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FINITE ELEMENT APPROXIMATION OF STOKES-LIKE SYSTEMS WITH IMPLICIT CONSTITUTIVE RELATION*

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Abstract. The paper deals with the numerical simulations of steady flows if incompressible fluids whose stress-strain relation is given through an implicit function. In particular, the stress-power law is studied and compared to the classical power law. We propose several formulations of the problem, their stable approximations and particular examples of finite element spaces. The method is demonstrated by numerical results.

Key words. incompressible fluid, non-Newtonian fluid, implicit constitutive relation, finite element method

AMS subject classifications. 35Q30, 65N30, 76D07

1. Introduction. It is well-known that for many fluids the Newton's postulate of linear relation between the traceless part $\mathbb{S}$ of the Cauchy stress $\mathbb{T}$ and the symmetric velocity gradient $\mathbb{D}$ is invalid. One of mostly used class of non-Newtonian models, taking into account the dependence of the viscosity on the shear rate, determines $\mathbb{S}$ as a possibly nonlinear function of $\mathbb{D}$. Despite the generality, there are some fluid materials for which this explicit dependence of the stress is not true, e.g. viscoplastic fluids. As pointed out in Málek et al. [5], the concept is also unnatural from the viewpoint of causality, namely that the force is expressed in terms of the kinematic quantities that reflect the deformation. They propose a solution through the stress-power laws that express $\mathbb{D}$ in terms of powers of $\mathbb{S}$. A general framework of Rajagopal and Srinivasa [7] unifies the above mentioned approaches allowing $\mathbb{T}$ and $\mathbb{D}$ to be related in an implicit way.

To be more specific, we consider steady flows of incompressible fluids whose motion is described by the equations

$$-\operatorname{div}\mathbb{S} + \nabla p = \rho\boldsymbol{f}, \quad \operatorname{div}\boldsymbol{v} = 0, \quad \mathbb{G}(\mathbb{S},\mathbb{D}) = 0 \text{ in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded domain with Lipschitz boundary, ρ, $\boldsymbol{v}$, $\boldsymbol{f}$ is the density, the velocity and the body force, respectively,

$$\mathbb{D} = \mathbb{D}(\boldsymbol{v}) := \frac{1}{2}\left(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^\top\right)$$

and $\mathbb{G} : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is a continuous tensor function. For example, $\mathbb{G}$ can take the following forms:

- Fluids with shear rate dependent viscosity (power-law fluids)

$$\mathbb{G}(\mathbb{S},\mathbb{D}) = \mathbb{S} - \mu(1+\lambda|\mathbb{D}|^2)^{\frac{r-2}{2}}\mathbb{D}; \tag{1.2}$$

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- Stress-power-law fluids

$$\mathbb{G}(\mathbb{S},\mathbb{D}) = \mu^{-1}(1+\lambda\mu^{-2}|\mathbb{S}|^2)^n\mathbb{S} - \mathbb{D},\ n = \frac{2-r}{2(r-1)}. \tag{1.3}$$

The scaling has been chosen in such a way that the asymptotic behaviour of both models is the same for given $\mu, \lambda > 0$ and $r \in (1,\infty)$. Observe that when $\mathbb{G}$ takes the form (1.2) or (1.3), the implicit relation $(1.1)_3$ automatically implies the incompressibility constraint

$$\operatorname{tr}\mathbb{D}\boldsymbol{v} = \operatorname{div}\boldsymbol{v} = 0.$$

The system has to be completed by an appropriate boundary condition, for instance by the no-slip condition

$$\boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega.$$

The difficulty of the general approach stems from the fact that the stress cannot be eliminated from the governing equations of motion. Hence the traditional velocity-pressure formulation does not apply. The aim of this paper is to propose several suitable formulations involving the stress and their finite element approximation.

Mathematical analysis of incompressible fluids with general implicit relation $(1.1)_3$ is far from being established, the author is aware only of the references [2, 5].

2. Weak formulations. We will present several formulations of problem (1.1) involving Cauchy stress or its deviatoric part as the unknown. In all cases the stress is assumed to be only integrable while the derivatives are left on the velocity. We refer to [3] for a different approach where the stress function space is a subspace of $\mathbb{H}(\Omega;\operatorname{div})$.

Throughout the paper the following notation will be used.

$$\begin{aligned}
L^q_0 &:= \left\{\psi \in L^q(\Omega);\ \int_\Omega \psi = 0\right\},\\
\boldsymbol{W}^{1,q}_0 &:= \left\{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,q}(\Omega);\ \operatorname{Tr}\boldsymbol{\varphi} = \boldsymbol{0} \ \text{ on } \partial\Omega\right\},\\
\mathbb{L}^q &:= \left\{\boldsymbol{\xi} \in L^q(\Omega;\mathbb{R}^{d\times d}_{sym});\ \operatorname{tr}\boldsymbol{\xi} \in L^q_0\right\},\\
\mathbb{L}^q_0 &:= L^q(\Omega;\mathbb{R}^{d\times d}_{sym,\mathrm{tr}=0}).
\end{aligned}$$

The first formulation reads:

Problem (A). Find $(\mathbb{S},\boldsymbol{v},p) \in \mathbb{L}^{r'}_0 \times \boldsymbol{W}^{1,r}_0 \times L^{r'}_0$ such that

$$\begin{aligned}
(\mathbb{S},\mathbb{D}\boldsymbol{\varphi}) - (p,\operatorname{div}\boldsymbol{\varphi}) &= (\boldsymbol{f},\boldsymbol{\varphi}) && \forall\boldsymbol{\varphi} \in \boldsymbol{W}^{1,r}_0,\\
(\psi,\operatorname{div}\boldsymbol{v}) &= 0 && \forall\psi \in L^{r'}_0,\\
(\mathbb{G}(\mathbb{S},\mathbb{D}\boldsymbol{v}),\boldsymbol{\xi}) &= 0 && \forall\boldsymbol{\xi} \in \mathbb{L}^{r'}_0.
\end{aligned}$$

If one considers the full Cauchy stress as an unknown then it is possible to eliminate the pressure as it can be seen from the next formulation:

Problem (B). Find $(\mathbb{T},\boldsymbol{v}) \in \mathbb{L}^{r'} \times \boldsymbol{W}^{1,r}_0$ such that

$$\begin{aligned}
(\mathbb{T},\mathbb{D}\boldsymbol{\varphi}) &= (\boldsymbol{f},\boldsymbol{\varphi}) && \forall\boldsymbol{\varphi} \in \boldsymbol{W}^{1,r}_0,\\
\big(\mathbb{G}(\mathbb{T}^\delta,\mathbb{D}\boldsymbol{v}),\boldsymbol{\xi}\big) &= 0 && \forall\boldsymbol{\xi} \in \mathbb{L}^{r'}.
\end{aligned}$$

Here $\mathbb{T}^\delta := \mathbb{T} - \frac{1}{d}(\operatorname{tr}\mathbb{T})\mathbb{I}$ stands for the deviatoric part of $\mathbb{T}$.

In the last case we relax the constitutive relation by adding the symmetric velocity gradient to the list of unknowns, which yields:

Problem (C). Find $(\mathbb{T}, \boldsymbol{v}, \mathbb{D}) \in \mathbb{L}^{r'} \times \boldsymbol{W}_0^{1,r} \times \mathbb{L}_0^r$ such that

$$\begin{aligned}
(\mathbb{T}, \mathbb{D}\boldsymbol{\varphi}) &= (\boldsymbol{f}, \boldsymbol{\varphi}) && \forall \boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,r}, \\
\big(\mathbb{G}(\mathbb{T}^\delta, \mathbb{D}), \eta\big) &= 0 && \forall \eta \in \mathbb{L}_0^r, \\
(\mathbb{D} - \mathbb{D}\boldsymbol{v}, \boldsymbol{\xi}) &= 0 && \forall \boldsymbol{\xi} \in \mathbb{L}^{r'}.
\end{aligned}$$

Let us note that for the Stokes problem, i.e.

$$\mathbb{G}(\mathbb{S}, \mathbb{D}) = \mathbb{S} - \mathbb{D},\ r = 2, \tag{2.4}$$

the existence of a unique weak solution is well known. To keep ideas clear, we will restrict to this case in the next section.

3. Numerical analysis of problems (A)-(C). In this section we investigate the general conditions on the stability of the finite element approximation of problems (A)-(C) under the assumption (2.4).

Let $\mathcal{T}_h$ be a partition of Ω into simplices with the norm $h := \max_{K \in \mathcal{T}_h} \operatorname{diam} K$. The discrete formulations are based on the Galerkin method, i.e. we seek the approximate solutions in finite dimensional spaces $\mathbb{L}_h \subset \mathbb{L}^2$, $\mathbb{L}_{0h} \subset \mathbb{L}_0^2$, $\boldsymbol{W}_h \subset \boldsymbol{W}_0^{1,2}$, and $L_h \subset L_0^2$, which are to be determined. In order to emphasize the structure of these problems, we define the operators $A : \mathbb{L}^2 \to (\mathbb{L}^2)^*$, $B_1 : \mathbb{L}^2 \to (\boldsymbol{W}_0^{1,2})^*$ and $B_2 : L_0^2 \to (\boldsymbol{W}_0^{1,2})^*$:

$$\begin{aligned}
\langle A\mathbb{T}, \boldsymbol{\xi} \rangle &:= \left(\mathbb{T}^\delta, \boldsymbol{\xi}^\delta\right), \\
\langle B_1\mathbb{T}, \boldsymbol{\varphi} \rangle &:= -\left(\mathbb{T}, \mathbb{D}\boldsymbol{\varphi}\right), \\
\langle B_2 p, \boldsymbol{\varphi} \rangle &:= \left(p, \operatorname{div} \boldsymbol{\varphi}\right),
\end{aligned}$$

and define:

Problem (A)$_h$. Find $(\mathbb{S}_h, \boldsymbol{v}_h, p_h) \in \mathbb{L}_{0h} \times \boldsymbol{W}_h \times L_h$ such that

$$\begin{bmatrix} A & 0 & B_1' \\ 0 & 0 & B_2' \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{S}_h \\ p_h \\ \boldsymbol{v}_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{f} \end{bmatrix} \quad \begin{array}{l} \text{in } (\mathbb{L}_{0h})^*, \\ \text{in } (L_h)^*, \\ \text{in } (\boldsymbol{W}_h)^*. \end{array}$$

Problem (B)$_h$. Find $(\mathbb{T}_h, \boldsymbol{v}_h) \in \mathbb{L}_h \times \boldsymbol{W}_h$ such that

$$\begin{bmatrix} A & B_1' \\ B_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{T}_h \\ \boldsymbol{v}_h \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{f} \end{bmatrix} \quad \begin{array}{l} \text{in } (\mathbb{L}_h)^*, \\ \text{in } (\boldsymbol{W}_h)^*. \end{array}$$

Problem (C)$_h$. Find $(\mathbb{D}_h, \boldsymbol{v}_h, \mathbb{T}_h) \in \mathbb{L}_{0h} \times \boldsymbol{W}_h \times \mathbb{L}_h$ such that

$$\begin{bmatrix} A & 0 & -A \\ 0 & 0 & -B_1 \\ -A & -B_1' & 0 \end{bmatrix} \begin{bmatrix} \mathbb{D}_h \\ \boldsymbol{v}_h \\ \mathbb{T}_h \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{f} \\ 0 \end{bmatrix} \quad \begin{array}{l} \text{in } (\mathbb{L}_{0h})^*, \\ \text{in } (\boldsymbol{W}_h)^*, \\ \text{in } (\mathbb{L}_h)^*. \end{array}$$

Due to the saddle-point structure of these problems the choice of the finite dimensional spaces is restricted by the requirement to satisfy appropriate inf-sup conditions. At this point we recall some results of the theory of abstract saddle-point problems. We start by the classical statement (see e.g. [1, 3]).

THEOREM 3.1. *Let U and P be reflexive Banach spaces, $A : U \to U^*$ be continuous, $B : P \to U^*$ continuous and linear. Let $Z := \ker B'$. Then the following are equivalent.*

1. *(Existence of solutions) For all $f \in U^*$ and $g \in P^*$ there exists $(u,p) \in U \times P$ such that*
   (a) $A(u) - Bp = f$ *in* U^*, $\quad B'u = g$ *in* P^*,
   (b) $\|u\|_{U/Z} \le \frac{1}{c}\|g\|_{P^*}$.
2. (a) *For all $f \in U^*$ and $u_g \in U$ there exists $u \in U$ such that $u - u_g \in Z$ and*

$$A(u) = f \text{ in } Z^*,$$

   (b) *There exists $c > 0$ such that*

$$\|Bp\|_{U^*} \ge c\|p\|_P. \tag{3.1}$$

Note that (3.1) is equivalent to

$$\inf_{p\in P\setminus\{0\}} \sup_{u\in U} \frac{\langle Bp, u\rangle}{\|u\|_U \|p\|_P} \ge c,$$

usually called the *inf-sup condition* or the *Babuška-Brezzi condition.*

If the problem consists of more constraints, e.g. $P = P_1 \times P_2$, the verification of the inf-sup condition from the above theorem may become troublesome. For this reason we mention the following result for the so-called twofold saddle-point problems.

Theorem 3.2. *Let U, P_1, P_2 be reflexive Banach spaces, $B_1 : U \to P_2^*$, $B_2 : P_1 \to P_2^*$ be continuous and linear. Define $B : U \times P_1 \to P_2^*$ by $B(u,p_1) := B_1u + B_2p_1$. Then the following are equivalent.*

1. *There exist constants $C, c > 0$ such that*

$$\|B'p_2\|_{(U\times P_1)^*} \ge c\|p_2\|_{P_2} \ \ \forall p_2 \in P_2 \text{ and } \|p_1\|_{P_1} \le C\|u\|_U \ \ \forall (u,p_1) \in \ker B.$$

2. *There exists $c > 0$ such that*

$$\|B_2p_1\|_{P_2^*} \ge c\|p_1\|_{P_1} \ \ \forall p_1 \in P_1 \text{ and } \|B_1'p_2\|_{U^*} \ge c\|p_2\|_{P_2} \ \ \forall p_2 \in \ker B_2'.$$

3. *There exists $c > 0$ such that*

$$\|B'p_2\|_{(U\times P_1)^*} \ge c\|p_2\|_{P_2} \ \ \forall p_2 \in P_2 \text{ and } \|B_2p_1\|_{P_2^*} \ge c\|p_1\|_{P_1} \, \forall p_1 \in P_1.$$

In the following subsections we will apply these abstract results on the particular problems and suggest examples of finite element spaces.

3.1. Approximation of problem (A). In this subsection we establish the well-posedness of problem $(\mathrm{A})_h$ under requirements on the finite element spaces.

Theorem 3.3. *Let $\mathbb{L}_{0h}$, L_h, $\boldsymbol{W}_h$ be finite dimensional subspaces of $\mathbb{L}^2_0$, L^2_0, $\boldsymbol{W}^{1,2}_0$, respectively, which satisfy the following conditions:*

(i) *There exists $c > 0$ such that*

$$\sup_{\boldsymbol{\varphi}\in\boldsymbol{W}_h} \frac{\langle B_2p, \boldsymbol{\varphi}\rangle}{\|\boldsymbol{\varphi}\|_{1,2}} \ge c\|p\|_2 \ \ \forall p \in L_h;$$

(ii) $\{\mathbb{D}\boldsymbol{\varphi};\ \boldsymbol{\varphi} \in \boldsymbol{W}_h\} \subset \mathbb{L}_{0h}$.

Then for every $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$ problem $(A)_h$ has a unique solution $(\mathbb{S}_h, p_h, \boldsymbol{v}_h) \in \mathbb{L}_{0h} \times L_h \times \boldsymbol{W}_h$. Further there exists a constant $C > 0$ independent of $\boldsymbol{f}$ such that

$$\|\mathbb{S}_h\|_2 + \|p_h\|_2 + \|\boldsymbol{v}_h\|_{1,2} \le C\|\boldsymbol{f}\|_2.$$

A particular choice of spaces that satisfy the assumptions of Theorem 3.3 is based on the $\mathcal{P}_{k+1}/\mathcal{P}_k$ Taylor-Hood velocity-pressure elements.

COROLLARY 3.4. *Let $k \geq 1$. Then*

$$\begin{aligned}\mathbb{L}_{0h} &:= \{\mathbb{S} \in \mathbb{L}^2_0;\ \mathbb{S}_{|K} \in \mathcal{P}_k(K)^{d\times d}\ \forall K \in \mathcal{T}_h\},\\ L_h &:= \{p \in L^2_0 \cap C(\overline{\Omega});\ p_{|K} \in \mathcal{P}_k(K)\ \forall K \in \mathcal{T}_h\},\\ \boldsymbol{W}_h &:= \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}_0;\ \boldsymbol{v}_{|K} \in \mathcal{P}_{k+1}(K)^d\ \forall K \in \mathcal{T}_h\}\end{aligned}$$

satisfy the hypothesis of Theorem 3.3.

3.2. Approximation of problem (B). Sufficient conditions for the well-posedness of problem $(\mathrm{B})_h$ are given in the following theorem.

THEOREM 3.5. *Let $\mathbb{L}_h$, $\boldsymbol{W}_h$ be finite dimensional subspaces of $\mathbb{L}^2$, $\boldsymbol{W}^{1,2}_0$, respectively, which satisfy the following conditions:*

(i) $\{\mathbb{D}\boldsymbol{\varphi};\ \boldsymbol{\varphi} \in \boldsymbol{W}_h\} \subset \mathbb{L}_h$;

(ii) There exists $c > 0$ such that

$$\sup_{\boldsymbol{\varphi}\in \boldsymbol{W}_h} \frac{\langle B_2(\operatorname{tr}\mathbb{T}), \boldsymbol{\varphi}\rangle}{\|\boldsymbol{\varphi}\|_{1,2}} \geq c\,\|\operatorname{tr}\mathbb{T}\|_2\ \ \forall \mathbb{T} \in \mathbb{L}_h.$$

Then for every $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$ problem $(B)_h$ has a unique solution $(\mathbb{T}_h, \boldsymbol{v}_h) \in \mathbb{L}_h \times \boldsymbol{W}_h$. Further there exists a constant $C > 0$ independent of $\boldsymbol{f}$ such that

$$\|\mathbb{T}_h\|_2 + \|\boldsymbol{v}_h\|_{1,2} \leq C\,\|\boldsymbol{f}\|_2\,.$$

The assumption (ii) arises from the fact that A is not elliptic on $\mathbb{L}_2$. The ellipticity can be fixed if the estimate

$$\|\operatorname{tr}\mathbb{T}\|_2 \leq c\left\|\mathbb{T}^\delta\right\|_2\ \ \forall \mathbb{T} \in \mathbb{L}_h \text{ s.t. } B_1\mathbb{T} = 0 \text{ in } \boldsymbol{W}^*_h$$

holds true, which follows from (ii).

We again describe several finite element combinations that satisfy the hypothesis of Theorem 3.5. The first one is based on the $\mathcal{P}_2+$*bubble*$/\mathcal{P}_1-$*discontinuous* velocity-pressure approximation.

COROLLARY 3.6. *The spaces*

$$\begin{aligned}\mathbb{L}_h &:= \{\mathbb{T} \in \mathbb{L}^2;\ \mathbb{T}_{|K} \in \mathcal{P}_1(K)^{d\times d}\ \forall K \in \mathcal{T}_h\},\\ \boldsymbol{W}_h &:= \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}_0;\ \boldsymbol{v}_{|K} \in \mathcal{P}_2(K)^d \oplus \mathcal{B}_{d+1}(K)^d\ \forall K \in \mathcal{T}_h\},\end{aligned}$$

where $\mathcal{B}_{d+1}(K)$ denotes the one-dimensional space of bubble functions on K, satisfy the hypothesis of Theorem 3.5.

Another case with a discontinuous pressure space is the $\mathcal{P}_{k+1}/\mathcal{P}_k-$*discontinuous* Scott-Vogelius element, which in general leads to unstable approximations, however under some additional assumptions on the topology of $\mathcal{T}_h$ the stability has been established [8, 6, 9].

COROLLARY 3.7. *Let $\mathcal{T}_h$ be a barycentrically refined triangulation and $k \geq d$. Then*

$$\begin{aligned}\mathbb{L}_h &:= \{\mathbb{T} \in \mathbb{L}^2;\ \mathbb{T}_{|K} \in \mathcal{P}_k(K)^{d\times d}\ \forall K \in \mathcal{T}_h\},\\ \boldsymbol{W}_h &:= \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}_0;\ \boldsymbol{v}_{|K} \in \mathcal{P}_{k+1}(K)^d\ \forall K \in \mathcal{T}_h\}\end{aligned}$$

satisfy the assumptions of Theorem 3.5.

The last example is based on the edge stabilization of $\operatorname{tr}\mathbb{T}$ and $\operatorname{div}\boldsymbol{v}$. Let $\mathcal{E}_h$ denote the set of edges in $\mathcal{T}_h$ and $[\![f]\!]$ the jump of f across a given edge.

COROLLARY 3.8. *Let $k \geq 1$ and*

$$\mathbb{L}_h := \{\mathbb{T} \in \mathbb{L}^2;\ \mathbb{T}_{|K} \in \mathcal{P}_k(K)^{d\times d}\ \forall K \in \mathcal{T}_h\},$$
$$\boldsymbol{W}_h := \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,2};\ \boldsymbol{v}_{|K} \in \mathcal{P}_{k+1}(K)^d\ \forall K \in \mathcal{T}_h\}.$$

Define the operators $C_1 : \mathbb{L}^2 \to (\mathbb{L}^2)^*$ and $C_2 : \boldsymbol{W}_0^{1,2} \to (\boldsymbol{W}_0^{1,2})^*$ by the formulae

$$\langle C_1\mathbb{T}, \boldsymbol{\xi}\rangle := \sum_{E\in\mathcal{E}_h} \left([\![\operatorname{tr}\mathbb{T}]\!], [\![\operatorname{tr}\boldsymbol{\xi}]\!]\right)_E,$$
$$\langle C_2\boldsymbol{v}, \boldsymbol{\varphi}\rangle := \sum_{E\in\mathcal{E}_h} \left([\![\operatorname{div}\boldsymbol{v}]\!], [\![\operatorname{div}\boldsymbol{\varphi}]\!]\right)_E.$$

Then for every $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$ and $\gamma > 0$ the problem

$$\begin{bmatrix} A+\gamma C_1 & B_1' \\ B_1 & \gamma C_2 \end{bmatrix} \begin{bmatrix} \mathbb{T}_h \\ \boldsymbol{v}_h \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{f} \end{bmatrix} \quad \begin{matrix} \text{in } (\mathbb{L}_h)^* \\ \text{in } (\boldsymbol{W}_h)^* \end{matrix}$$

has a unique solution $(\mathbb{T}_h, \boldsymbol{v}_h) \in \mathbb{L}_h \times \boldsymbol{W}_h$. Further there exists a constant $C > 0$ independent of $\boldsymbol{f}$ and γ such that

$$\|\mathbb{T}_h\|_2 + \|\boldsymbol{v}_h\|_{1,2} + \gamma \sum_{E\in\mathcal{E}_h} \left(\|[\![\operatorname{tr}\mathbb{T}_h]\!]\|_{2,E} + \|[\![\operatorname{div}\boldsymbol{v}_h]\!]\|_{2,E}\right) \leq C\,\|\boldsymbol{f}\|_2.$$

3.3. Approximation of problem (C). In the case of problem $(\mathrm{C})_h$ the situation is similar as in the previous section.

THEOREM 3.9. *Let $\mathbb{L}_{0h}$, $\boldsymbol{W}_h$, $\mathbb{L}_h$ be finite dimensional subspaces of $\mathbb{L}_0^2$, $\boldsymbol{W}_0^{1,2}$, $\mathbb{L}^2$, respectively, which satisfy the following conditions:*

(i) $\{\mathbb{D}\boldsymbol{\varphi};\ \boldsymbol{\varphi} \in \boldsymbol{W}_h\} \subset \mathbb{L}_h$;

(ii) $\{\mathbb{T}^\delta;\ \mathbb{T} \in \mathbb{L}_h\} \subset \mathbb{L}_{0h}$;

(iii) There exists $c > 0$ such that

$$\sup_{\boldsymbol{\varphi}\in\boldsymbol{W}_h} \frac{\langle B_2(\operatorname{tr}\mathbb{T}), \boldsymbol{\varphi}\rangle}{\|\boldsymbol{\varphi}\|_{1,2}} \geq c\,\|\operatorname{tr}\mathbb{T}\|_2\ \ \forall \mathbb{T} \in \mathbb{L}_h.$$

Then for every $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$ problem $(C)_h$ has a unique solution $(\mathbb{D}_h, \boldsymbol{v}_h, \mathbb{T}_h) \in \mathbb{L}_{0h} \times \boldsymbol{W}_h \times \mathbb{L}_h$. Further there exists a constant $C > 0$ independent of $\boldsymbol{f}$ such that

$$\|\mathbb{D}_h\|_2 + \|\boldsymbol{v}_h\|_{1,2} + \|\mathbb{T}_h\|_2 \leq C\,\|\boldsymbol{f}\|_2.$$

For the particular choice of the finite dimensional spaces, one can use the examples of $\mathbb{L}_h$, $\boldsymbol{W}_h$ from Section 3.2 together with $\mathbb{L}_{0h} := \{\mathbb{T}^\delta;\ \mathbb{T} \in \mathbb{L}_h\}$.

4. Numerical results. The discrete problems $(\mathrm{A})_h$–$(\mathrm{C})_h$ lead to a system of nonlinear algebraic equations

$$\boldsymbol{R}(\boldsymbol{X}) = \boldsymbol{0}, \tag{4.1}$$

where $\boldsymbol{X} \in \mathbb{R}^N$ contains the degrees of freedom. Each component of $\boldsymbol{R}$ is of the form

$$R_i(\boldsymbol{X}) = \sum_{K \in \mathcal{T}_h} \int_K r_K(\boldsymbol{X}, \boldsymbol{\varphi}_i) + \sum_{E \in \mathcal{E}_h} \int_E r_E(\boldsymbol{X}, \boldsymbol{\varphi}_i), \; i = 1, \dots N,$$

where $\boldsymbol{\varphi}_i$ is the appropriate test function. System (4.1) is linearized by the Newton-Raphson method and the resulting system is solved by the sparse direct solver UMFPACK. The numerical method has been implemented in an in-house C++ code, where the entries of the linearized matrix $\nabla \boldsymbol{R}$ are obtained from the definition of r_K and r_E by means of a simple automatic differentiation.

The described approximations have been tested on several model problems. The constitutive relations under considerations were either the power law or the stress-power law.

4.1. Stress-power law in a stenosed channel. The first example models the flow of a stress-power-law fluid through a stenosed channel, whose geometry is depicted in Figure 4.1. For fixed material parameters $\rho = \mu = \lambda = 1$, different values

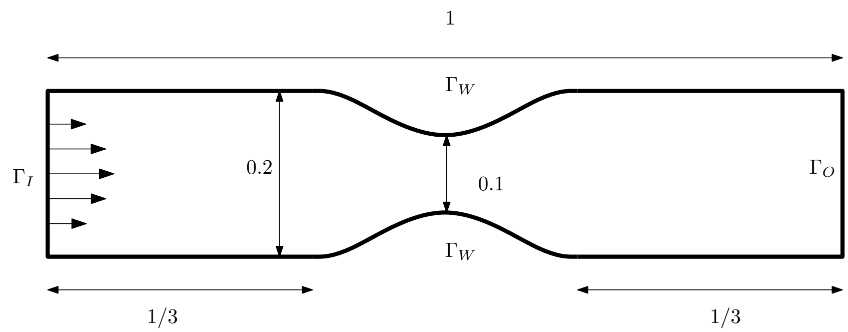

FIG. 4.1. *Geometry of the stenosed channel.*

of r have been used. The following boundary conditions are prescribed:

$$\begin{aligned} \boldsymbol{v} &= (100x_2(0.2 - x_2), 0) && \text{on } \Gamma_I, \\ \boldsymbol{v} &= \boldsymbol{0} && \text{on } \Gamma_W, \\ \mathbb{T}\boldsymbol{n} \cdot \boldsymbol{n} = -p + \mathbb{S}\boldsymbol{n} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_O, \\ \boldsymbol{v} \times \boldsymbol{n} &= 0 && \text{on } \Gamma_O. \end{aligned}$$

We used a computational mesh consisting of 2657 triangles and approximation of problems (A)-(C) with the FE spaces described in Table 4.1. In the cases (B) and (C) either the edge stabilization or the barycentric refinement were used. Apparently,

(A)		(B)		(C)	
$\mathbb{S}$	$\mathcal{P}_1^{disc}$	$\mathbb{T}$	$\mathcal{P}_1^{disc}$	$\mathbb{T}$	$\mathcal{P}_1^{disc}$
$\boldsymbol{v}$	$\mathcal{P}_2$	$\boldsymbol{v}$	$\mathcal{P}_2$	$\boldsymbol{v}$	$\mathcal{P}_2$
p	$\mathcal{P}_1$			$\mathbb{D}$	$\mathcal{P}_1^{disc}$

TABLE 4.1
Tested finite element spaces.

all discrete problems yielded results with negligible differences. Some characteristics of the results are illustrated in Figures 4.2–4.3.

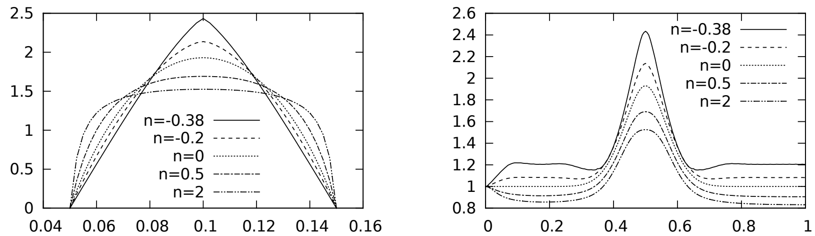

FIG. 4.2. *Velocity in the middle cross-section (left), along the channel (right).*

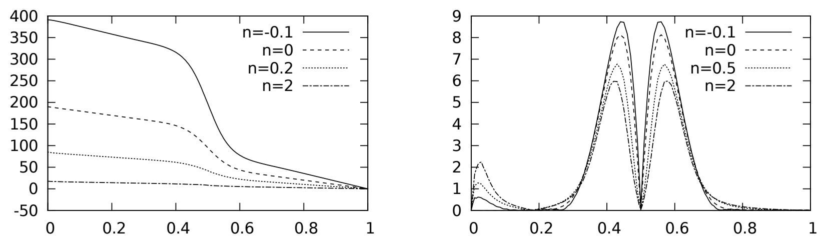

FIG. 4.3. *Pressure (left) and norm of* $\mathbb{D}\boldsymbol{v}$ *(right) along the channel.*

4.2. Benchmark problem: Flow around cylinder. In the next example we compared the classical power-law and stress-power-law models on the benchmark problem of flow around a cylinder [4]. The geometry is depicted in Figure 4.4. The

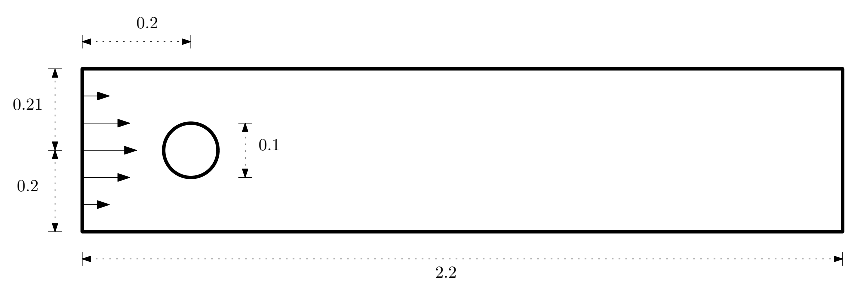

FIG. 4.4. *Flow around a cylinder.*

following boundary conditions were prescribed:

$$
\begin{aligned}
\boldsymbol{v} &= \left(0.3\frac{4x_2(0.41-x_2)}{0.41^2}, 0\right) && \text{on } \Gamma_I,\\
\boldsymbol{v} &= \boldsymbol{0} && \text{on } \Gamma_W,\\
\mathbb{T}\boldsymbol{n}\cdot\boldsymbol{n} = -p + \mathbb{S}\boldsymbol{n}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_O,\\
\boldsymbol{v}\times\boldsymbol{n} &= 0 && \text{on } \Gamma_O.
\end{aligned}
$$

The power-law model was solved using the usual velocity-pressure formulation and $\mathcal{P}_2/\mathcal{P}_1$ Taylor-Hood elements:

$$\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \mu\,\mathrm{div}\left((1+\lambda|\mathbb{D}\boldsymbol{v}|^2)^{\frac{r-2}{2}}\mathbb{D}\boldsymbol{v}\right) + \nabla p = \rho\boldsymbol{f}, \quad \mathrm{div}\,\boldsymbol{v} = 0.$$

For the stress-power-law model we used the formulation $(\mathrm{B})_h$ and $\mathcal{P}_1^{disc}/\mathcal{P}_2$ elements with jump stabilization:

$$\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \operatorname{div}\mathbb{T} = \rho\boldsymbol{f}, \quad \mathbb{D}\boldsymbol{v} = \mu^{-1}(1+\lambda\mu^{-2}|\mathbb{T}^\delta|^2)^n\mathbb{T}^\delta,\ n = \frac{2-r}{2(r-1)}.$$

The material parameters are $\rho = \lambda = 1$, $\mu = 2\cdot 10^{-3}$. Due to the small value of μ it is convenient to substitute $\tilde{\mathbb{T}} := \mu^{-1}\mathbb{T}$ which leads to the system

$$\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \mu\operatorname{div}\tilde{\mathbb{T}} = \rho\boldsymbol{f}, \quad \mathbb{D}\boldsymbol{v} = (1+\lambda|\tilde{\mathbb{T}}^\delta|^2)^n\tilde{\mathbb{T}}^\delta.$$

We evaluated the following quantities:

- pressure drop $\Delta p := p(A) - p(B)$,
- drag coefficient $C_D := 500\displaystyle\int_{\Gamma_S}\mathbb{T}\boldsymbol{n}\cdot(1,0)^\top$,
- lift coefficient $C_L := 500\displaystyle\int_{\Gamma_S}\mathbb{T}\boldsymbol{n}\cdot(0,1)^\top$,

where $A = (0.15, 0.2)$, $B = (0.25, 0.2)$, Γ_S is the surface of the cylinder. The results obtained on a mesh consisting of 3926 triangles are given in Figures 4.5–4.6. For $r = 2$ they are compared to the reference values computed in [4] for the Navier-Stokes system.

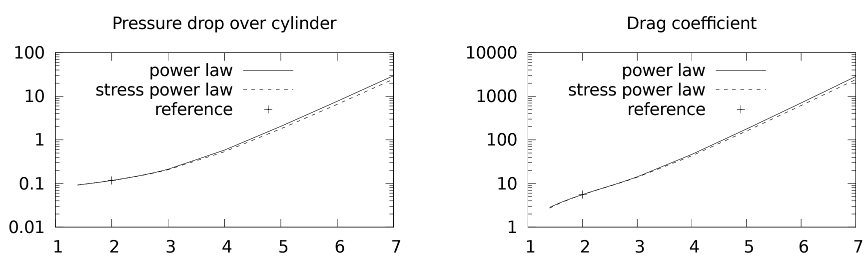

FIG. 4.5. *Flow around cylinder: Pressure drop (left), drag coefficient (right).*

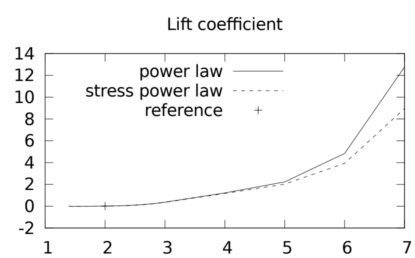

FIG. 4.6. *Flow around cylinder: Lift coefficient.*

For given $r \neq 2$ the velocity fields of power-law and stress-power-law model show no differences while the pressure behaves differently. In particular, in the zone before the cylinder the pressure of the later model is always smaller, as can be seen in Figure 4.8.

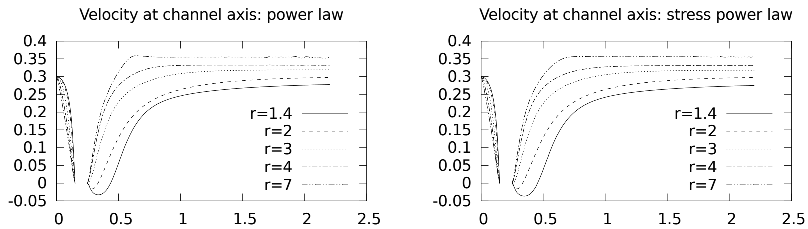

Fig. 4.7. *Velocity $\boldsymbol{v}_1$ at $x_2 = 0.2$: power law (left), stress-power law (right).*

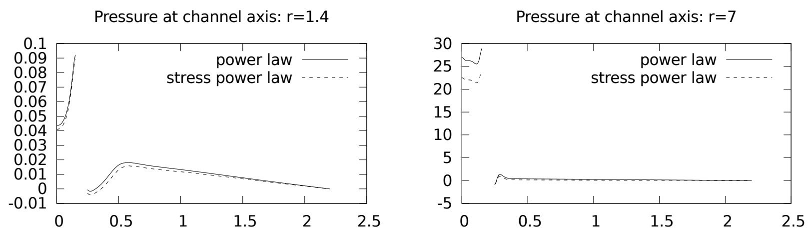

Fig. 4.8. *Pressure p at $x_2 = 0.2$: $r = 1.4$ (left), $r = 7$ (right).*

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