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3D BEM APPLICATION TO NEUMANN GEODETIC BVP USING THE COLLOCATION WITH LINEAR BASIS FUNCTIONS

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Abstract. This paper presents improved numerical solution of Neumann geodetic boundary value problem (NGBVP) applying boundary element method (BEM). NGBVP is an exterior oblique derivative problem for Laplace equation. First numerical experiments were presented in [4]. Now the collocation with linear basis functions is applied for deriving the system of linear equations from boundary integral equations. With respect to a giant size of the Earth and in order to get accuracy as high as possible a computing on high-speed parallel computers is necessary.

In first experiment the Earth's surface is approximated by 44 378 nodes and 88 752 triangles. A global quasigeoid model as a result of the BEM application to NGBVP is compared with Earth geopotential model EGM-96 that is computed by spherical harmonics and geopotential coefficients. Local refinement in Europe is presented in second experiment.

1. Introduction. The determination of gravity field is usually formulated in terms of BVP for Laplace equation [8,12]. In the previous article [4] we formulated NGBVP

$$
\begin{aligned}
&\Delta T(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in R^3 - \Omega\,,\\
&\langle \nabla T(\boldsymbol{x}), \boldsymbol{n}_e(\boldsymbol{x})\rangle = \delta g(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma\,,\\
&T(\boldsymbol{x}) \to 0 \quad \text{for } \boldsymbol{x} \to \infty
\end{aligned} \tag{1}
$$

where $T(\boldsymbol{x})$ is the disturbing potential (a difference between the actual W and normal U gravity potentials), $\delta g(\boldsymbol{x})$ is the absolute value of the surface gravity disturbance, i.e. $|\nabla W - \nabla U|$ which is a measurable quantity, and $\boldsymbol{n}_e(\boldsymbol{x})$ is the normal to the geocentric equipotential ellipsoid of revolution [6]. $\langle .,. \rangle$ represents the scalar product of vectors. Equations (1) represent the exterior oblique derivative BVP for the Laplace equation with the Neumann boundary condition (BC). The domain Ω represents the body of the Earth and the boundary surface Γ is the Earth's surface. The normal to the Earth's surface Γ doesn't coincide with the normal to ellipsoid $\boldsymbol{n}_e$.

BEM as a numerical method based on variational formulation of PDE is suitable for solving exterior BVPs. The collocation is one of the simplest techniques for deriving the linear system of equations from boundary integral equations. Thanks its simplicity this method is very popular in engineer applications while it has some drawbacks [10]. In great majority of applications constant basis functions are used for approximating boundary functions on each panel of the boundary surface. In our experiments we decided to use linear basis functions. Main reasons were:

- Discretizing the Earth's surface in the way described later a number of triangles is equal to $2(N-2)$ where N is a number of triangulation nodes. It means that the assembly of the linear system of equations using constant basis functions needs about 4 times more memory storage than applying linear basis functions.

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- The Earth's surface is approximated by the triangulation of the topography with input data in vertices of triangles. In case of constant basis functions the nodes (collocation points) are in centres of triangles and further approximations are necessary.
- In case of linear basis functions nodes lie in vertices of triangles. It seems to be better for computing singular elements.

Although the assemble of the linear system of equations applying linear basis functions is more complicated and takes more time a numerical solution leads to more precise results using the same internal memory storage.

2. Direct BEM formulation and C^1 collocation method. In the direct BEM formulation a boundary integral equation is derived from Laplace equation (1) through application of Green's second theorem [3]. In 3D it has the following form

$$4\pi T\,(p) + \int_\Gamma \frac{\partial G}{\partial n_q}(p,q)\,T(q)\,d\Gamma_q = \int_\Gamma G(p,q)\frac{\partial T}{\partial n_q}(q)\,d\Gamma_q\,, \quad p,q \in \Gamma \tag{2}$$

where $\boldsymbol{n}_q$ is a normal to the boundary Γ. A kernel function G represents the fundamental solution of the Laplace equation

$$G(p,q) = \frac{1}{4\pi|p-q|}\,, \quad p,q \in R^3\,. \tag{3}$$

Since the directions of ∇T and $\boldsymbol{n}_e$ are almost identical we can approximate $\langle \nabla T(\boldsymbol{x}), \boldsymbol{n}_q(\boldsymbol{x})\rangle$ by $\delta g(\boldsymbol{x})\cos\alpha$ where α is an angle $\angle(\boldsymbol{n}_q, \boldsymbol{n}_e)$. Thus we can replace $\partial T/\partial n_q$ in (2) by these corresponding quantities. In this way the oblique derivative BC (1) is incorporated into BEM formulation (2).

The collocation method with linear basis functions is used for deriving the linear system of equations from the boundary integral equation (2). The Earth's surface as a boundary surface is approximated by the triangulation of the topography – expressed as a set of panels $\Delta\Gamma_j$. Vertices $x_i, \ldots, x_N$ of triangles represent the nodes – collocation points. The C^1 collocation method involves representing the boundary function by a linear function on each triangle panel using linear basis functions

$$\begin{array}{lll} \{\psi_1, \psi_2, \ldots, \psi_N\} & \psi_j(x_i) = 1 & x_i = x_j \in R^3 \\ & \psi_j(x_i) = 0 & x_i \neq x_j \end{array} \tag{4}$$

where N is a number of nodes. It allows to reduce the boundary integral equation (2) to a discrete form for each collocation point i

$$c_i T_i \psi_i + \sum_{j=1}^{N} \int_{supp\,\psi_j} \frac{\partial G_{ij}}{\partial n_q} T_j \psi_j\, d\Gamma_j = \sum_{j=1}^{N} \int_{supp\,\psi_j} G_{ij}\delta g_j \psi_j d\Gamma_j\,,\ i = 1 \ldots N \tag{5}$$

where $\operatorname{supp}\psi_j$ is a support of the j-th basis function. The function c_i represents "spatial angle" bounded by the panels joined in the node i [1]. In case of linear basis functions

$$c_i = \sum_{l=1}^{L} \frac{\varphi_{i_l}}{4\pi}(1 - \cos\phi_{i_l}) \tag{6}$$

where φ_{i_l} is an angle between two planes intersecting in $\boldsymbol{n}_e$ in node i and which create two edges of the l-th triangle of the supp ψ_i and ϕ_{i_l} is an angle between $\boldsymbol{n}_e$ and the l-th triangle. L represents a number of triangles in the supp ψ_i. Equations (5) can be written in the matrix-vector form

$$\boldsymbol{MT} = \boldsymbol{Q}\delta\boldsymbol{g}\,. \tag{7}$$

Coefficients of matrices $\boldsymbol{M}, \boldsymbol{Q}$ represent the approximations of integrals in (5) in the collocation points. Regular integrals can be approximated by the Gaussian quadrature rules for a triangle

$$M_{ij} = \frac{1}{4\pi}\sum_{l=1}^{L} A_{j_l} k_{ij_l} \sum_{k=1}^{K} \frac{1}{l_{ik_l}^3}\psi_k w_k\,, \tag{8}$$

$$Q_{ij} = \frac{1}{4\pi}\sum_{l=1}^{L} A_{j_l}\cos\alpha_{j_l} \sum_{k=1}^{K} \frac{1}{l_{ik_l}}\psi_k w_k \quad i \neq j$$

where A_{j_l} is the area of the l-th triangle of the supp ψ_j, k_{ij_l} is a perpendicular from node i to this triangle and w_k are weights. L represents a number of triangles in the supp ψ_j and K is a number of used points for the Gaussian quadrature. $\cos\alpha_{j_l}$ represents a projection of $\boldsymbol{n}_e$ in the node j to the normal $\boldsymbol{n}_q$ of the l-th triangle. Then the j-th component of the vector $\delta\boldsymbol{g}$ in (7) represents the input value of measured gravity disturbance $\delta\boldsymbol{g}$ in the node j.

Non-regular integrals (singular elements) that arise because of the singularity of the kernel function (3) need a special treatment. Thanks the orthogonality of the normal to its triangle the kernel functions in integrals on the left hand side in (5) are equal to zero and

$$M_{i,i} = c_i \tag{9}$$

Diagonal coefficients $Q_{i,i}$ can be evaluated analytically using the software *Mathematica*® [16].

In case of Neumann BC a known vector on the right hand side in (7) is given. Solving this linear system of equations we obtain values of the unknown disturbing potential in collocation points. Then the disturbing potential is transformed to quasigeoidal heights above the ellipsoid using the Bruns formula in an iterative way described in [4].

3. First numerical experiment. Global Quasigeoid Model. In the first numerical experiment we approximated the Earth's surface by 44 378 nodes and 88 752 triangles (latitude interval: $\Delta\mathrm{B} = 1.0227^\circ$). A triangulation of the topography is based on a subdivision of triangular faces of a "12-hedron". Each triangle is subdivided into 4 congruent sub-triangles by halving the sides until required level (fig.1).

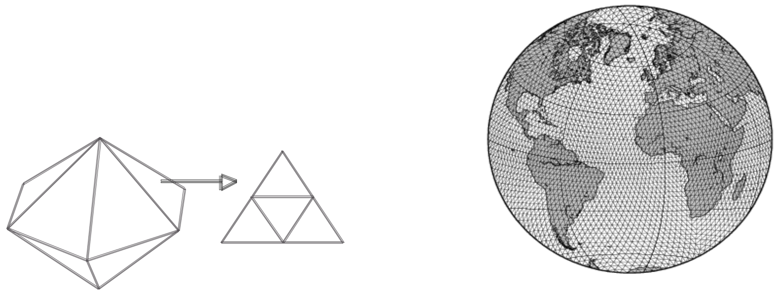
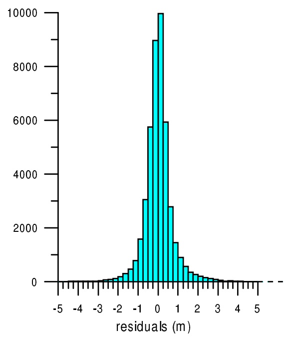


FIG. 1. *The triangulation of the topography*

To obtain ellipsoidal heights as vertical information of nodes positions we used Global Digital Elevation Model GTOPO-30 [5] and Earth Geopotential Model EGM-96. EGM-96 is formulated as the spherical harmonics with geopotential coefficients determined from satellite, altimetric and gravimetric measurements [13]. EGM-96 allows to generate input surface gravity disturbances in our nodes.



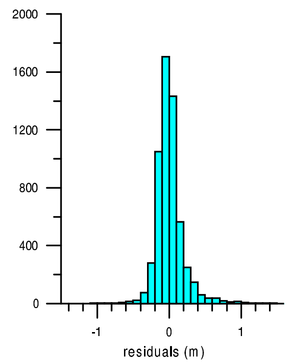
STATISTICS	
Nodes	44 378
Mean	0.026 m
Max	6.161 m
Min	-15.147 m
St.dev.	0.710 m

TABLE 1
"residuals = EGM-96 – BEM".

Computing was accomplished at ICM Warsaw on high-speed parallel computer TAJFUN: CRAY SV1-1/32 with 32 processors and 32 GB internal (shared) memory. The sparse linear system of equations was solved by non-stationary iterative method BiConjugate Gradient Stabilized (BiCGSTAB) [2] without preconditioning. Computing used about 17 GB of the internal memory. 16 BiCGSTAB iterations were necessary to keep error lower than the prescribed tolerance in absolute residual error. Iterations took only several seconds while the matrices assembly several hours. Total CPU time took 25 169.30 s. The Global Quasigeoid model as a result of the 3D BEM application to NGBVP is presented in fig.2.

The Global Quasigeoid Model is compared with EGM-96 (BEM application versus spherical harmonics). Residuals are presented in fig.3 and statistic characteristics in tab.1.

4. Second numerical experiment. Local refinement in Europe. A possibility of the local refinement depicted the second experiment. The Earth's surface is approximated by 43 733 nodes; 38 402 from the global triangulation ($\Delta B = 1.125^\circ$) and 5331 in the local refinement ($\Delta B = 0.28125^\circ$) (fig.4). Computing on TAJFUN used about 16.5 GB of the internal memory. 17 BiCGSTAB iterations were necessary to keep error lower than the prescribed tolerance in absolute residual error. Total CPU time took 24 592.52 s.



STATISTICS (only local refinement)	
Nodes	5 764
Mean	0.006 m
Max	1.530 m
Min	-1.035 m
St.dev.	0.195 m

TABLE 2

The European Quasigeoid Model (fig.5) as a result of the local refinement is compared with EGM-96 (fig.6). Statistic characteristics are presented in tab.2.

5. Conclusions and perspectives for geodesy. A definition of Neumann BC in form of surface gravity disturbances and the BEM application to NGBVP represents a new approach in the gravity field modelling. Numerical results show the evident correlation and agreement with EGM96 (spherical harmonics). A main perspective of this approach is in the global gravity field modelling. Increasing a number of nodes the numerical solutions can yield to more precise results. Inputting real gravity data a solution will become independent from EGM-96 and it can be a suitable test based on the different mathematical background.

A need of gigantic internal memory storage is the main disadvantage of this method. Therefore a possibility of local refinement that leads to a precise local gravity field modelling is limited. Nevertheless second numerical experiment confirms a possibility to apply BEM for a local modelling. To reduce differences in extreme zones (in Alps) another local refinements would be necessary.

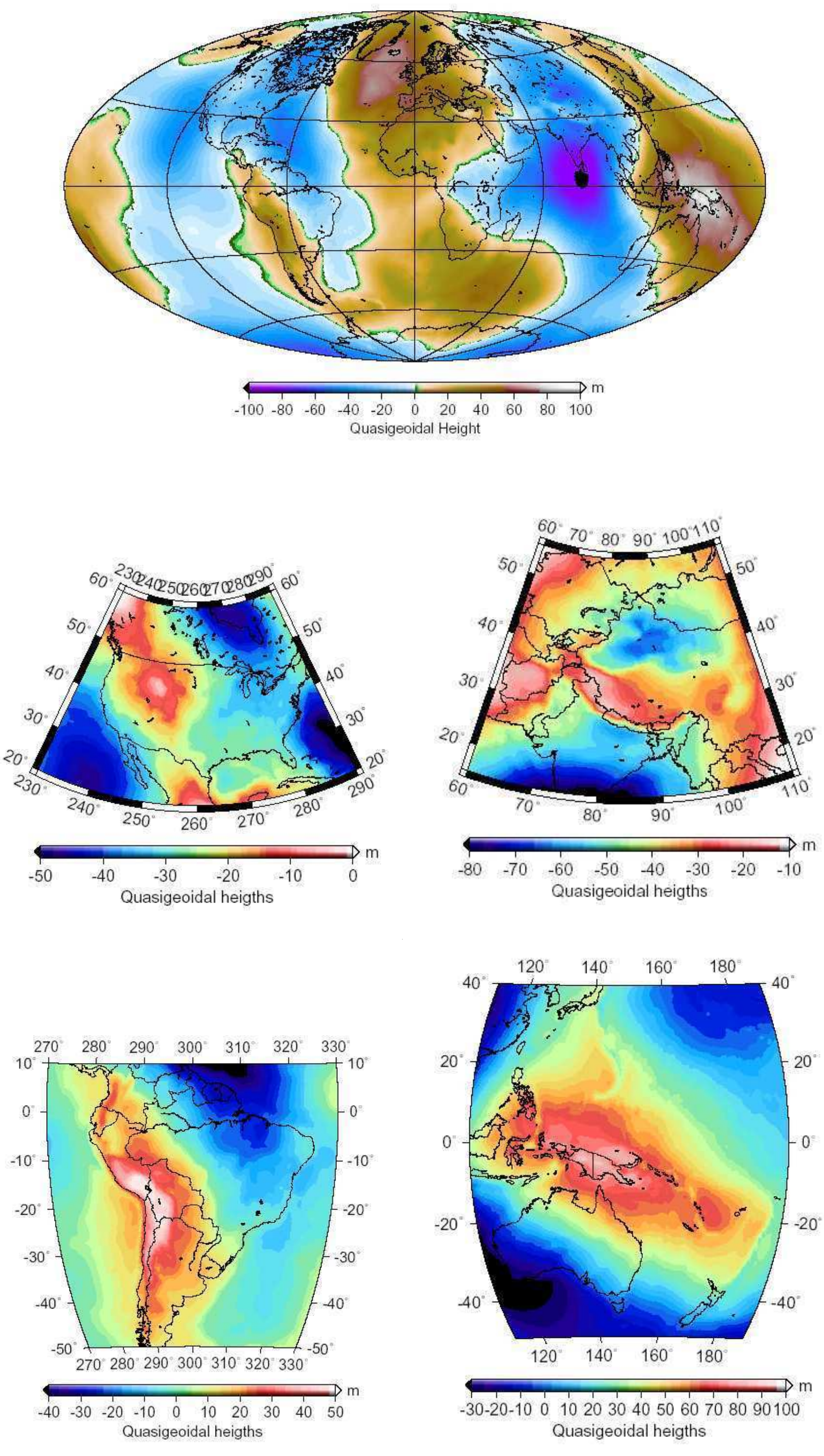


FIG. 2. *Global Quasigeoid Model - 3D BEM application to Neumann geodetic BVP*

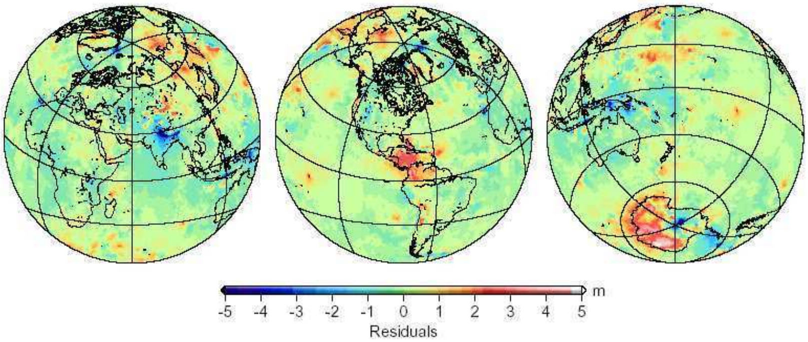


FIG. 3. *The comparison between the Global Quasigeoid Model and EGM-96*

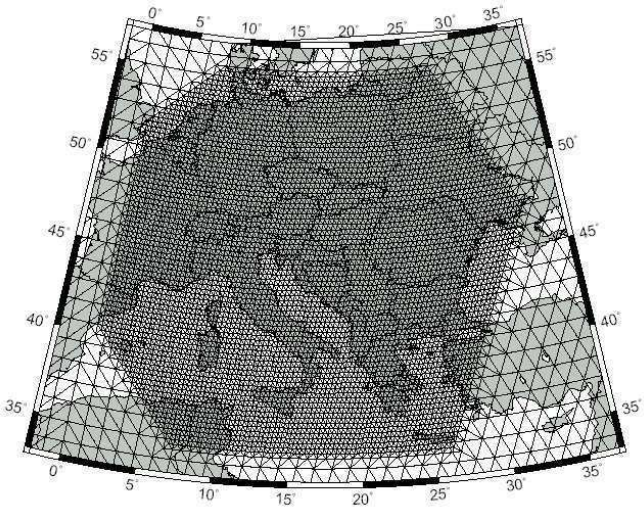


FIG. 4. *The local refinement in Europe*

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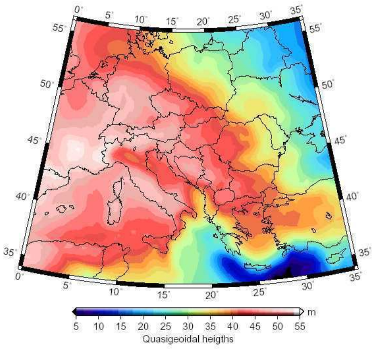

FIG. 5. *The European Quasigeoid Model*

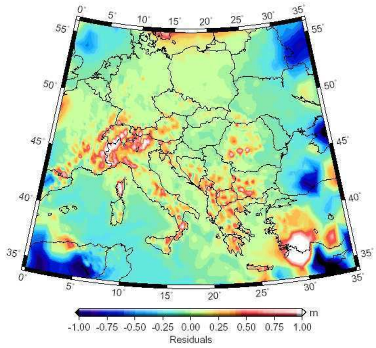

FIG.6. *The comparison with EGM-96*

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