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MODIFIED EXPLICIT FINITE VOLUME SCHEME FOR PERONA-MALIK EQUATION *

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Abstract. We propose modified explicit finite volume computational method for the numerical solution of the modified (in the sense of Catté, Lions, Morel and Coll) Perona–Malik nonlinear image selective smoothing equation (called *anisotropic diffusion*) in the image processing. This access reduces the computational effort considerably, because we compute difusion coefficients not in every scale step. Convergence of the method and numerical examples are presented.

Key words. Perona Malik equation, image processing, explicit scale discretization, convergence of numerical solution, finite volume method

AMS subject classifications. 35K55, 65M12

1. Mathematical model and computational method.

1.1. Mathematical model of the problem. We are dealing with Perona-Malik type problem suggested by [1] in the following form

$$\partial_t u - \nabla.(g(|\nabla G_\sigma * u|)\nabla u) = 0 \quad \text{in } Q_T \equiv I \times \Omega, \tag{1.1}$$

$$\partial_\nu u = 0 \quad \text{on } I \times \partial\Omega, \tag{1.2}$$

$$u(0,\cdot) = u_0 \quad \text{in } \Omega, \tag{1.3}$$

where $\Omega \subset I\!R^d$ is a rectangular domain, $I = [0, T]$ is a scaling interval, and

$$g(s) \text{ is a Lipschitz continuous decreasing function,}$$

$$g(0) = 1, 0 < g(s) \to 0 \text{ for } s \to \infty, \tag{1.4}$$

$$G_\sigma \in C^\infty(I\!R^d) \text{ is a smoothing kernel with } \int_{I\!R^d} G_\sigma(x)dx = 1 \tag{1.5}$$

$$\text{and } G_\sigma(x) \to \delta_x \text{ for } \sigma \to 0,\ \delta_x \text{ - Dirac function at point } x,$$

$$u_0 \in L^2(\Omega). \tag{1.6}$$

1.2. Formulation of explicit finite volume scheme. Let τ_h be a uniform mesh of Ω with cells p of measure $m(p)$ (we assume rectangular cells here). For every cell p we consider set of neighbours $N(p)$ consisting of all cells $q \in \tau_h$ for which common interface of p and q, denoted by e_{pq}, is of non-zero measure $m(e_{pq})$. It is assumed that for every p, there exists representative point $x_p \in p$, such that for every pair $p, q, q \in N(p)$, the vector $\frac{x_q - x_p}{|x_q - x_p|}$ is equal to unit vector n_{pq} which is normal to e_{pq} and oriented from p to q. In simple case of uniform grid we can take x_p just as center of the pixel. Then, let x_{pq} be the point of e_{pq} intersecting the segment $\overline{x_p x_q}$. We define

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$$T_{pq} := \frac{m(e_{pq})}{|x_q - x_p|} \tag{1.7}$$

Our modification of the usual explicit scheme is in computing diffusion coefficients not in every scale step. To do this we first define:
$R < \infty$ - fixed real number (usually 4 or 5) i.e. the number of scale steps at which diffusion coefficients are constants.

The modified finite volume explicit scheme on uniform grid is then written as follows:

Let $0 = t_0 \le t_1 \le ... \le t_{nR+i} .... \le t_{N_{\max}R}$, $N_{\max} \cdot R = T$ *denote the scale discretization steps with* $t_l = t_{l-1} + k$, *where* k *is the discrete scale step,* $l = 1, 2, ...N_{\max}R$.
For $n = 0, ..., N_{\max} - 1$ *and* $i = 0, ..., R-1$ *we look for* $\overline{u}_p^{nR+i+1}$, $p \in \tau_h$, *satisfying the identities*

$$\left(\overline{u}_p^{nR+i+1} - \overline{u}_p^{nR+i}\right) m(p) = k \sum_{q \in N(p)} g_{pq}^{\sigma,nR} T_{pq} \left(\overline{u}_q^{nR+i} - \overline{u}_p^{nR+i}\right) \tag{1.8}$$

$$g_{pq}^{\sigma,nR} := g\left(|\nabla G_\sigma * \tilde{u}\,(x_{pq})|\right) \tag{1.9}$$

where $\tilde{u}$ is a periodic extension of discrete image computed in nR-th scale step.

2. Numerical aspects and convergence results.

2.1. Computing diffusion coefficients. For computing the term (1.9), i.e. the vector

$$\nabla G_\sigma * \tilde{u}\,(x_{pq}) = \left(\frac{\partial (G_\sigma * \tilde{u})}{\partial x}\,(x_{pq}), \frac{\partial (G_\sigma * \tilde{u})}{\partial y}\,(x_{pq})\right),$$

which is an input of the Perona-Malik function g. For that goal, we use the following property of convolution

$$\frac{\partial (G_\sigma * \tilde{u})}{\partial x}\,(x_{pq}) = (\frac{\partial G_\sigma}{\partial x} * \tilde{u})\,(x_{pq})\,.$$

Then one gets

$$(\frac{\partial G_\sigma}{\partial x} * \tilde{u})(x_{pq}) = \int_{\mathbb{R}^d} \frac{\partial G_\sigma}{\partial x}(x_{pq} - s)\tilde{u}\,(s)\,ds = \sum_r \overline{u}_r^n \int_r \frac{\partial G_\sigma}{\partial x}(x_{pq} - s)ds \tag{2.1}$$

and thus

$$\nabla G_\sigma * \tilde{u}\,(x_{pq}) = \sum_r \overline{u}_r^n \int_r \nabla G_\sigma\,(x_{pq} - s)\,ds \tag{2.2}$$

where the sum is restricted to control volumes r inside $B_\sigma\,(x_{pq})$, the ball centered at x_{pq} with radius σ. The ball B_σ is given either by a support of compactly supported smoothing kernel or it can represent a "numerical support" of the Gauss function (a domain in which values of Gauss function are above some treshold given e.g. by a computer precision). In any case just a finite sum in (2.2) is evaluated and coefficients of this sum, namely $\int_r \nabla G_\sigma(x_{pq} - s)\, ds$ can be precomputed in advance using a computer algebra system, e.g. Mathematica. It is worth noting that such approach for evaluation of diffusion coefficient $g_{pq}^{\sigma,n}(\overline{u}_{h,k})$ avoids explicit computation of gradients.

2.2. Weak formulation of the problem. Discrete approximation of a solution of partial differential equation are considered to be piecewise constant in control volumes, which in image processing corresponds to pixel structure of a discrete image.

We define a weak solution to the problem (1.1)-(1.3) equation (1.1) is multiplied by a test function $\varphi \in \Psi$, where Ψ is the space of smooth test functions

$$\Psi = \{\varphi \in C^{2,1}(\overline{\Omega} \times [0,T]), \nabla\varphi.\vec{n} = 0 \text{ on } \partial\Omega \ \times \ (0,T), \varphi(.,T) = 0\}.$$

After integrating over $[0,T]$ and Ω and applying per partes and properties of a test function, we come to a definition of a weak solution.

Definition 2.1. *A weak solution of the regularized Perona-Malik problem (1.1)-(1.3) is a function $u \in L_2(I, H^1(\Omega))$ satisfying the identity*

$$\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t}(x,t)\, dx\, dt + \int_\Omega u_0(x)\varphi(x,0)\, dx -$$

$$\int_0^T \int_\Omega (g(|\nabla G_\sigma * u|)\nabla u \nabla \varphi\, dx\, dt = 0 \tag{2.3}$$

for all $\varphi \in \Psi$.

We can write the modified discrete scheme in its discrete weak form analogous to the identity (2.3), i.e.

$$\begin{aligned}\sum_{n=0}^{N_{\max}-1} \sum_{i=1}^{R} k \sum_{p \in \tau_h} \overline{u}_p^{nR+i} \frac{\varphi(x_p, t_{nR+i}) - \ \varphi(x_p, t_{nR+i-1})}{k} m(p) + \\ + \sum_{p\in\tau_h} \overline{u}_p^0 \varphi(x_p, 0)\, m(p) - \\ -\frac{1}{2} \sum_{n=0}^{N_{\max}-1} \sum_{i=0}^{R-1} k \sum_{(p,q)\in\mathcal{E}} g_{pq}^{\sigma,nR} T_{pq} \left(\overline{u}_q^{nR+i} - \overline{u}_p^{nR+i}\right) \\ \left(\varphi(x_q, t_{nR+i}) - \varphi(x_p, t_{nR+i})\right) = 0.\end{aligned} \tag{2.4}$$

2.3. Convergence results. To prove convergence of disrete scheme to the continuous one we need the following stability estimates:

Lemma 2.2 (A priori estimates in $L_2(Q_T)$). *We make following* **stability condition assumption***:*

$$k \leq (1-\xi) \frac{m(p)}{\sum\limits_{q\in N(p)} g_{pq}^{\sigma,nR} T_{pq}} \qquad \textit{for all } \ p \in \tau \textit{ and } \varepsilon \in (0,1) \tag{2.5}$$

Under this assumption it holds, that there exist positive constants C_1, C_2 such that
(i) $\max_{0\le l\le N_{\max}R} \sum_{p\in\tau_h} \left(\overline{u}_p^l\right)^2 m(p) \le C_1$
(ii) $\sum_{l=0}^{N_{\max}R} k \sum_{(p,q)\in\mathcal{E}} \frac{\left(\overline{u}_p^l - \overline{u}_q^l\right)^2}{d_{pq}} m\left(e_{pq}\right) \le C_2$
and the constants C_1, C_2 do not depend on the h, k.

Proof. Proof is the same as in [3]. □

For some fixed space and scale mesh h and k let us denote finite volume numerical solution $\overline{u}_{h,k}$. This solution is piecewise constant on each finite volume and each time step as is ussual for finite volume numerical schemes of parabolic type. Then we have:

LEMMA 2.3. *(Convergence of $\overline{u}_{h,k}$)*
There exists $u \in L^2\left(Q_T\right)$ a weak solution of (2.3) such that

$$\overline{u}_{h,k} \to u \text{ in } L^2\left(Q_T\right)$$

as $h, k \to 0$. Furthermore, the convergence is pointwise.

Proof. Convergence to some $u \in L^2\left(Q_T\right)$ is proved in ([3]). To show that this solution is really a weak sloution of a (2.3) we must prove the convergence of each term of (2.4) to its continuous analogy in (2.3) for all test functions $\varphi \in \Psi$ and is similar as in ([3]). The only exception is in one term (we use the same notation as in ([3]), which is proved in the following lemma and this complete the whole proof. □

LEMMA 2.4. *We denote*

$$R_3 = I_3 - I_4 \tag{2.6}$$

where

$$I_3 = \frac{1}{2} \sum_{n=0}^{N_{\max}-1} \sum_{i=0}^{R-1} \sum_{(p,q)\in\mathcal{E}} g_{pq}^{\sigma,nR} \left(\overline{u}_q^{nR+i} - \overline{u}_p^{nR+i}\right) \int_{t_{nR+i}}^{t_{nR+i+1}} \int_{e_{pq}} \nabla\varphi(x,t)\vec{n}_{pq}\, dx\, dt,$$

$$I_4 = \frac{1}{2} \sum_{n=0}^{N_{\max}-1} \sum_{i=0}^{R-1} \sum_{(p,q)\in\mathcal{E}} \int_{t_nR+i}^{t_{nR+i+1}} \int_{e_{pq}} g(|\nabla G_\sigma * \tilde{u}_{h,k}(x,t)|) \left(\overline{u}_q^{nR+i} - \overline{u}_p^{nR+i}\right) \nabla\varphi(x,t)\vec{n}_{pq}\, dx\, dt,$$

then $R_3 \to 0$.

Proof. Now, we denote

$$G_{pq}^{nR} = g(|\nabla G_\sigma * \tilde{u}_{h,k}\left(x_{pq}, t_{nR}\right)|) - g(|\nabla G_\sigma * \tilde{u}_{h,k}\left(x,t\right)|) \tag{2.7}$$

and then we have

$$R_3 = \frac{1}{2} \sum_{n=0}^{N_{\max}-1} \sum_{i=0}^{R-1} \sum_{(p,q)\in\mathcal{E}} \left(\overline{u}_q^{nR+i} - \overline{u}_p^{nR+i}\right) \int_{t_{nR+i}}^{t_{nR+i+1}} \int_{e_{pq}} G_{pq}^{nR} \nabla\varphi\vec{n}_{pq}\, d\nu\, dt. \tag{2.8}$$

To prove the convergence of R_3 to 0, first we bound G_{pq}^{nR}. For that purpose, we use the fact that g is Lipschitz continuous. Let L_g be the Lipschitz constant of g, i.e., for any positive real numbers ζ_1 and ζ_2 holds

$$|g(\zeta_1) - g(\zeta_2)| \le L_g |\zeta_1 - \zeta_2| . \tag{2.9}$$

Then we have after using triangular inequality for the Euclidean norm

$$\left|G_{pq}^{nR}\right| \le L_g \left|\nabla G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_{nR}) - \nabla G_\sigma * \tilde{u}_{h,k}(x,t)\right| .$$

Using the form of the convolution as given in (2.2) and as $t \in (t_{nR+i}, t_{nR+i+1})$, we can show that for any $x \in e_{pq}$ holds

$$\left|G_{pq}^{nR}\right| \le L_g \sum_{i=0}^{R-1} \sum_r \left|\overline{u}_r^{nR+i}\right| \int_r \left|\nabla G_\sigma(x_{pq} - s) - \nabla G_\sigma(x-s)\right| ds \le$$

$$\le L_g C \sum_{i=0}^{R-1} \sum_r \left|\overline{u}_r^{nR+i}\right| h m(r) \le L_g C R h. \tag{2.10}$$

The sum is evaluated only on control volumes $r \in \tau_h$ (as well as in reflexion of τ_h through boundary of Ω) intersecting $B_\sigma(x_{pq}) \cup B_\sigma(x)$, the balls centered at x_{pq} resp. x with radius σ. Thanks to the hypotheses on G_σ, which is in $C^\infty(I\!R^d)$, the Cauchy-Schwarz inequality and the apriori estimate (i) for discrete scheme we obtain

$$\left|G_{pq}^{n}\right| \le hC$$

with a positive constant C. Since $\nabla\varphi$ is a continuous function, $S = \sup_{Q_T} |\nabla\varphi| < \infty$, we have that

$$|R_3| \le h\frac{CS}{2} \sum_{n=0}^{N_{\max}-1} \sum_{i=0}^{R-1} k \sum_{(p,q)\in\mathcal{E}} \left|\overline{u}_q^{nR+i} - \overline{u}_p^{nR+i}\right| m(e_{pq})$$

which together with Cauchy-Schwarz inequality and stability estimate (ii) leads to the desired result

$$|R_3| \to 0 \text{ as } h, k \to 0. \tag{2.11}$$

□

3. Numerical experiments. In this section we present experiments with some real as well as artificial images perturbed by various type of noise. In simulations, we use the function

$$g(s) = \frac{1}{1 + Ks^2}$$

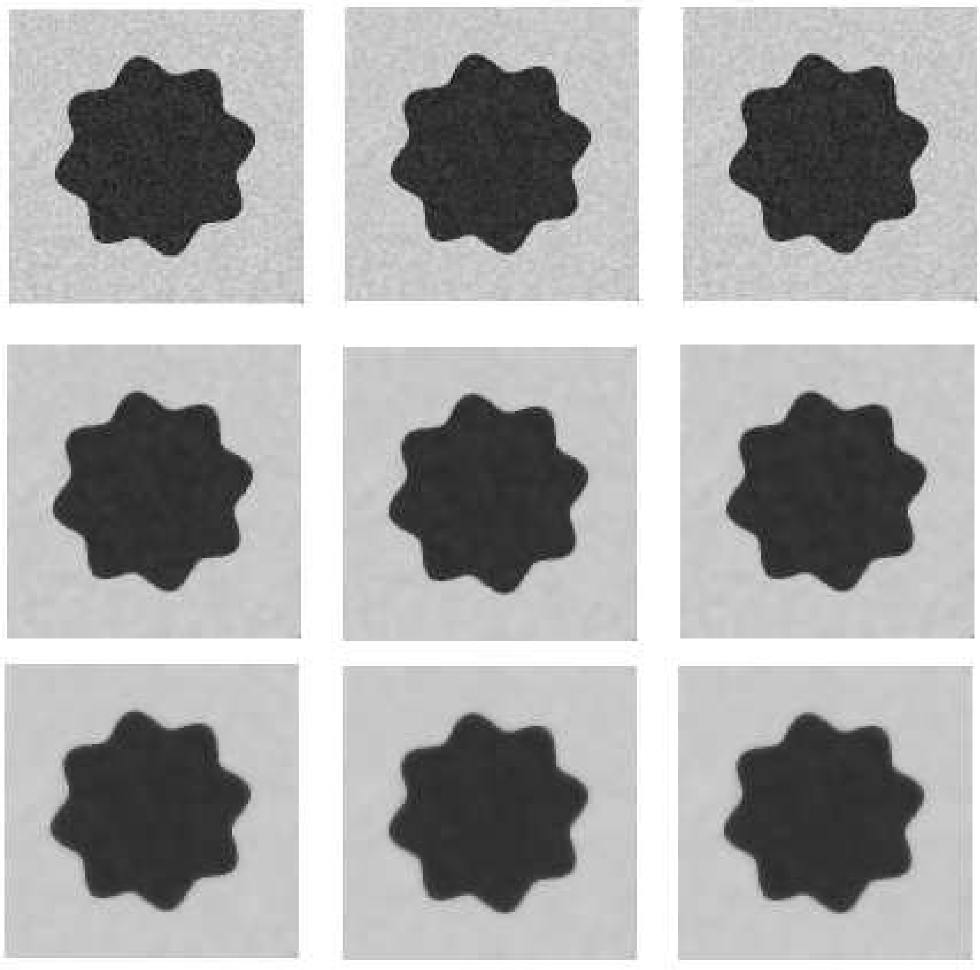
and the convolution is realized with the kernel

$$G_\sigma(x) = \frac{1}{Z} e^{\frac{|x|^2}{|x|^2 - \sigma^2}},$$

where the constant Z is chosen so that G_σ has unit mass.
Example 1. To every position of the initial image we apply a noise function f defined as follows: if $\psi(x)$ is a random function generating values in $[0, 2C]$, then for every position x

$$f(u_0(x)) = MIN(255, MAX(0, u_0(x) - C + \psi).$$

$C = 50$ and the difference in intensity between the two values of the initial image is 150. We compare the work using semi-implicit method without SOR, with SOR, explicit method and our modified explicit method. For this example we have $h = 1, T = 10$. For semi-implicit schemes we choose $k = 1$, i.e. 10 scale steps, for explicit schemes we must choose scale step $k = \frac{1}{4}$ i.e. 40 scale steps.



a) b) c)

FIG. 3.1. *The column a) shows the work of semi-implicit scheme($3^{rd}, 6^{th}$ and 10^{th} scale steps), b) shows the work of explicit scheme and the column c) the work of modified explicit scheme for Example 1*

method	CPU time	number of iterations
semi-implicit without SOR	4.74 s	$12 - 15$
semi-implicit with SOR ($\omega = 1.25$)	4.00 s	$9 - 13$
explicit	3.63 s	
modified explicit	2.18 s	

Example 2. This example compares the work of explicit and modified explicit schemes applied to medical data, which has to be prepared for segmentation algorithms. The significant improvement of CPU time is caused by the fact, that we used slowing down of diffusion in some regions of the image and computing the diffusion coefficients takes more time. Visual effect of both algorithms is similar. In this experiment $T = 5$ and $k = \frac{1}{4}$.

method	CPU time
explicit	17.05 s
modified explicit	4.69 s

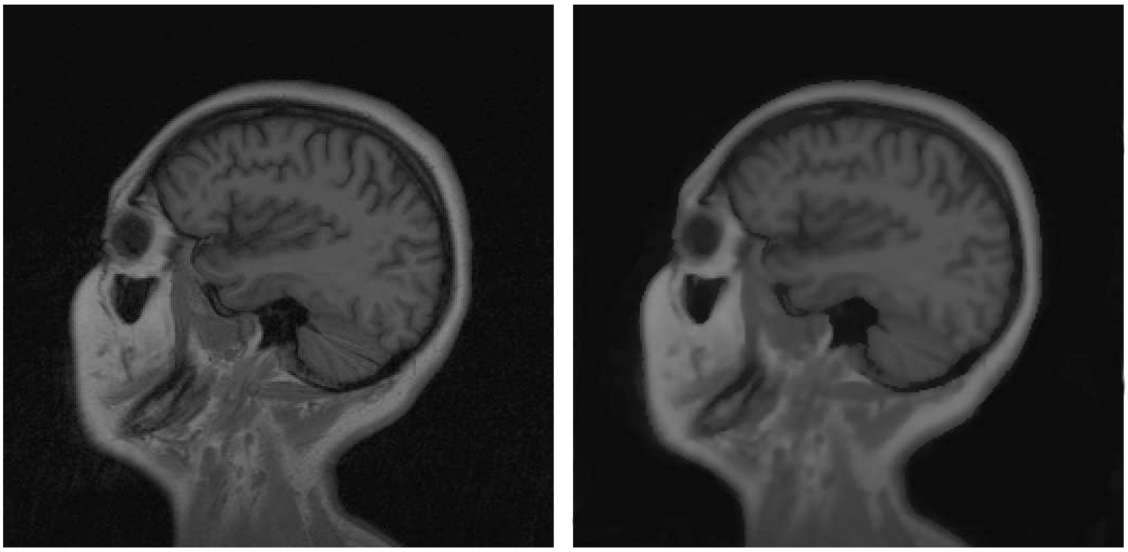


FIG. 3.2. *The initial noisy image and the result of smoothing for Example 2*

Example 3. The artificial data for this example are given by a double-valued function $\hat{u}(x)$ with intensity difference set to 50. In order to obtain $u_0(x)$, the initial data is perturbed by the additive noise with $C = 38$. We compare the work of the semi-implicit, explicit and modified explicit schemes for 3D data and show the visual results of algorithms which are different in this case. $T = 11$ with $k = 1$ for the semi-implicit scheme and $k = \frac{1}{4}$ for the explicit schemes.

method	CPU time
semi-implicit with SOR ($\omega = 1.20$)	159.16 s
explicit	92.46 s
modified explicit	66.70 s

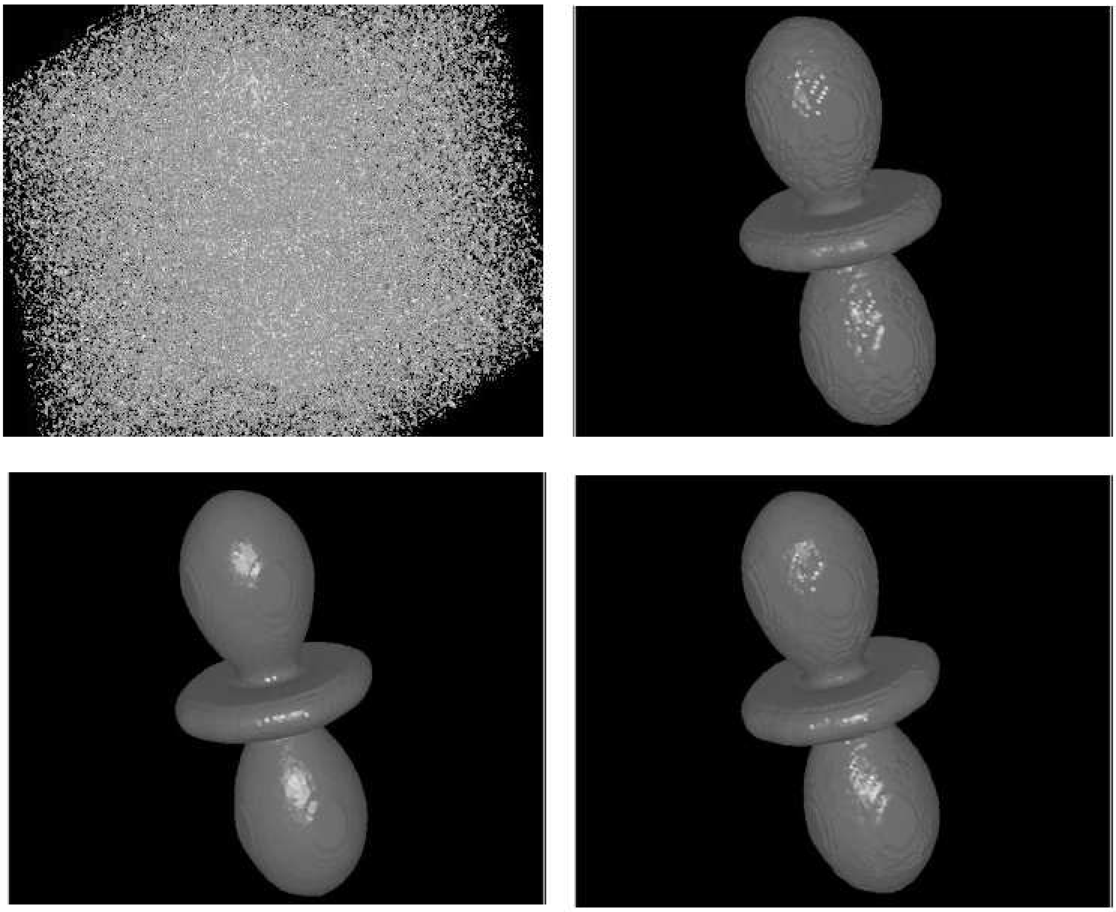


FIG. 3.3. *The picture shows the initial noisy image and results of smoothing by semi-implicit scheme at the top and results of smoothing by explicit and modified explicit schemes at the bottom for Example 3*

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