of the solution operator. This is the basis of

most numerical multiscale techniques. A priori error estimates are meanwhile available for all established classes of numerical multiscale methods in the case of linear elliptic or parabolic equations in the case of periodic or stochastic homogenization problems. Moreover, there are also results for the multiscale finite element method and the heterogeneous multiscale method for quasilinear and monotone elliptic homogenization problems [12, 19, 4, 14].

On the other hand, in an a posteriori approach, the choice of discretization spaces reflect a posteriori error estimates of the form

$$||u_\mu^\epsilon - u_{\mu,h}^\epsilon|| \leq \eta(u_{\mu,h}^\epsilon) \tag{3.3}$$

where $\eta(u_{\mu,h}^\epsilon)$ denotes an a posteriori indicator that can be computed from the approximate solution. Different from the a priori approach, the estimate in this case incorporates approximate information of the underlying problem and is not purely related to approximation properties of function spaces. For grid based discretization methods, such a posteriori error estimates enable to construct approximation spaces that are specifically tailored to approximate the solution of the particular partial differential equation. The construction is usually done by $h$, $p$, or $r$-refinement, or a combination of these. In the context of multiscale problems, such approach naturally induces an adaptive construction of macroscopic meshes, and a correlated adaptive construction of locally adapted basis functions. Also here, the resulting approximation spaces are specifically constructed to approximate one particular solution of the underlying variational problem.

A posteriori error estimates for multiscale methods have been derived in the context of the variational multiscale method (cf. [24, 26] and references therein), and for the heterogeneous multiscale method (cf. [30, 3, 17] and references therein). To our knowledge, so far no a posteriori results are available for the multiscale finite element method or multiscale finite volume approaches. In the particular case of the above defined heterogeneous multiscale method for linear elliptic multiscale problems (2.5), (2.7) an a posteriori error estimate was first derived for the periodic homogenization case in [30], based on a comparison of the heterogeneous multiscale method with a standard finite element approximation with quadrature of the corresponding two scale homogenized variational formulation. The a posteriori error estimate is localized and contains error indicators that allow to construct adapted function space, both at the macro scale and the fine scale. The construction of the function spaces is based on local grid adaptivity. Moreover, the error indicators also give information on how to choose the fine grid size with respect to the coarse grid size.

If not only one variational problem is considered, but rather a whole class of parametrized problems, a classical a posteriori approach based on local refinement would result in the construction of different approximation spaces for different choices of parameters. On the other hand, an a priori approach would allow to stay with one approximation space for the whole class of problems at the price of a usually very high computational complexity, as this function space would not take into account the particular structure of the solution manifold. Model reduction techniques such as the reduced basis approach try to combine the benefits of a priori and a posteriori approaches in the context of parametrized problems. The basic idea is to start from an a priori choice of the approximation space (usually very high dimensional) and to construct a reduced subspace (usually very low dimensional) in an a posteriori manner

that is tailored to approximate the solution manifold of the underlying parametrized problem.

In the next Section we review the construction process of reduced approximation spaces in the reduced basis framework in more detail and finally discuss suitable adjustments in the multiscale setting in Section 5.

**4. Model reduction with reduced basis techniques.** Let us look at parametrized multiscale problems of type (1.1) for fixed $\epsilon$ in the classical reduced basis framework [31, 32] where $\mu \in \mathcal{P} \subset \mathbb{R}^p$ denotes a parameter vector. For simplicity of presentation, let us suppose that the trial and test spaces for problem (1.1) coincide, i.e. $U = V$ and that we are given an a priori chosen high dimensional approximation space $V_h$ such that the discretization error for the discrete problem (3.1) is negligible for arbitrary $\mu \in \mathcal{P}$.

Reduced basis methods can now be summarized as follows: For any subspace $V_{red} \subset V_h$, define $u^\epsilon_{\mu_0,red} \in V_{red}$ as solution of the reduced problem

$$R^\epsilon_\mu[u^\epsilon_{\mu,red}](v_{red}) = 0 \qquad \forall v_{red} \in V_{red} \tag{4.1}$$

and let us suppose that we are given an a posteriori error estimate

$$||u^\epsilon_{\mu,h} - u^\epsilon_{\mu,red}|| \leq \eta_{red}(u^\epsilon_{\mu,red}). \tag{4.2}$$

Based on such a posteriori error estimate a sequence of reduced spaces $V_N \subset V_h$, $N = 0, 1, \ldots$ with dimension $N$ are constructed from (3.1) with the following iterative Greedy algorithm:

1. Let a tolerance $TOL > 0$ be given and define a suitable finite training set $\Sigma_{train} \subset \mathcal{P}$.
2. Choose a parameter $\mu_0 \in \Sigma_{train}$, compute $u^\epsilon_{\mu_0,h} \in V_h$ from (3.1) and define $V_0 := \text{span}(u^\epsilon_{\mu_0,h})$.
3. Let $V_{N-1}$ be given. To construct $V_N$, first compute

   $$\mu_N := \arg\max_{\mu \in \Sigma_{train}} \eta_{red}(u^\epsilon_{\mu,N-1})$$

   where $u^\epsilon_{\mu,N-1} \in V_{N-1}$ is a solution of (4.1) for $V_{red} = V_{N-1}$. Compute $u^\epsilon_{\mu_N,h} \in V_h$ from (3.1) and define

   $$V_N := \text{span}(V_{N-1}, u^\epsilon_{\mu_N,h}).$$

4. If $\eta_{red}(u^\epsilon_{\mu_N,N-1}) > \text{TOL}$, set $N := N + 1$ and continue with step 3, else stop and return $V_N$.

The resulting space $V_N$ is called reduced space and a basis $\Phi_N$ of $V_N$ reduced basis.

Given an arbitrary $\mu \in \mathcal{P}$, the reduced basis approximation $u^\epsilon_{\mu,N} \in V_N$ is defined through (4.1) for $V_{red} = V_N$.

It is clear from construction that the dimension of the reduced problem is given by $N := \dim(V_N)$ and it can be expected form a priori analysis [9] that $N$ can be chosen very small in comparison to $\dim(V_h)$. However, the calculation of the reduced Fréchet derivatives still has high dimensional complexity of order $\mathcal{O}(\dim(V_h))$. To deal with this complexity, the calculation of reduced Fréchet derivatives is split into an offline phase with high dimensional parameter independent calculations and an online phase with only low dimensional parameter dependent calculations. Such procedure,

however, is only possible if the residual $R^\epsilon_\mu$ (and its Fréchet derivatives) allow an affine parameter decomposition of the following form

$$R^\epsilon_\mu[v](w) = \sum_{q=1}^{Q} a_q(\mu) R^\epsilon_q[v](w) \tag{4.3}$$

with parameter independent residuals $R^\epsilon_q, q = 1, \ldots, Q$. If such a decomposition is given, $R^\epsilon_q[v](w)$ can be computed once with high complexity in an offline phase for all base function $v, w$ of $\Phi_N$, while in the online phase only the coefficients $a_q$ have to be evaluated for the chosen parameter $\mu$ and the summation in (4.3) can be evaluated with low complexity of order $\mathcal{O}(Q \times N^2)$.

Note that for simple linear elliptic problems of type (1.2), (1.3) a decomposition of the form (4.3) is naturally given, while in more general (in particular nonlinear) situations, a decomposition of form (4.3) has to be constructed in an approximate sense which can be achieved by an empirical operator interpolation [10].

A crucial assumption in the construction process of the reduced basis space $V_N$ is that we can define a relatively small finite training set $\Sigma_{train}$ that accurately samples the parameter space $\mathcal{P}$. Such assumption usually restricts the application of the reduced basis method to moderate dimensions of the parameter spaces with typical dimensions up to about ten. In our multiscale setting (1.1), however, we also want to treat the case of parameter functions $\mu : \Omega \to \mathbb{R}^p, p \in \mathbb{N}$ instead of parameter vectors $\mu \in \mathbb{R}^p$ that can be interpreted as only constant functions in the more general setting. It is clear that in the general case of parameter functions, one needs additional structural assumptions on the class of admissible parameter functions. In the context of multiscale problems, we naturally assume that the parameter functions are smooth functions on the macroscopic scale, i.e. they can be represented well in $U_H$ and that they are chosen from some compact low dimensional manifold that is embedded in a high dimensional or even infinite dimensional function space.

**5. Model reduction approach for multiscale problems.** Let us now focus on applying the reduced basis approach to our multiscale setting (1.1) including the generalization to smooth parameter functions $\mu : \Omega \to \mathbb{R}^p, p \in \mathbb{N}$. The classical reduced basis approach corresponds to constant parameter functions and is based on a linear expansion

$$u^\epsilon_{\mu,N}(x) = \sum_{i=1}^{N} a_i \phi_i(x), \qquad x \in \Omega \tag{5.1}$$

where $\phi_i \in \Phi_N$ are basis functions of $V_N$. Hence, the spatial variation of the solution is represented by the globally defined basis functions only. The principal idea to generalize the reduced basis approach in our multiscale setting is to replace the linear combination of reduced basis functions in (5.1) by a suitable generalized nonlinear combination

$$u^\epsilon_{\mu,N} = S(\phi_1, \ldots, \phi_N) \tag{5.2}$$

with the hope to significantly reduce the number $N$ of reduced basis functions needed to represent the solution manifold of the underlying parametrized problem.

Particular realizations of this approach are the local reduced basis discontinuous Galerkin method [23], the localized reduced basis multiscale method [5], and also the

reduced basis element approach [25] can be seen in this context. Another approach that fits into this context is the mixed multiscale finite element method using limited global information [1]. In all these cases the nonlinear combination is chosen as

$$S(\phi_1, \dots, \phi_N) = \sum_{i=1}^{N} a_i(x)\phi_i(x), \tag{5.3}$$

where the coefficients $a_i, i = 1, \dots N$ are now supposed to be macroscopic functions that are able to take care of the macroscopic spatial variation of the solution manifold. Coming back to the notation introduced in Subsection 2, let us assume $a_i \in U_H$, while $\phi_i \in V_N \subset U_h$. The reduced multiscale solution space $U_{H,N}$ is then defined as

$$U_{H,N} := \{u_{H,N}(x) = \sum_{i=1}^{N} a_i(x)\phi_i(x) | a_i \in U_H, \phi_i \in \Phi_N\}. \tag{5.4}$$

If $U_{H,N}$ is a conforming subset of $U$, a multiscale reduced solution is immediately defined as Galerkin projection into this space. This would be for instance the case for $U_H \subset U_h \subset U$. A prominent example is obtained, if $U_H$ is chosen as a globally continuous piecewise linear finite element space on a coarse grid $\mathcal{T}_H$ that is embedded in a fine grid $\mathcal{T}_h$ associated with the globally continuous piecewise linear finite element space $U_h$. In this case the resulting method can be seen as a particular variant of the generalized finite element method [8, 27] or an equivalent partition of unity method [7].

In the most simple case, however, $U_H$ is chosen as piecewise constant functions. As in that case $U_{H,N}$ is non-conforming, a corresponding reduced multiscale approximation can be defined e.g. via discontinuous Galerkin projection. If $U_h$ is chosen as the globally continuous piecewise linear finite element space corresponding to the fine grid $\mathcal{T}_h$, the resulting method is the local reduced basis discontinuous Galerkin method [23]. If also $U_h$ is a non-conforming DG space, the resulting method corresponds to the localized reduced basis multiscale method [5].

As an particular example of the new conceptual approach, let us now present the local reduced basis discontinuous Galerkin method [23] in more detail where we assume that $U_H$ is chosen as a piecewise constant function space on a coarse mesh $\mathcal{T}_H$. Let us also assume that a localized reduced basis on each coarse grid cell $T \in \mathcal{T}_H$ is given, i.e. $\Phi_T := \{\varphi_T^1, \dots, \varphi_T^{N_T}\} \subset S_{h,k}(T)$ where $S_{h,k}(T)$ denotes the restriction of $U_h := S_{h,k}$ to the coarse grid cell $T$. Here $S_{h,k}$ denotes the globally continuous finite element space with local polynomials of degree $k$ on a fine mesh $\mathcal{T}_h$. In the simplest form $\Phi_T$ is obtained from a global reduced basis $\Phi_N$ by restriction to $T$. In that case we have $N_T = N$, but also more localized choices are possible as discussed in Section 6 below. We then define the coarse scale reduced broken space $U_{h,N}$ by

$$U_{H,N} = \{u_{H,N} \in L^2(\Omega) | u_{H,N}|_T \in \operatorname{span}(\Phi_T) \, \forall T \in \mathcal{T}_H\}$$

where $N = \dim(U_{H,N}) = \sum_{T \in \mathcal{T}_H} N_T$ denotes the degrees of freedom of the macroscopic discontinuous reduced space. The reduced basis multiscale discontinuous Galerkin scheme is then defined for the simple model problem (1.2), (1.3) as follows. Find $u_{H,N} \in U_{H,N}$ such that

$$R_{DG_\mu^\epsilon}[u_{H,N}](v_{H,N}) = 0, \quad \forall v_{H,N} \in U_{H,N} \tag{5.5}$$

where the DG residual $R_{DG_\mu^\epsilon}$ is e.g. defined as $R_{DG_\mu^\epsilon}[u] := \mathcal{A}_{DG_\mu^\epsilon}[u] - F_\mu$ with

$$\mathcal{A}_{DG_\mu^\epsilon}[u](v) := \sum_{T\in\mathcal{T}_H} \int_T \mu_1 A^\epsilon \nabla u \cdot \nabla - \sum_{e\in\mathcal{E}} \int_e \{\mu_1 A^\epsilon \nabla v \cdot n_e\}[w] - \sum_{e\in\mathcal{E}} \int_e \{\mu_1 A^\epsilon \nabla w \cdot n_e\}[v] + J_{\sigma,\beta}(v,w). \tag{5.6}$$

Here the penalty contribution is given through $J_{\sigma,\beta}(v,w) = \sum_{e\in\mathcal{E}} \frac{\sigma}{|e|^\beta} \int_e [v][w]$ with stabilization parameters $\sigma > 0$, $\beta > 0$. As usual in the context of discontinuous Galerkin we denote averages across edges as $\{v\}$, jumps as $[v]$, and a normal with defined direction on an edge $e \in \mathcal{E}$ with $n_e$. For further details and other possible choices of discontinuous Galerkin methods for elliptic equations we refer to [6].

For an a posteriori analysis of (5.5), (5.6) and a numerical evaluation of its efficiency we refer to [23]. Note that also for this localized reduced basis method the a posteriori error estimate is used in the reduced basis construction step with a modified version of the Greedy algorithm introduced in Section 4 above.

A particular difference in the model reduction based approach for multi-scale problems in comparison to classical numerical multiscale methods, is the multiplicative splitting of macroscopic and fine scale approximation spaces (5.4) instead of an additive splitting as depicted in (2.3), (2.4). Another difference is the offline/online splitting, although such ideas could be also incorporated in classical multiscale approximations. In the next section we will discuss complexity issues in multi-query scenarios and address the issue of localization in combination with the model reduction approach for multiscale problems to further reduce computational complexity.

**6. Offline versus online computational complexity and localization.** Depending on the choice of $U_H$ and $V_N$ in the definition of $U_{H,N}$, our approach of model reduction for multiscale problems is able to interpolate between a resolved standard discretization on a fine grid ($U_H := U_h, V_N := \mathbb{P}_0$) and the classical reduced basis method ($U_H := \mathbb{P}_0, V_N$ as defined in Section 4). It is clear from these extreme scenarios that the optimal choice of a multiscale method in this class depends on the given multi-query scenario. While the offline complexity is very high in the classical RB framework, the online complexity is maximal for a standard discretization on a fine grid. Let us now look at the complexities in more detail. In particular, let us denote $N_H := \dim(U_H)$, $N_h := \dim U_h$ , and $N := \dim(V_N)$. Then the offline complexity is of polynomial order in $(NN_h)$, while the online complexity is of polynomial order in $(NN_H)$ where we assume that an offline/online decomposition as discussed in Section 4 is possible. As $N = 1$ for $N_H = N_h$ and $N = N_{\max}$ for $N_H = 1$, the choice of $N_H$ has a direct influence in the ratio between offline and online complexity and thus can be chosen in dependence of the number of problem evaluations that are expected in the online phase in the context of a multi-query scenario. For a more detailed numerical study of actual CPU-times for varying choices of $N_H$ we refer to [5] where such analysis is done for large scale problems in the context of heterogeneous flow in porous media for the localized reduced basis multiscale method.

In contrast to traditional multiscale methods, our model reduction based approach requires a certain number of global fine scale solutions in the offline phase. Such global solutions are also incorporated in traditional multiscale methods (see e.g.

[1]), but might be relaxed by suitable localization, e.g. in the context of nonlinear domain decomposition strategies [28, 29, 22]. For model reduction based multiscale approaches as introduced above, there are also several possibilities for localization. A first approach relies on an adaptive local coarsening of the reduced space $V_N$ which can be efficiently obtained by an successive application of local POD projections. Such ideas are investigated in [23, 5]. A second approach would be the usage of classical localized numerical multiscale methods in the offline construction process of the reduced basis space or an adaption of nonlinear domain decomposition in this step. However, to our knowledge such combination of methods has not been presented so far. Let us summarize, the such additional localization procedures will result in a further reduction of offline, as well as online computational complexity and will be subject of further investigations.

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