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GENERALIZED FRONTS PROPAGATION ON WEIGHTED GRAPHS*

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Abstract. In this paper, we propose a general formulation and an algorithm for simultaneous propagation of several fronts evolving on a weighted graph. This formulation is an adaptation of the continuous level set formulation for front propagation and uses a Partial difference Equations (PdEs) framework. The proposed algorithm is a graph-based version of the Fast Marching algorithm that allows simultaneous inward or outward propagation of several fronts. Experiments illustrate the behavior of the algorithm and show some application results on several types of data.

Key words. Multiple fronts propagation, weighted graphs, Partial difference Equations, level sets, inward/outward, Fast Marching.

AMS subject classifications. 05C22, 90C35, 39A14

1. Introduction.

Many applications involve data defined on topologically complex domains. These data can be defined on manifolds or irregularly shaped domains, defined on network-like structures, or defined as high dimensional point clouds such as collections of features vectors. Such organized or unorganized data can be conveniently represented as graphs, where the vertices represent initial data and the edges represent interactions between them. Hence, it is very important to transfer many tools which were initially developed on usual Euclidean spaces and proven to be efficient for many problems, to graphs. In this paper, we consider the level set method for front propagation, as it was introduced by Osher-Sethian [1] and extend it to weighted graphs. This transcription is based on a framework of PdEs [2] along with a family of weighted gradients. Then, using the advantages of our graph-based front representation, we extend the initial formulation (that considers a single front evolving outward), to obtain a very general formulation that allows to consider the simultaneous propagation of several fronts evolving inward or outward.

2. Partial difference Equations on graphs.

All operators and definitions presented in this Section were previously introduced in [2] and [3].

A weighted graph $G = (V, E, w)$ consists in a finite set V of vertices and a finite set $E \subseteq V \times V$ of weighted edges. An edge $(u, v) \in E$ connects two adjacent (neighbor) vertices u and v. The neighborhood of a vertex u is noted $N(u) = \{v \in V \setminus \{u\} : (u, v) \in E\}$. The weight $w(u, v)$ of an edge (u, v) can be defined with a function $w : V \times V \to \mathbb{R}^+$ if $(u, v) \in E$, and $w(u, v) = 0$ otherwise. For the sake of simplicity, $w(u, v)$ will be denoted by w_{uv}. Graphs are assumed to be simple, connected and undirected implying that function w is symmetric. Let $f : V \to \mathbb{R}$ be a real-valued function that assigns a real value $f(u)$ to each vertex $u \in V$. We denote by $\mathcal{H}(V)$ the Hilbert space of such functions.

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2.1. Differences and gradients operators on weighted graphs. Let $f : V \to \mathbb{R}$ be a function of $\mathcal{H}(V)$. The *gradient or difference operator* of f, noted $\mathcal{G}_w : \mathcal{H}(V) \to \mathcal{H}(E)$, is defined on an edge $(u,v) \in E$ by $(\mathcal{G}_w f)(u,v) \overset{def.}{=} \gamma\,(w_{uv})\,(f(v) - f(u))$, where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ depends on the weight function (in the sequel we denote $\gamma(w_{uv})$ by γ_{uv}). Based on the previous definitions, the two *upwind gradients* $\mathcal{G}_w^{\pm} : \mathcal{H}(V) \to \mathcal{H}(E)$, are expressed by $\mathcal{G}_w^{\pm} f(u,v) \overset{def.}{=} \gamma_{uv}\big(f(v) - f(u)\big)^{\pm}$, with the notation $(x)^+ = \max(0,x)$ and $(x)^- = -\min(0,x)$.

The *discrete weighted gradient* of a function $f \in \mathcal{H}(V)$, noted $\nabla_w : \mathcal{H}(V) \to \mathbb{R}$, is defined on a vertex $u \in V$ as the vector of all *differences* with respect to the set of edges $(u,v) \in E$ by $(\nabla_w f)(u) \overset{def.}{=} \Big((\mathcal{G}_w f)(u,v)\Big)^T_{v \in V}$. Similarly, *discrete upwind weighted gradients* are defined as $(\nabla_w^{\pm} f)(u) \overset{def.}{=} \Big(\big((\mathcal{G}_w f)^{\pm} f\big)(u)\Big)^T_{v \in V}$.

The $\mathcal{L}_p$ norms, $1 \leqslant p < \infty$ of these gradients : $\|\nabla_w f\|_p$ and $\|\nabla_w^{\pm} f\|_p$, allow to define the notion of the regularity of a function around a vertex. They are expressed as : $\|\big(\nabla_w^{\pm} f\big)(u)\|_p = \big[\sum_{v \in V} \gamma_{uv}^p \big(f(v) - f(u)\big)^{\pm p}\big]^{1/p}$. With the property $\|\big(\nabla_w f\big)(u)\|_p^p = \|\big(\nabla_w^{+} f\big)(u)\|_p^p + \|\big(\nabla_w^{-} f\big)(u)\|_p^p$. Similarly, the $\mathcal{L}_\infty$ norm of these gradients is expressed as : $\|\big(\nabla_w^{\pm} f\big)(u)\|_\infty = \max_{v \in V}\big(\gamma_{uv}|\big(f(v) - f(u)\big)^{\pm}|\big)$. With the property $\|\big(\nabla_w f\big)(u)\|_\infty = \|\big(\nabla_w^{+} f\big)(u)\|_\infty + \|\big(\nabla_w^{-} f\big)(u)\|_\infty$.

All these definitions are provided according to our definition of a weighted graph.

2.2. PdE-based morphological processes. Let $\mathcal{A}$ be a set of connected vertices with $\mathcal{A} \subset V$ such that for all $u \in \mathcal{A}$, there exists a vertex $v \in \mathcal{A}$, with $(u,v) \in E$. We denote by $\partial^+\mathcal{A}$ and $\partial^-\mathcal{A}$: the *external* and *internal* boundary sets of $\mathcal{A}$, respectively $\partial^+\mathcal{A} = \{u \in \overline{\mathcal{A}} : \exists v \in \mathcal{A} \text{ with } (u,v) \in E\}$ and $\partial^-\mathcal{A} = \{u \in \mathcal{A} : \exists v \in \overline{\mathcal{A}} \text{ with } (u,v) \in E\}$, where $\overline{\mathcal{A}} = V \setminus \mathcal{A}$ is the complement of $\mathcal{A}$.

Let $f : V \to \mathbb{R}$ be a function of $\mathcal{H}(V)$. Morphological dilation and erosion processes on f are defined using the previously introduced gradients as the following Partial difference Equations (PdEs) . Respectively $\delta(f)(u) \overset{def.}{=} \partial_t f(u) = \|(\nabla_w^+ f)(u)\|_p$ and $\varepsilon(f)(u) \overset{def.}{=} \partial_t f(u) = -\|(\nabla_w^- f)(u)\|_p$. Intuitively, given a set of vertices $\mathcal{A} \subset V$ and using external and internal graph boundaries equation of dilation over $\mathcal{A}$ can be interpreted as a growth process that adds vertices from $\partial^+\mathcal{A}$ to $\mathcal{A}$. By duality, erosion over $\mathcal{A}$ can be interpreted as a contraction process that removes vertices from $\partial^-\mathcal{A}$ to $\mathcal{A}$.

3. Outward fronts propagation on weighted graphs. In this Section, we will introduce our graph-based equation for outward front propagation. This equation came from the classical continuous level set formulation and can be easily linked with the particular case of the eikonal equation. An efficient algorithm to solve the proposed equation is presented, and an extension to multiple fronts, using the advantages of the graph-based front representation (as a discrete subset), will also be given.

3.1. Front representation. Let $G = (V, E, w)$ be a weighted graph. At time t, a front Γ evolving on G is defined as a subset $\Omega_t \subset V$, and can be implicitly represented by a level set function $\phi_t = \mathcal{X}_{\Omega_t} - \mathcal{X}_{\overline{\Omega}_t}$, where $\mathcal{X} : V \to \{0,1\}$ is the indicator function. In other words, ϕ_t equals 1 in Ω_t and -1 on its complementary.

Let $\mathcal{F} : V \to \mathbb{R}$ be a speed function, such that the sign of $\mathcal{F}$ controls the direction

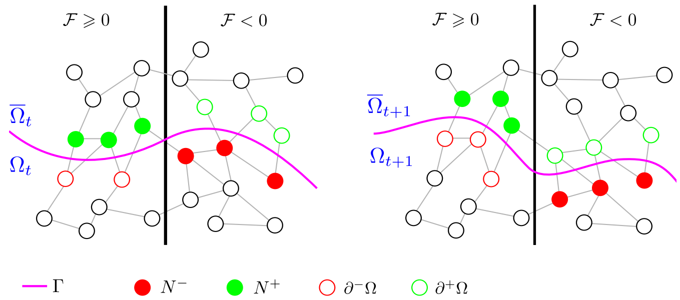

FIG. 3.1. *Illustration of narrow bands on an arbitrary weighted graph. Left and right: position of front, subsets and narrow bands at step t, respectively $t+1$.*

of the front evolution. The front moves *inward* when the speed is negative (i.e., vertices are removed from Ω_t) or *outward* when the speed is positive (i.e., vertices are added to Ω_t).

3.2. Level set formulation. The level set formulation, to describe a front evolution, has been introduced by Osher-Sethian [1], as

$$\frac{\partial \phi}{\partial t} = \mathcal{F}\|\nabla\phi\|, \tag{3.1}$$

such that the 0 level set of $\phi(x,t)$ provides the position of the front at time t. Transposed on a weighted graph G, and using the framework of discrete operators introduced in [2], the level set formulation (3.1) can be rewritten by the following PdE

$$\begin{cases} \frac{\partial \phi}{\partial t}(u,t) = \mathcal{F}(u)\|(\nabla_w\phi)(u,t)\|_p \\ \phi(u,0) = \phi_0. \end{cases} \tag{3.2}$$

Let Γ be a front represented by the subset Ω_0. We denote by N_0^+ the narrow band of vertices $u \in \partial^+\Omega_0$ likely to be added to Ω_0 (i.e., $\mathcal{F}(u) > 0$). Similarly, we denote by N_0^- the narrow band of vertices $u \in \partial^-\Omega_0$ likely to be added to $\overline{\Omega}_0$ (i.e., $\mathcal{F}(u) < 0$). See Fig. 3.1 for an illustration of these concepts.

PROPOSITION 3.1. *On narrow bands of Ω_0, one has the following property*

$$\|(\nabla_w\phi)(u,t)\|_p = \begin{cases} \|\big(\nabla_w^+\phi\big)(u,t)\|_p & \forall\ u\ \in\ N_0^+ \\ \|\big(\nabla_w^-\phi\big)(u,t)\|_p & \forall\ u\ \in\ N_0^-. \end{cases} \tag{3.3}$$

Proof. According to the relation between gradient norms, one has

$$\|(\nabla_w\phi)(u,t)\|_p = \Big[\|\big(\nabla_w^+\phi\big)(u,t)\|_p^p + \|\big(\nabla_w^-\phi\big)(u,t)\|_p^p\Big]^{1/p}. \tag{3.4}$$

In the case where $u \in N_0^+$, we have $\phi(u,0) = -1$. Then according to the gradient norms definition, we have $\|\big(\nabla_w^-\phi\big)(u,0)\|_p = 0$. Similarly, in the case where $u \in N_0^-$ we have $\|\big(\nabla_w^+\phi\big)(u,0)\|_p = 0$. $\square$

In order to take advantages of Prop.3.1, we will now consider a front propagation as a succession of very small steps of evolution. Then, the propagation of front Γ on a thin narrow band $NB = \partial^+\Omega_0 \cup \partial^-\Omega_0$ can be represented by the level set function $\phi : \partial^+\Omega_0 \cup \partial^-\Omega_0 \times [0, \tau] \to \mathbb{R}$ and described by the following morphological process

$$\begin{cases} \frac{\partial \phi}{\partial t}(u,t) = \begin{cases} \mathcal{F}(u)\|(\nabla_w^+\phi)(u,t)\|_p & \forall u \in N_0^+ \\ \mathcal{F}(u)\|(\nabla_w^-\phi)(u,t)\|_p & \forall u \in N_0^- \\ 0 & \text{otherwise (the front doesn't move)} \end{cases} \\ \phi(u,0) = \phi_0, \end{cases} \tag{3.5}$$

where τ is a small enough step of time to ensure that Γ does not step out of $\partial^+\Omega_0 \cup \partial^-\Omega_0$. At time τ, the relation between the function ϕ and the front is given by

$$u \in \begin{cases} \Omega_\tau & \text{if } \phi(u,\tau) \geqslant 0 \\ \overline{\Omega}_\tau & \text{if } \phi(u,\tau) < 0. \end{cases} \tag{3.6}$$

Then, the next step of evolution is performed by replacing Ω_0 by Ω_τ in equation (3.5) and a function $\phi : \partial^+\Omega_\tau \cup \partial^-\Omega_\tau \times [\tau, 2\tau]$. This, until the front is fully propagated. A general iterative scheme to solve this morphological process has been introduced in [4].

3.3. Link with the eikonal equation and Fast Marching algorithm. Considering the case where the sign of $\mathcal{F}$ is always non negative, equation (3.5) can be rewritten as

$$\frac{\partial \phi(u,t)}{\partial t} = \mathcal{F}(u)\|\big(\nabla_w^+\phi\big)(u,t)\|_p \quad \forall u \in N_0^+. \tag{3.7}$$

Let $\mathcal{T} : V \to \mathbb{R}$ be the arrival time function of Γ (i.e., $\mathcal{T}(u)$ is the arrival time of Γ at u). By analogy to the continuous case, where the relation between the level set formulation and the eikonal equation stems from the change of variable $\phi(x,t) = t - \mathcal{T}(x)$, equation (3.7) can be rewritten as

$$\begin{aligned} \frac{\partial \phi(u,t)}{\partial t} &= \mathcal{F}(u)\|\big(\nabla_w^+(t-\mathcal{T})\big)(u)\|_p \\ &= \mathcal{F}(u)\|\big(\nabla_w^-\mathcal{T}\big)(u)\|_p, \end{aligned} \tag{3.8}$$

which is a discrete adaptation of the eikonal equation on weighted graphs. Finally, with $P = 1/\mathcal{F}$, the static outward front evolution equation is given by

$$\begin{cases} \|\big(\nabla_w^-\mathcal{T}\big)(u)\|_p = P(u) & \forall u \in \overline{\Omega}_0 \\ \mathcal{T}(u) = 0 & \forall u \in \Omega_0, \end{cases} \tag{3.9}$$

where P is a potential function. Numerical schemes and an efficient algorithm to solve this equation have been previously provided in [5]. The algorithm is a generalized version of the Fast Marching method proposed by Sethian in [1], which has the advantage to be monotonic so that each vertex is visited only once.

On an arbitrary graph, the Fast Marching consists in an active list (A) of vertices for which the solution is already known and fixed, and in a narrow band (NB) of vertices which are not yet fixed and have at least one neighbor in the active list. The active list and narrow band are respectively initialized with vertices of Ω_0 and N_0^+.

Vertices that are neither active nor in the narrow band are said far away (FA). The narrow band is built as a sorted heap, so that vertices are added in the active list in the order of increasing arrival time. When a vertex is added to the active list, the arrival times of it's not yet fixed neighbors are updated, and the narrow band is refreshed. One can remark that at any time t we have $NB = N_t^+$. This is iterated until the narrow band is empty. More details on this algorithm can be found in [5].

3.4. Multiple outward fronts propagation. Now, we will study the case where many fronts evolve simultaneously on G. We consider that all fronts move outward, can't overlap and are mutually blocking. We will show that all these fronts can be represented and handled as a single global front associated with a marker function (to distinguish each of the fronts).

Let $\mathcal{T}$ be the arrival time function of a front Γ (represented by subset Ω_0) and let $\mathcal{T}_v$ be the arrival time function of a front starting from a single vertex v.

LEMMA 3.2. *For any vertex $u \in \overline{\Omega}_0$, one has the following property :*

$$\mathcal{T}(u) = \min_{v \in \Omega_0} (\mathcal{T}_v(u)).$$

Proof. The Lemma is a direct consequence of the Fast Marching algorithm which visits vertices from the smallest to the greatest distance. □

Let $\Gamma^1, .., \Gamma^N$ be N fronts simultaneously evolving on G, respectively represented by subsets $\Omega_0^1, ..\Omega_0^N$, with $\Omega_0^i \cap \Omega_0^j = \emptyset$, $\forall i \neq j$ (we recall that fronts move outward, can't overlap and are mutually blocking). We define the global front Γ that represents the simultaneous evolution of every fronts Γ^i as the subset $\Omega_0 = \Omega_0^1 \cup .. \cup \Omega_0^N$. Let $\mathcal{T}$, respectively, $\mathcal{T}^i$, be the arrival time function of Γ, respectively Γ^i.

PROPOSITION 3.3. *For any vertex $u \in \overline{\Omega}_0$ reached by a front Γ^i, one has $\mathcal{T}(u) = \mathcal{T}^i(u)$.*

Proof. According to Lem. 3.2 and with the knowledge that u is reached by Γ^i, one has $\mathcal{T}(u) = \min\limits_{v \in \Omega_0^i} (\mathcal{T}_v(u)) = \mathcal{T}^i(u)$. □

Thus, according to Prop. 3.3, the arrival time function $\mathcal{T}(u)$ of the global front Γ gives the arrival time of the front Γ^i that reaches u at first. Function $\mathcal{T}$ can then be seen as a global arrival time function, which provides at each point u the exact arrival time of the first front that reaches u.

In order to track every fronts individually, we introduce a marker function $L : V \to \{0, .., N\}$. At initial time, L is defined by $L(u) = 0$ $\forall u \in \overline{\Omega}_0$ and $L(u) = i$ $\forall u \in \Omega_0^i$. Then, each time a vertex u is visited by the algorithm, the vertex is marked by the label of the incoming front, i.e., the label of it's most similar neighbor with the smallest arrival time, according to the following equation

$$L(u) = L(v) \ \ | \ \ \mathcal{T}(v) w_{uv} = \max_{z \in N(u)} (\mathcal{T}(z) w_{uz}). \tag{3.10}$$

Finally, simultaneous outward evolution of every fronts Γ^i can be performed as the simple evolution of the global front Γ, using the graph-based version of the Fast Marching algorithm associated with the marker function L. The no-overlap condition is ensured by the algorithm, since a vertex cannot be visited more than once.

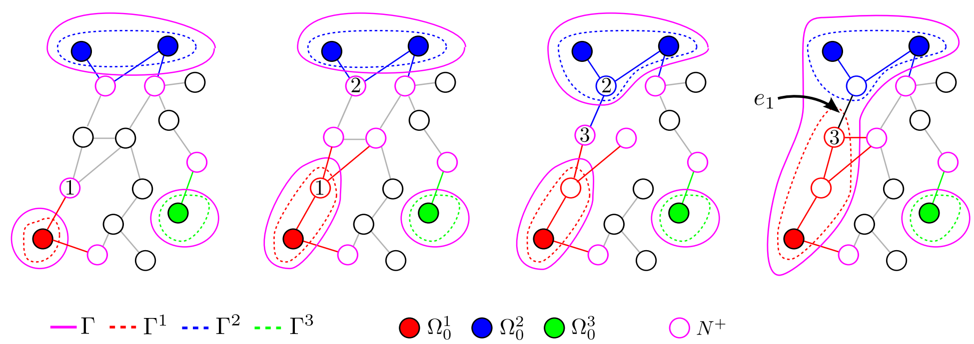

FIG. 3.2. *Multiple fronts propagation on an arbitrary weighted graph. Initial step: three fronts ($\Gamma^1, \Gamma^2, \Gamma^3$) are represented by a unique front Γ. Second step: vertex 1 passes from the narrow band to the active set, and is marked by label of front Γ^1. Third step: vertex 2 passes from the narrow band to the active set, and is marked by label of front Γ^1. Fourth step: vertex 3 passes from the narrow band to the active set, and because the nearest incoming front (in sense of distance and weight function) is Γ^1, the label of Γ^1 is set to vertex 3. One can remark that fronts Γ^1 and Γ^2 collapse on edge e_1.*

Due to the multitude of fronts, we have to define one speed per front where the speed of a front Γ^i is denoted $\mathcal{F}_i$. Consequently, each time a vertex u is reached by a front Γ^i, its neighbors v such that $v \in A$ are updated by :

$$\|(\nabla_w^- \mathcal{T})(v)\| = P_i(v), \tag{3.11}$$

where $P_i(v) = 1/\mathcal{F}_i(v)$.

The process is illustrated step by step in Fig. 3.2 with the simultaneous propagation of three fronts.

Remark: The tracking (in position or time) of each front can be easily performed by a simple thresholding of L, respectively of $\mathcal{T}$.

4. Generalized fronts propagation on graphs. In the case where speeds $\mathcal{F}_i$ can be either positive or negative, the outward equation for fronts evolution and associated fast algorithm become insufficient to perform the fronts evolutions. To overcome this outward limitation, we propose to take advantages of both graph-based front representation and multi-fronts form of the fast algorithm, to transform an outward/inward front evolution to two outward fronts evolutions.

4.1. Link between outward and inward propagation. In this Section, for the sake of clarity, we consider the case of a single front to present the idea. Let Γ be an evolving front (according to a speed function $\mathcal{F}$) defined as a subset $\Omega \subset V$ and represented by the level set function ϕ. Let $\overline{\Omega}$ be the complementary of Ω, represented by the level set function ϕ^c. We denote by Γ^c, the front defined by the subset $\overline{\Omega}$.

In order to simplify the problem of inward/outward front propagation, we propose to express an inward evolution equation considering set Ω as an outward one considering the complementary of Ω ($\overline{\Omega}$), and then only consider outward evolution equations.

PROPOSITION 4.1. *The inward evolution of Γ (when $\mathcal{F} < 0$) can be expressed as an outward evolution of Γ^c.*

Proof. In the case where $\mathcal{F} < 0$, according to Prop.3.1 and as $\phi(u) = -\phi^c(u)$ the

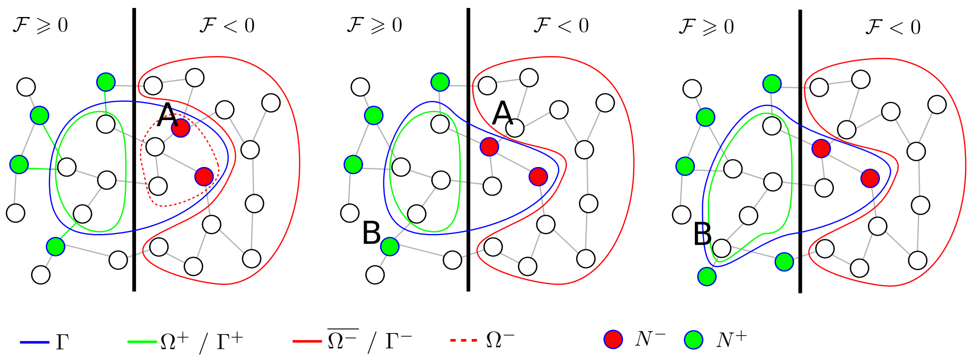

FIG. 4.1. *Illustration of inward/outward front propagation for a single front. Initial step: the narrow band is initialized with vertices of $N^+ \cup N^-$. Subset Ω^- is ignored and the two only considered fronts are Γ^+ and Γ^-. Second step: vertex A is reached by Γ^- and removed from Ω. This corresponds to an inward evolution of Γ. Third step: vertex B is reached by Γ^+ and added to Ω. This corresponds to an outward evolution of Γ.*

level set formulation (3.5) can be rewritten as

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \mathcal{F}\|\big(\nabla_w^- \phi\big)\|_p = \mathcal{F}\|\big(\nabla_w^+ \phi^c\big)\|_p\\ &= \mathcal{F}\|\big(\nabla_w^- \mathcal{T}^c\big)\|_p,\end{aligned}$$

with $\mathcal{T}^c : V \to \mathbb{R}$ is the arrival time of Γ^c, defined by $\phi^c = t - \mathcal{T}^c$. Finally, with $P = 1/|\mathcal{F}|$, we obtain

$$\|\big(\nabla_w^- \mathcal{T}^c\big)(u)\|_p = P(u),$$

which is the outward equation for front evolution. □

Thus, according to Prop. 4.1, and using the duality between fronts and subsets of V, inward or outward fronts evolution can be simply expressed by the single static outward front evolution equation (3.9).

Let V^- be the subset of V where $\mathcal{F}$ is negative and V^+ the subset of V where $\mathcal{F}$ is non-negative. Let Ω^+ be a subset of V^+, and Ω^- a subset of V^-. Let $\overline{\Omega^-}$ be the subset of V^- such that $\Omega^- \cap \overline{\Omega^-} = \emptyset$ and $\Omega^- \cup \overline{\Omega^-} = V^-$.

PROPOSITION 4.2. *Inward/outward evolution of a front Γ defined by $\Omega = \Omega^+ \cup \Omega^-$ is equivalent to outward evolution of Γ' defined by $\Omega' = \Omega^+ \cup \overline{\Omega^-}$.*

Proof. According to Prop. 4.1, inward evolution of a front defined by Ω^- is equivalent to outward evolution of a front defined by $\overline{\Omega^-}$. □

Then, considering fronts Γ^+ defined by the subset Ω^+ and Γ^- defined by the subset $\overline{\Omega^-}$, and according to Prop. 4.2, the propagation of front Γ can be performed by the simultaneous propagation of fronts Γ^+ and Γ^-. Such a process is illustrated in Fig. 4.1. In next Section, we extend this formulation to the case of several fronts and present a detailed version of the graph-based inward/outward Fast Marching algorithm.

4.2. General formulation and algorithm. Finally, we will consider the most general case, where several inward/outward fronts evolve simultaneously on a weighted graph.

Let $\Gamma_1, .., \Gamma_N$ be N fronts simultaneously evolving on G, respectively represented by $\Omega_0^1, .., \Omega_0^N$ and driven by speeds $\mathcal{F}_1, .., \mathcal{F}_N$ without restriction on their sign. We consider that all these fronts are mutually blocking and can't overlap. According to Prop. 3.3 and Prop. 4.2, the propagation of these fronts can be performed as the propagation of a unique front Γ defined as

$$\Omega_0 = \Omega_0^{1+} \cup \overline{\Omega_0^{1-}} \bigcup .. \bigcup \Omega_0^{N+} \cup \overline{\Omega_0^{N-}},$$

where Ω_0^{i+} is the subset of Ω_0^i where $\mathcal{F}_i$ is non negative and $\overline{\Omega_0^{i-}}$ is the subset of $\overline{\Omega_0^i}$ where $\mathcal{F}_i$ is negative. The arrival time function $\mathcal{T}$ associated to front Γ gives the exact arrival time of each inward/outward fronts evolving on the graph.

Remark: Pseudo fronts Γ^{i-} can overlap with real fronts Γ^k (with $k \neq i$). This does not contradict the no-overlap condition for real fronts Ω^i.

The propagation is performed using the graph-based version of the Fast Marching algorithm with some adaptations to consider the case where a vertex u is reached by an *inward* front Γ^{i-}. Indeed, on the contrary of an *outward* front Γ^{i+} that adds vertices to Ω^i, an *inward* front Γ^{i-} removes vertices from Ω^i.

Let $u \in \Omega^i$ be a vertex reached by Γ^{i-}. First, the vertex u is leaved by front Γ^i and removed from Ω^i but is still eligible for another front. Therefore, u is not added to the active list A, but added to FA (or inserted in the narrow band if there is another front in it's neighborhood). Second, in order to track the position of front Γ^i, the marker function has to unmark u (instead of marking it) and becomes

$$L(u) = \begin{cases} L(v) \ \mid \ \mathcal{T}(v)w_{uv} = \max\limits_{z \in N(u)} (\mathcal{T}(z)w_{uz}) & \text{if } \mathcal{F}_{L(v)}(u) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

as the label 0 means "not marked". The entire algorithm is detailed in Algo. 1.

The general formulation and the algorithm for multiple fronts propagation on weighted graphs, presented in this paper, provide several advantages. First, the graph-based formulation of the algorithm extends applications of the Fast Marching algorithm to data defined on non-Euclidean domains (as social newtorks). Moreover, most types of discrete data (images, meshes, networks, unorganized data, etc.) can be represented as weighted graphs, and then can be processed by this single general algorithm. Second, the proposed algorithm allows to propagate simultaneously several fronts with the only one condition that fronts do not overlap. Finally, the last advantage is the efficiency of the algorithm since its complexity does not depend on the number of fronts and is given by $\mathcal{O}(cNlog(N))$ with $c \ll N$.

A similar approach that proposes an adaptation of the Fast Marching algorithm for a single front with velocity-changing sign can be found in [6]. Interested readers can also find an extension of the Fast Marching algorithm to the particular case of triangulated domains in [7].

4.3. Experiments. Before concluding this paper, we present some experiments involving the generalized front propagation formulation and algorithm on weighted graphs. First, Fig. 5.1 illustrates the behavior of the algorithm to propagate several fronts on a weighted graph, with different combinations of weight and speeds functions

Algorithm 1 Generalized fronts propagation algorithm

0. List of variables:

Ω_0 : Initial subset of global front Γ;
A : the set of *active* vertices; NB : the set of vertices in the *narrow band*;
FA : the set of vertices said as *far away*; L : the label indicator function;
$\mathcal{F}^i, N_0^{i+}, N_0^{i-}$: speed function, initial outer and inner narrow bands of front Γ^i;

1. Initialization:

$A = \{u \mid u \in N_0^{i+} \text{ and } \mathcal{F}_i(u) < 0\} \cup \{u \mid u \in N_0^{i-} \text{ and } \mathcal{F}_i(u) > 0\}$
$NB = \{u \mid \exists\, v \in A \cap N(u) \text{ and } \mathcal{F}_i(v) \times \mathcal{F}_i(u) > 0\}$
$FA = V \setminus (A \cup NB)$
$\mathcal{T}(u) = 0;\ L(u) = i\ \forall\, u \in A;\ \mathcal{T}(u) = \min(1/w_{uv}),\ v \in A \cap N(u)$
$K(u) = i;\ s(u) = \mathcal{T}(u)\ \forall\, u \in NB$
$\mathcal{T}(u) = +\infty;\ L(u) = 0;\ s(u) = -\infty\ \forall\, u \in FA$

2. Process:

while $FA \neq \emptyset$ **do**
  $u \leftarrow$ first element of NB
  remove u from NB and add u in A.
  for all $v \in N(u) \cap \overline{A}$ and $\mathcal{F}_i(v) \times \mathcal{F}_i(u) > 0$ **do**
    compute local solution $t \leftarrow \mathcal{T}(v)$
    if $t < \mathcal{T}(v)$ **then**
      $\mathcal{T}(v) = t$
      if $v \in FA$ **then**
        remove v from FA and add v in NB
      else
        update position of v in NB
      end if
      if $\mathcal{T}(u)w_{uv} > s(v)$ **then**
        $s(v) = \mathcal{T}(u)w_{uv}$
        $L(v) = L(u)$
      end if
    end if
  end for
end while

(for the sake of visual clarity, the used graph is a regular grid graph). Then, Fig. 5.2 presents direct applications of the algorithm, applied to image segmentation or data clustering. In both cases, propagation is performed using $p = 2$.

5. Conclusion. In this paper, we have proposed a general formulation and an algorithm for simultaneous propagation of several fronts evolving on a weighted graph. The algorithm extends the Fast Marching algorithm to weighted graphs, what enables the process of many kinds of data. Experiments have shown the behavior of this approach so as applications to image segmentation and data clustering.

REFERENCES

[1] J. A. Sethian, *Level Set Methods and Fast Marching Methods*, Cambridge University Press, 1999.

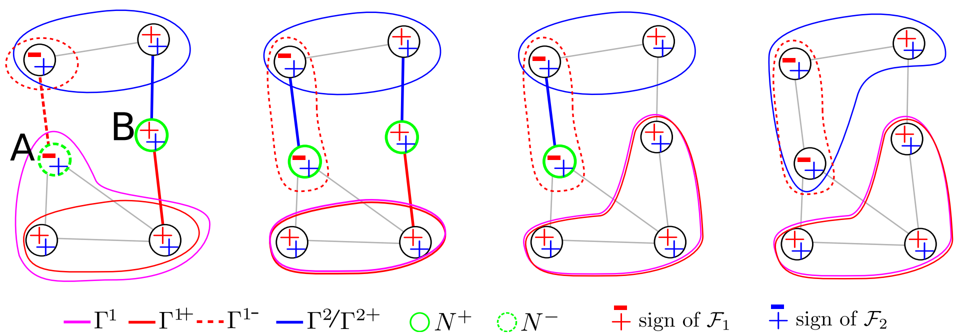

FIG. 4.2. *Generalized fronts propagation with two fronts. Initial step, the narrow band is initialized with vertices of $N^{1+} \cup N^{1-} \cup N^{2+}$ (in this case vertices A and B). Because $\mathcal{F}_2$ is always non-negative, front Γ^2 is represented by the single front Γ^2+. On the contrary, front Γ^1 is represented by the two fronts Γ^{1-} and Γ^{1+}. Second step, A is reached by Γ^{1-} and removed from Ω^1 (inward evolution of Γ^1). Because Γ^{1-} is an inward front, vertex A is unmarked and becomes eligible for other fronts. Therefore, A is added to the narrow band by front Γ^{2+}. Third step, vertex B is reached by Γ^{1+} and added to Ω^1 (outward evolution of Γ^1). Final step, vertex A which was previously removed from Ω^1 is reached by Γ^{2+} and added to Ω^2.*

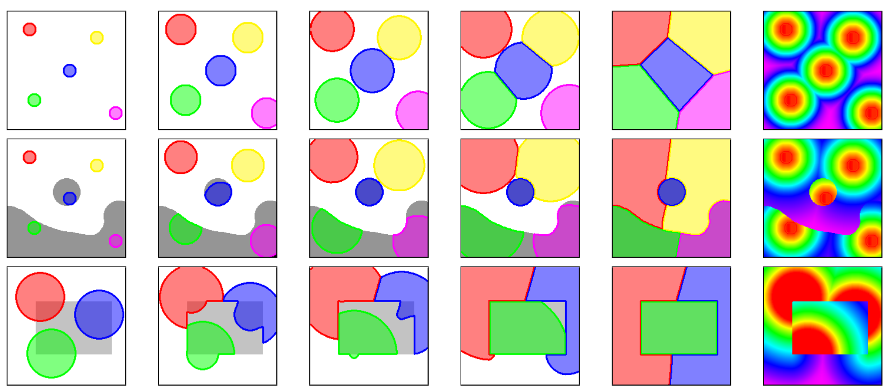

FIG. 5.1. *Multiple fronts propagation on a regular grid-graph. First line: the weight and speeds functions are constant ($\mathcal{F} > 0$) and the five fronts are propagated until they collapse. Second line: the weight function is $w = 1$ where background is constant and $w = 0$ on discontinuities, speeds functions are constant ($\mathcal{F} > 0$). The propagation is then also stopped on background discontinuities. Third line: the weight function is constant, but speeds functions are defined as follows. Red and blue fronts have positive, respectively negative speeds on white, respectively grey backgrounds. Green front has positive, respectively negative, speed on grey, respectively white backgrounds. Last column (on the right) shows the global arrival time map in both cases.*

[2] A. ELMOATAZ, O. LÉZORAY, AND S. BOUGLEUX, *Nonlocal discrete regularization on weighted graphs: A framework for image and manifold processing*, IEEE Trans. Image Process., 17, (2008), pp. 1047–1060.

[3] V.-T. TA, A. ELMOATAZ, AND O. LÉZORAY, *Partial difference equations over graphs: Morphological processing of arbitrary discrete data*, Proc. of ECCV, vol. LNCS 5304, (2008), pp. 668–680.

[4] X. DESQUESNES, A. ELMOATAZ, AND O. LÉZORAY, *PDEs Level Sets on Weighted Graphs,*, Proc. of ICIP, (2011), pp. 3377–3380.

[5] X. DESQUESNES, A. ELMOATAZ, O. LÉZORAY AND V.-T. TA, *Efficient Algorithms for Image and high Dimensional Data Processing Using Eikonal Equation on Graphs*, Proc. of ISVC, vol. LNCS 6454, (2010), pp. 647–658.

[6] E. CARLINI, M. FALCONE, N. FORCADEL, R. MONNEAU, *Convergence of a generalized fast-*

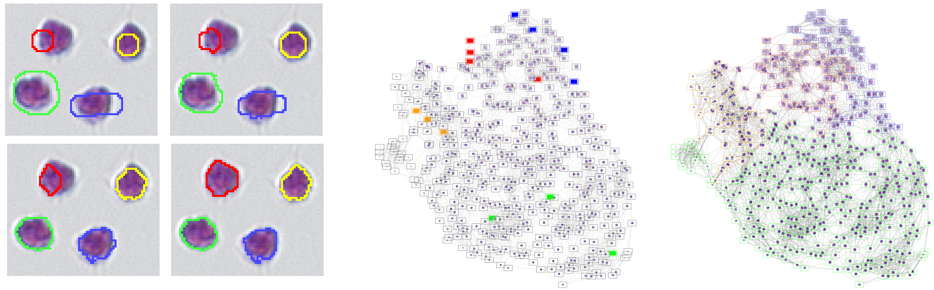

Fig. 5.2. *Application of the proposed algorithm to image segmentation (left) and unorganized data clustering (right).*

marching method for an eikonal equation with a velocity-changing sign, SIAM J. Numer. Anal., (2008), 46, pp. 2920–2952.

[7] R. Kimmel, J. A. Sethian, *Computing Geodesic Paths on Manifolds*, Proc. of Nat. Acad. Sc., (1998), 95, pp. 8431–8435.