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LOCAL QUADRATIC APPROXIMATION IN VERTICES OF PLANAR TRIANGULATIONS*

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Abstract. For every strongly regular triangulation $\mathcal{T}_h$ in 2D, we describe a class of local sets of vertices in which the least–squares approximations of smooth functions by quadratic polynomials are of optimal order. As an application of this result, we prove that for any inner vertex a with affine neighbourhood $b^1, \ldots, b^6$, the least–squares quadratic approximation Q of any smooth function u in the points $b^1, \ldots, b^6, a$ has the following relation to the globally continuous and piecewise linear projection Πu of u. The gradient $\operatorname{grad} Q(a)$ is equal to the arithmetic mean of constant gradients $\operatorname{grad} \Pi u/_T$ on the triangles $T \in \mathcal{T}_h$ meeting a.

Key words. plane triangulations, poised sets of vertices, quadratic least–squares approximation

AMS subject classifications. 41A05, 41A10, 65D05

1. Introduction and basic notions. This paper concerns the classical topic of discrete approximation of smooth functions in two variables by polynomials, see Beresin, Shidkov [1], whose dynamical recent development can be illustrated by the papers Sauer, Xu [2] and Gasca, Sauer [3]. We study the least–squares approximation by quadratic polynomials under the condition that points of approximation are vertices of a given triangulation. In this section, we describe a class of six–tuples of vertices of a given strongly regular triangulation $\mathcal{T}_h$ in which, due to Dalík [7], the problem of interpolation by quadratic polynomials does always have a unique solution. In Section 2, we prove a theorem saying that quadratic discrete least–squares approximations in vertices from specified sets are of optimal order. In Section 3, we prove the following statement for any affine neighbourhood $b^1, \ldots, b^6$ of an inner vertex a with triangles $T_1 = \overline{ab^6b^1}$, $T_2 = \overline{ab^1b^2}, \ldots, T_6 = \overline{ab^5b^6}$ in $\mathcal{T}_h$. For any function $u \in \mathbf{C}^3(T_1 \cup \ldots \cup T_6)$, its projection $\Pi u \in \mathbf{C}(T_1 \cup \ldots \cup T_6)$ linear on $T_1, \ldots, T_6$, and for the quadratic least–squares approximation Q of u in $b^1, \ldots, b^6, a$, we have $\operatorname{grad} Q(a) = \frac{1}{6}\,(\operatorname{grad} \Pi u/_{T_1} + \ldots + \operatorname{grad} \Pi u/_{T_6})$.

We denote by (x_1, x_2) the cartesian coordinates of a point $x \in \mathbf{R}^2$ and put

$$D(abc) = \begin{vmatrix} a_1 - c_1 & a_2 - c_2 \\ b_1 - c_1 & b_2 - c_2 \end{vmatrix}$$

for arbitrary points $a, b, c \in \mathbf{R}^2$. It is known that $D(abc) > 0$ if and only if the ordered triple (a, b, c) is oriented positively and $A(\overline{abc}) = |D(abc)|$ is the area of triangle $\overline{abc}$. We denote by $\mathcal{P}^2$ the space of polynomials in the variables x_1, x_2 of total degree less than or equal to two.

Definition 1.1. *Points $b^1, \ldots, b^6 \in \mathbf{R}^2$ are said to be poised if there exists a unique $P \in \mathcal{P}^2$ such that*

$$P(b^i) = f_i \quad \text{for} \quad i = 1, \ldots, 6 \tag{1.1}$$

for arbitrary given $f_1, \ldots, f_6 \in \mathbf{R}$.

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Definition 1.2. *For arbitrary points $b^1, \ldots, b^6 \in \mathbf{R}^2$, we put*

$$l(b^1, \ldots, b^6) = D(b^1b^5b^6)D(b^1b^2b^3)D(b^4b^5b^3)D(b^4b^6b^2) + D(b^1b^3b^5)D(b^1b^6b^2)D(b^4b^5b^6)D(b^4b^2b^3).$$

The following statement gives us the basic tool for the decision whether a six–tuple of points is poised.

Theorem 1.3. *Points $b^1, \ldots, b^6 \in \mathbf{R}^2$ are poised if and only if $l(b^1, \ldots, b^6) \neq 0$.*

Proof. See [5]. □

Definition 1.4. *We say that a finite nonempty set $\mathcal{T}$ of triangles is a triangulation if the intersection of any two different triangles T_1, T_2 is either a common side of T_1, T_2 or a common vertex of T_1, T_2 or an empty set.*

The symbol $\mathcal{T}_h$ denotes a triangulation with the largest length of sides of triangles equal to h. Further, we denote by Ω_h the union $\bigcup_{T \in \mathcal{T}_h} T$ and by $\mathcal{V}_h$ the set of vertices of triangles from $\mathcal{T}_h$. We put

$$\mathcal{N}_h(a) = \{b \in \mathcal{V}_h;\ \overline{ab} \text{ is a side of a triangle from } \mathcal{T}_h\}$$

for every $a \in \mathcal{V}_h$.

Definition 1.5. *Let Ω be an open and bounded set in $\mathbf{R}^2$ (a domain in $\mathbf{R}^2$) and $\mathcal{T}_h$ be a triangulation. We denote by $\delta\Omega$ the boundary of Ω and call $\mathcal{T}_h$ a triangulation of Ω if $\mathcal{V}_h \subseteq \overline{\Omega}$ and $\mathcal{V}_h \cap \delta\Omega_h = \mathcal{V}_h \cap \delta\Omega$.*

Definition 1.6. *A system $\mathbf{F} = (\mathcal{T}_h)_{h \in I}$ of triangulations of a domain Ω in $\mathbf{R}^2$ is called strongly regular if*

a) I is a set of positive real numbers satisfying $0 \in \overline{I}$ and

b) there exists a $\kappa > 0$ such that all triangles $T \in \mathcal{T}_h \in \mathbf{F}$ contain a disc with radius κh.

It can be shown that for every strongly regular system $\mathbf{F}$ there exist $\kappa_0 > 0$, $\alpha_0 > 0$ such that each triangle from $\mathcal{T}_h$ has all sides longer than $\kappa_0 h$ and all inner angles greater than α_0.

Notations. We denote by $\mathbf{F}$ a strongly regular system of triangulations of a fixed domain Ω characterized by the parameters κ, κ_0, α_0 and reserve the symbols $C, C_1, \ldots$ for generic constants independent of the parameter h.

We describe two basic types of local poised sets of vertices of a triangulation.

Definition 1.7. *We call mutually different vertices $b^1, \ldots, b^6$ of a triangulation $\mathcal{T}_h \in \mathbf{F}$ a neighbourhood of a triangle $T_1 \in \mathcal{T}_h$ if $T_1 = \overline{b^1b^3b^5}$ and triangles*

$$T_2 = \overline{b^1b^2b^3}, \quad T_3 = \overline{b^3b^4b^5}, \quad T_4 = \overline{b^5b^6b^1}$$

belong to $\mathcal{T}_h$.

The following theorem says that neighbourhoods of triangles are poised.

Theorem 1.8. *There exists a constant $C > 0$ satisfying*

$$l(b^1, \ldots, b^6) > C\, A(T_k)\, A(T_2)\, A(T_3)\, A(T_4) \quad \text{for some} \quad k \in \{1, \ldots, 4\}$$

for all $\mathcal{T}_h \in \mathbf{F}$ and all $T_1 \in \mathcal{T}_h$ with a neighbourhood $b^1, \ldots, b^6$ such that $T_1, \ldots, T_4$ have no inner angles obtuse.

Proof. This is the content of Dalík [6]. □

Now we relate poised sets to vertices of triangulations from $\mathbf{F}$.

Agreement. For arbitrary points $x^1, \ldots, x^n \in \mathbf{R}^2$, the operations $+$ and $-$ are addition and subtraction modulo n on the set $\{1, \ldots, n\}$ of indices.

Definition 1.9. *Let* $a \in \mathcal{V}_h$ *and* $b^1, \dots, b^k \in \mathcal{N}_h(a)$. *We put* $T_i = \overline{ab^{i-1}b^i}$, $\alpha_i = \angle b^{i-1}ab^i$, $\beta_i = \angle b^ib^{i-1}a$, $\gamma_i = \angle ab^ib^{i-1}$ *for* $i = 1, \dots, k$ *and call* $b^1, \dots, b^k$ *an oriented neighbourhood of* a *whenever* $D(ab^{i-1}b^i) > 0$ *for* $i = 1, \dots, k$ *and* $\alpha_1 + \dots + \alpha_k = 2\pi$. *See Fig. 1.*

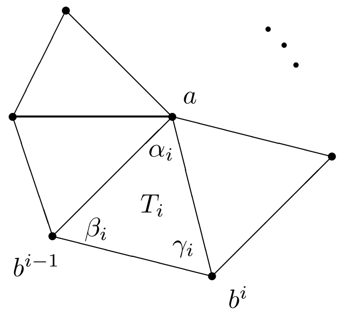

Figure 1

Theorem 1.10. *There exists a constant* $C > 0$ *such that*

$$l(b^1, \dots, b^6) > C\, h^8$$

for all $\mathcal{T}_h \in \mathbf{F}$, $b^6 = a \in \mathcal{V}_h$ *and oriented neighbourhoods* $b^1, \dots, b^5$ *of* a *with the following properties*

a) $\max(\alpha_1, \dots, \alpha_5, \pi/2) \le \alpha_i + \alpha_{i+1}$ *for* $i = 1, \dots, 5$,

b) $\alpha_i \le \frac{2}{3}\pi$ *for* $i = 1, \dots, 5$,

c) $\pi < \alpha_i + \alpha_{i+1}$ *for at most one index* i,

d) $\beta_i \le \pi/2$, $\gamma_i \le \pi/2$ *for* $i = 1, \dots, 5$.

Proof. See Dalík [7]. □

In [7], a simple reduction procedure is described which selects an oriented neighbourhood with the properties a) – d) from the set $\mathcal{N}_h(a)$ of any inner vertex a of a triangulation $\mathcal{T}_h \in \mathbf{F}$ such that $|\mathcal{N}_h(a)| \ge 5$ and the inner angles of all triangles from $\mathcal{T}_h$ meeting a are less than or equal to $\pi/2$.

Definition 1.11. *Let us assume that* $\mathcal{T}_h \in \mathbf{F}$ *and either*

a) $b^1, \dots, b^6$ *is a neighbourhood of a triangle from* $\mathcal{T}_h$ *satisfying the assumptions of Theorem 1.8 or*

b) $b^1, \dots, b^5$ *is an oriented neighbourhood of a vertex* $a = b^6$ *of* $\mathcal{T}_h$ *satisfying the assumptions a) – d) of Theorem 1.10.*

Then we call $\{b^1, \dots, b^6\}$ *a local poised set (in* $\mathcal{T}_h$*).*

2. Quadratic least squares approximation. In this section we describe certain supersets of local poised sets and prove that discrete quadratic least–squares approximations in points from these supersets are of optimal order.

Definition 2.1. *Let* $b^1, \dots, b^6$ *be a local poised set in* $\mathcal{T}_h \in \mathbf{F}$. *We put* $B = \{b^1, \dots, b^6\}$ *in the case a),* $B = \{a\} \cup \mathcal{N}_h(a)$ *in the case b) from 1.11 and call the set*

$$\mathcal{E}(b^1, \dots, b^6) = \{x \in \mathcal{V}_h \mid \overline{xyz} \in \mathcal{T}_h \ \text{ for some } \ y, z \in B\}$$

an extension of $\{b^1, \dots, b^6\}$.

For every nonempty set $E \subseteq \mathcal{V}_h$, *we denote by* $\mathcal{C}E$ *the convex closure of* E. *Instead of* $\mathcal{C}E(b^1, \dots, b^6)$, *we briefly write* $\mathcal{C}\mathcal{E}$.

If $\{b^1, \dots, b^6\}$ is a local poised set in a triangulation $\mathcal{T}_h \in \mathbf{F}$ and we put

$$l_1(x) = l(x, b^2, \dots, b^6)$$

then $l_1(b^1) \neq 0$ by Theorems 1.8, 1.10 and it is easy to see that $l_1(b^j) = 0$ for $j = 2, \ldots, 6$. By symmetry, analogous properties share the following quadratic polynomials $l_2, \ldots, l_6$. See [5] for more details.

DEFINITION 2.2. *Let be $\{b^1, \ldots, b^6\}$ a local poised set in $\mathcal{T}_h$ and K a closed convex set such that $\mathcal{C}\{b^1, \ldots, b^6\} \subseteq K \subseteq \mathcal{CE}$. We put*

$$l_i(x) = l(x, b^{i+1}, \ldots, b^{i+5}) \ \text{ and } \ L_i(x) = \frac{l_i(x)}{l_i(b^i)} \quad \text{for} \quad i = 1, \ldots, 6.$$

We have $L_i(b^j) = \delta_{ij}$ for $i, j = 1, \ldots, 6$, so that $L_1, \ldots, L_6$ is a *Lagrange basis* in $\mathcal{P}^2$ related to the points $b^1, \ldots, b^6$ and

$$L(x) = \sum_{i=1}^{6} u(b^i)\, L_i(x)$$

is the *Lagrange interpolation polynomial* of any function $u \in \mathbf{C}(K)$ in $b^1, \ldots, b^6$.

LEMMA 2.3. *There exists a constant $\nu_1 > 0$ such that*

$$|L_i(x)| \leq \nu_1, \quad \left| \frac{\partial L_i}{\partial x_\iota}(x) \right| \leq \nu_1 h^{-1} \tag{2.1}$$

for all triangulations $\mathcal{T}_h \in \mathbf{F}$, all local poised sets $\{b^1, \ldots, b^6\}$ in $\mathcal{T}_h$, all $x \in \mathcal{C}E$, $i = 1, \ldots, 6$ and $\iota = 1, 2$.

Proof. Due to Theorems 1.8, 1.10 and to the strong regularity of $\mathbf{F}$, there exist positive constants C, C_1, C_2 such that $C\,h^8 < |l_i(b^i)|$, $|l_i(x)| < C_1 h^8$ and $|\partial l_i(x)/\partial x_\iota| < C_2 h^7$ for all local poised sets $\{b^1, \ldots, b^6\}$, all $x \in \mathcal{C}E$, $i = 1, \ldots, 6$ and $\iota = 1, 2$. The statements follow immediately. □

As a direct consequence, we obtain the following result.

LEMMA 2.4. *Assume that $\{b^1, \ldots, b^6\}$ is a local poised set in $\mathcal{T}_h \in \mathbf{F}$ and $P \in \mathcal{P}^2$ satisfies*

$$|P(b^i)| \leq c\,h^3 \quad \text{for} \quad i = 1, \ldots, 6$$

for some $c \geq 0$. Then

$$|P(x)| \leq 6\,\nu_1 c\,h^3 \ \forall\, x \in \mathcal{C}E.$$

In [7], we have proved the following two statements.

THEOREM 2.5. *Assume that $\{b^1, \ldots, b^6\}$ is a local poised set in $\mathcal{T}_h \in \mathbf{F}$, K is a closed convex set with $\mathcal{C}\{b^1, \ldots, b^6\} \subseteq K \subseteq \mathcal{C}E$ and functions $u \in \mathbf{C}^3(K)$, $P \in \mathcal{P}^2$ satisfy $|(u - P)(x)| < C_1 h^3$ for all $x \in K$. Then there exists a constant $C > 0$ such that*

$$\left| \frac{\partial^{|m|}(u - P)}{\partial x^m}(x) \right| \leq C\,h^{3-|m|} \ \forall\, x \in K$$

for all multiindices m, $|m| \leq 2$.

THEOREM 2.6. *Let $\mathcal{T}_h \in \mathbf{F}$, $\{b^1, \ldots, b^6\}$ be a local poised set in $\mathcal{T}_h$ and $u \in \mathbf{C}^3(\mathcal{C}E)$. Then there exist a unique interpolation polynomial $L \in \mathcal{P}^2$ of u in $b^1, \ldots, b^6$ and a constant $C > 0$ such that*

$$\left| \frac{\partial^{|m|}(u - L)}{\partial x^m}(x) \right| \leq C\,h^{3-|m|} \ \forall x \in \mathcal{CE}$$

for all multiindices m, $|m| \le 2$.

Now, we prove our result concerning the local discrete least–squares approximation.

Theorem 2.7. *Let us assume that $\{b^1,\dots,b^6\}$ is a local poised set in $\mathcal{T}_h \in \mathbf{F}$, $\{b^1,\dots,b^6\} \subseteq B \subseteq \mathcal{E}(b^1,\dots,b^6)$ and $u \in \mathbf{C}^3(\mathcal{C}E)$. Then there exist a unique discrete least–squares approximation $Q \in \mathcal{P}^2$ of u in the vertices from B and a constant $C > 0$ such that*

$$\left|\frac{\partial^{|m|}(u-Q)}{\partial x^m}(x)\right| \le C\,h^{3-|m|}\ \forall x \in \mathcal{C}E$$

for all multiindices m, $|m| \le 2$.

Proof. Consider the Lagrange basis functions $L_1,\dots,L_6$ related to $b^1,\dots,b^6$ and denote $B = \{b^1,\dots,b^6,\dots,b^k\}$. Then

$$Q(x) = \sum_{i=1}^{6} q_i L_i(x)$$

is a discrete least–squares approximation of u in $b^1,\dots,b^k$ if and only if

$$M^\top M q = M^\top b$$

for $M = (L_j(b^i))_{i=1,\dots,k}^{j=1,\dots,6}$, $q = (q_1,\dots,q_6)^\top$, $b = (u(b^1),\dots,u(b^k))^\top$. Because $L_j(b^i) = \delta_{ij}$ for $i,j = 1,\dots,6$, the columns of M are linearly independent. This guaranties existence and unicity of Q by Bjőrck [4], Theorem 1.1.3.

The interpolant $L(x) = \sum_{j=1}^{6} u(b^j)\,L_j(x)$ satisfies

$$|(u-L)(b^i)| \le C\,h^3 \quad \textit{for} \quad i = 1,\dots,k$$

by Theorem 2.6. Then

$$\sum_{i=1}^{k}(u-Q)(b^i)^2 \le \sum_{i=1}^{k}(u-L)(b^i)^2 \le C\,h^6$$

and we have

$$|(u-Q)(b^i)| \le C\,h^3 \quad \textit{for} \quad i = 1,\dots,k.$$

Of course

$$|(L-Q)(b^j)| \le C\,h^3 \quad \textit{for} \quad j = 1,\dots,6,$$

so that $|(L-Q)(x)| \le C\,h^3\ \forall x \in \mathcal{C}E$ by Lemma 2.4. This result and Theorem 2.6 give us

$$|(u-Q)(x)| \le C\,h^3\ \forall x \in \mathcal{C}E$$

and we finish the proof by an application of Theorem 2.5. □

3. Approximation of $\operatorname{grad} u$**.** Now, we prove a special property of $\operatorname{grad} Q(a)$ for any vertex a with affine neighbourhood $b^1, \dots, b^6$ in $\mathcal{T}_h \in \mathbf{F}$, any smooth function u and for the least–squares approximation Q of u in the points $b^1, \dots, b^6, a$.

DEFINITION 3.1. *We call* Π *a piecewise linear projection related to triangles* $T_1, \dots, T_k$ *whenever for every* $u \in C(T_1 \cup \dots \cup T_k)$, $\Pi u(b) = u(b)$ *in all vertices* b *of triangles* $T_1, \dots, T_k$ *and* Πu *is linear on each of the triangles* $T_1, \dots, T_k$.

DEFINITION 3.2. *Let us put* $\hat{b}^1 = (-1,1)$, $\hat{b}^2 = (-1,0)$, $\hat{b}^3 = (0,-1)$, $\hat{b}^4 = (1,-1)$, $\hat{b}^5 = (1,0)$, $\hat{b}^6 = (0,1)$, $\hat{b}^7 = \hat{a} = (0,0)$, $\hat{T}_i = \overline{\hat{a}\hat{b}^{i-1}\hat{b}^i}$ *for* $i = 1, \dots, 6$ *and* $\hat{\Theta} = \hat{T}_1 \cup \dots \cup \hat{T}_6$. *See Fig. 2.*

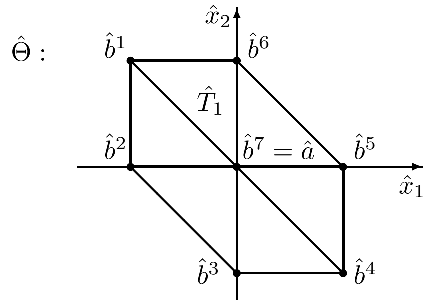

Figure 2

We denote by $\hat{\Pi}$ *the piecewise linear projection related to* $\hat{T}_1, \dots, \hat{T}_6$.

LEMMA 3.3. *Let* $\hat{u} \in C(\hat{\Theta})$ *and* $\hat{Q} \in \mathcal{P}^2$ *be a discrete least–squares approximation of* $\hat{u}$ *in* $\hat{b}^1, \dots, \hat{b}^7$. *Then*

$$\operatorname{grad} \hat{Q}(\hat{a}) = \frac{1}{6} \sum_{i=1}^{6} \operatorname{grad} \hat{\Pi}\hat{u}/_{\hat{T}_i}.$$

Proof. The polynomial $\hat{Q}(\hat{x}) = c_1\hat{x}_1^2 + c_2\hat{x}_1\hat{x}_2 + c_3\hat{x}_2^2 + c_4\hat{x}_1 + c_5\hat{x}_2 + c_6$ is a discrete least–squares approximation of $\hat{u}$ in $\hat{b}^1, \dots, \hat{b}^7$ if and only if the sum

$$\sum_{i=1}^{7} \left(\hat{u}_i - \hat{Q}(\hat{b}^i) \right)^2$$

is minimal for $\hat{u}_i = \hat{u}(\hat{b}^i)$. This is equivalent to the system $M\vec{c} = N\vec{u}$ of normal equations, where $\vec{c} = (c_1, \dots, c_6)^\top$, $\vec{u} = (\hat{u}_1, \dots, \hat{u}_7)^\top$ and

$$M = \begin{pmatrix} 4 & -2 & 2 & & & 4 \\ -2 & 2 & -2 & & & -2 \\ 2 & -2 & 4 & & & 4 \\ & & & 4 & -2 & \\ & & & -2 & 4 & \\ 4 & -2 & 4 & & & 7 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & & 1 & 1 & & \\ -1 & & & -1 & & & \\ 1 & & 1 & 1 & & 1 & \\ -1 & -1 & & 1 & 1 & & \\ 1 & & -1 & -1 & & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This system has a unique solution

$$\begin{array}{llrrrrrrl} c_1 & = (& & \hat{u}_2 & & & +\hat{u}_5 & & -2\hat{u}_7)/2 \\ c_2 & = (& -\hat{u}_1 & +\hat{u}_2 & +\hat{u}_3 & -\hat{u}_4 & +\hat{u}_5 & +\hat{u}_6 & -2\hat{u}_7)/2 \\ c_3 & = (& & & \hat{u}_3 & & & +\hat{u}_6 & -2\hat{u}_7)/2 \\ c_4 & = (& -\hat{u}_1 & -2\hat{u}_2 & -\hat{u}_3 & +\hat{u}_4 & +2\hat{u}_5 & +\hat{u}_6 &)/6 \\ c_5 & = (& \hat{u}_1 & -\hat{u}_2 & -2\hat{u}_3 & -\hat{u}_4 & +\hat{u}_5 & +2\hat{u}_6 &)/6 \\ c_6 & = (& & & & & & & \hat{u}_7) \end{array},$$

so that $\operatorname{grad}\hat{Q}(\hat{a}) = \begin{pmatrix} c_4 \\ c_5 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -\hat{u}_1 - 2\hat{u}_2 - \hat{u}_3 + \hat{u}_4 + 2\hat{u}_5 + \hat{u}_6 \\ \hat{u}_1 - \hat{u}_2 - 2\hat{u}_3 - \hat{u}_4 + \hat{u}_5 + 2\hat{u}_6 \end{pmatrix}$. On the other hand, we have

$$\begin{aligned}
\operatorname{grad}\hat{\Pi}\hat{u}/_{\hat{T}_1} &= \begin{pmatrix} \hat{u}_6 - \hat{u}_1 \\ \hat{u}_6 - \hat{u}_7 \end{pmatrix}, & \operatorname{grad}\hat{\Pi}\hat{u}/_{\hat{T}_2} &= \begin{pmatrix} \hat{u}_7 - \hat{u}_2 \\ \hat{u}_1 - \hat{u}_2 \end{pmatrix},\\
\operatorname{grad}\hat{\Pi}\hat{u}/_{\hat{T}_3} &= \begin{pmatrix} \hat{u}_7 - \hat{u}_2 \\ \hat{u}_7 - \hat{u}_3 \end{pmatrix}, & \operatorname{grad}\hat{\Pi}\hat{u}/_{\hat{T}_4} &= \begin{pmatrix} \hat{u}_4 - \hat{u}_3 \\ \hat{u}_7 - \hat{u}_3 \end{pmatrix},\\
\operatorname{grad}\hat{\Pi}\hat{u}/_{\hat{T}_5} &= \begin{pmatrix} \hat{u}_5 - \hat{u}_7 \\ \hat{u}_5 - \hat{u}_4 \end{pmatrix}, & \operatorname{grad}\hat{\Pi}\hat{u}/_{\hat{T}_6} &= \begin{pmatrix} \hat{u}_5 - \hat{u}_7 \\ \hat{u}_6 - \hat{u}_7 \end{pmatrix}
\end{aligned}$$

and $\frac{1}{6}\sum_{i=1}^{6}\operatorname{grad}\hat{\Pi}\hat{u}/_{\hat{T}_i} = \operatorname{grad}\hat{Q}(\hat{a})$ follows immediately. □

Now, we extend this statement to all inner vertices with an affine neighbourhood.

Definition 3.4. *Let $\mathcal{T}_h \in \mathbf{F}$, $a = b^7$ be an inner vertex of $\mathcal{T}_h$ and $b^1, \dots, b^6$ be an orientation of $\mathcal{N}_h(a)$. If there exist a regular matrix B and a point c such that the affine map*

$$F:\ \hat{\Theta} \longrightarrow \Omega_h,\ F(\hat{x}) = B\,\hat{x} + c$$

satisfies $F(\hat{b}^i) = b^i$ for $i = 1, \dots, 7$, then we call $b^1, \dots, b^6$ an affine neighbourhood of $a = b^7$, put $T_i = F(\hat{T}_i)$ for $i = 1, \dots, 6$, $\Theta = F(\hat{\Theta})$ and relate a function u to every $\hat{u} \in C(\hat{\Theta})$ by

$$u(F(\hat{x})) = \hat{u}(\hat{x})\quad \forall \hat{x} \in \hat{\Theta}.$$

We denote by Π the piecewise linear projection related to $T_1, \dots, T_6$.

Because

$$\sum_{i=1}^{7}\left(P(b^i) - u(b^i)\right)^2 = \sum_{i=1}^{7}\left(\hat{P}(\hat{b}^i) - \hat{u}(\hat{b}^i)\right)^2$$

for all $P \in \mathcal{P}^2$, $\hat{u} \in \mathbf{C}(\hat{\Theta})$, a polynomial $Q \in \mathcal{P}^2$ is a discrete least–squares approximation of u in $b^1, \dots, b^7$ if and only if $\hat{Q} \in \mathcal{P}^2$ is a discrete least–squares approximation of $\hat{u}$ in $\hat{b}^1, \dots, \hat{b}^7$.

Theorem 3.5. *Let $\mathcal{T}_h \in \mathbf{F}$, $b^1, \dots, b^6$ be an affine neighbourhood of a vertex $a = b^7$ and $Q \in \mathcal{P}^2$ be a discrete least–squares approximation of $u \in C(\mathcal{C}E)$ in $b^1, \dots, b^7$. Then*

$$\operatorname{grad} Q(a) = \frac{1}{6}\sum_{i=1}^{6}\operatorname{grad}\Pi u/_{T_i}.$$

Proof. If F is an affine map sending $\hat{b}^i$ to b^i for $i = 1, \dots, 7$, then

$$\begin{aligned}
\operatorname{grad} Q(a) &= \operatorname{grad}\hat{Q}(F^{-1}(a)) = J\,F^{-1}(a)^T \operatorname{grad}\hat{Q}(\hat{a})\\
&= \tfrac{1}{6}\textstyle\sum_{i=1}^{6} J\,F^{-1}(a)^T \operatorname{grad}\hat{\Pi}\hat{u}/_{\hat{T}_i} = \tfrac{1}{6}\sum_{i=1}^{6}\operatorname{grad}\Pi u/_{T_i}
\end{aligned}$$

according to Lemma 3.3 and to the fact that $\hat{Q}$ is the discrete least squares approximation of $\hat{u}$ in $\hat{b}^1, \dots, \hat{b}^7$. Here $J\,F^{-1}(a)$ is the Jacobi matrix of the vector–function F^{-1} in the point a. □

Due to Dalík [7], $\operatorname{grad} P(a) = \frac{1}{6}\sum_{i=1}^{6}\operatorname{grad}\Pi P/_{T_i}$ is valid for all $P \in \mathcal{P}^2$, but Theorem 3.5 is a special property of the discrete least–squares approximation Q.

COROLLARY 3.6. *For all triangulations $\mathcal{T}_h \in \mathbf{F}$, all affine neighbourhoods $b^1, \dots, b^6$ of a vertex $a \in \mathcal{V}_h$, and all $u \in \mathbf{C}^3(\mathcal{C}E)$, there exists $C > 0$ such that*

$$\| \operatorname{grad} u(a) - \frac{1}{6} \sum_{i=1}^{6} \operatorname{grad} \Pi u/_{T_i} \| < C\, h^2$$

for any vector–norm $\| \cdot \|$.

Proof. This statement follows by Theorem 3.5 and by $\| \operatorname{grad} u(a) - \operatorname{grad} Q(a) \| < C\, h^2$, valid due to Theorem 2.7. □

As $\operatorname{grad} \Pi u/_{T_i}$ is an approximation of $\operatorname{grad} u(a)$ of order 1 for $i = 1, \dots, n$, Corollary 3.6 presents a superapproximation of $\operatorname{grad} u(a)$. This classical superapproximation formula, used by ingeneers routinely, has been analysed in Křížek,Neittaanmäki [8] theoretically for the first time.

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