Proceedings of ALGORITMY 2005
pp. 319–324

METHOD OF GENERATION OF ADAPTIVE ANALYTICAL GRIDS BASED ON POISSON'S EQUATION

NADEGDAA FEDOSENKO * AND EVGENY SOKOLOV †

Abstract. In [1], the new method of grid generation based on an analytical solution of some system of elliptic differential equations had been suggested. The functions that map computational space to physical one (mapping-functions, MF) were obtained there in exact form. Method of quality control of obtained grids was based on application of analytical solutions of a boundary value problem for the class of higher order equations. But, such approach allows to control grid properties only near boundaries of computational domain, not inside it. In order to overcome this disadvantage, exact analytical solution of Poisson's equation is obtained in this paper and than used for grid generation. Its right-side source term allows to control grid properties inside computational domain.

It is well known that solution of Poisson's equation is expressed in terms of Green's function. Derivation of its finite algebraic form is described in present paper. Finally, it allows to use solution as effective tool for grid generation and control.

Approach is illustrated by examples of grid under the control.

Key words. Analytical grids, mapping-functions, Poisson's equation, Green's function

AMS subject classifications. 35J05, 35J25

1. Introduction. First, let us remind briefly traditional method of generation of elliptical grids. Usually, some system of elliptical equations satisfying to an extremum principle is formulated and the isolines of functions obtaining from a solution are taken as new coordinate lines. Extremum principle guarantees a one-to-one mapping between the physical and transformed regions if equations are formulated in physical space. To solve a system, it has to be written in computing space that essentially complicates equations. It leads to a system of closure of set of non-linear equations in computational space instead of linear equation in physical space. This system converges slowly with increasing of complexity of calculating domain.

In this paper, in accordance with ideas [1], generation equations are formulated in computational space. Their solution gives a MF which maps points from computing space to physical, i.e. as it is required in the numerical methods. In [1], solutions of a boundary value problem for the Laplace equation for square and cube (including multidimensional ones) as final integrals had been presented. They allowed mapping an arbitrary physical area to a cube or square. One of advantages of such approach is simplicity of its extension on the case of multidimensional space.

The method of quality control of the obtained grid [1] was based on application of analytical solutions of a boundary value problem for the class of higher order equations (polyharmonic equation). The Laplace equation allows only setting the shape of physical area. Increasing of equation order enabled to receive a sufficient amount of free parameters permitting to control different properties of grids. Evidently, this way does not allow controlling grid's properties inside computational domain. In order to do it, other family of PDE has to be used. The Poisson equation seems as appropriate one for this objective. In this paper, general analytical solution of Poisson's equation

*Center for Advanced Studies of St. Petersburg Polytechnic University,124021 St. Petersburg, Russia
†Central R&D Institute of Robotics and technical Cybernetics, 194064 St. Petersburg, Russia

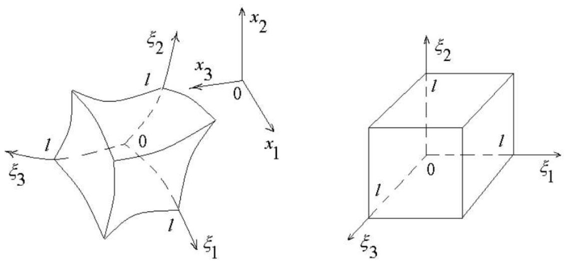

FIG. 2.1.

inside N-dimensional rectangle is introduced in finite algebraic form. Its right-side source term, which depends on coordinates, presents in solution explicitly. That is why it becomes possible to control grid behavior without difficulties.

2. Basic principle. As in [1], let us consider a solution of a boundary value problem of map of computational space on physical one (Fig. 2.1):

$$(x_1, x_2, ..., x_n) \Leftrightarrow (\xi_1, \xi_2, ..., \xi_n) \tag{2.1}$$

In the case under the study, the set of equations for a coordinate's determination is

$$\Delta x_i = \Phi(\xi_1, ...) \text{ where } \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial \xi_i^2},\ n - \text{dimension of space} \tag{2.2}$$

The equations (2.2) are solved in space $0 < \xi_i < l$.

The boundary conditions (BC) for a case of two dimensions (2D) are

$$\begin{aligned} x_1 &= f_1^{x_1}(\xi_2),\ x_2 = f_1^{x_2}(\xi_2),\ \text{at } \xi_1 = 0;\\ x_1 &= f_2^{x_1}(\xi_2),\ x_2 = f_2^{x_2}(\xi_2),\ \text{at } \xi_1 = l;\\ x_1 &= f_3^{x_1}(\xi_1),\ x_2 = f_3^{x_2}(\xi_1),\ \text{at } \xi_2 = 0;\\ x_1 &= f_4^{x_1}(\xi_1),\ x_2 = f_4^{x_2}(\xi_1),\ \text{at } \xi_2 = l; \end{aligned} \tag{2.3}$$

Functions f_i are coordinates of the nodes along the physical boundary.

These boundary conditions may be extended to any more than two-dimensional case. So, in three-dimensional case (3D) they will appear not as four BC on edges (as in 2D), but six BC on surfaces.

3. Green's function. Well-known form of analytical solution of Poisson's equation [2] rewritten in introduced above notations looks as:

$$\Delta T = -\Phi(\xi_1, \xi_2)$$

Here T corresponds to $x_i$in (2.2).

The first boundary value problem for it inside interval $0 \le \xi_1 \le l_1 \quad 0 \le \xi_2 \le l_2$ appears as:

$$T|_{\xi_1=0} = f_1(\xi_2), \quad T|_{\xi_1=l_1} = f_2(\xi_2), \quad T|_{\xi_2=0} = f_3(\xi_1), \quad T|_{\xi_2=l_2} = f_4(\xi_1)$$

The exact solution of this equation is of the form:

$$
\begin{aligned}
T(x,y) = & \int_0^{l_1}\int_0^{l_2} \Phi(\omega_1,\omega_2) G(\xi_1,\xi_2,\omega_1,\omega_2) d\omega_1 d\omega_2 \\
& + \int_0^{l_2} f_1(\omega_2)\left[\frac{\partial}{\partial\omega_1} G(\xi_1,\xi_2,\omega_1,\omega_2)\right]_{\omega_1=0} d\omega_2 \\
& - \int_0^{l_2} f_2(\omega_2)\left[\frac{\partial}{\partial\omega_1} G(\xi_1,\xi_2,\omega_1,\omega_2)\right]_{\omega_1=l_1} d\omega_2 \\
& + \int_0^{l_1} f_3(\omega_1)\left[\frac{\partial}{\partial\omega_2} G(\xi_1,\xi_2,\omega_1,\omega_2)\right]_{\omega_2=0} d\omega_1 \\
& - \int_0^{l_1} f_4(\omega_1)\left[\frac{\partial}{\partial\omega_2} G(\xi_1,\xi_2,\omega_1,\omega_2)\right]_{\omega_2=l_2} d\omega_1
\end{aligned}
\tag{3.1}
$$

where

$$
G(\xi_1,\xi_2,\omega_1,\omega_2) = \frac{4}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(p_n\xi_1)\sin(q_m\xi_2)\sin(p_n\omega_1)\sin(q_m\omega_2)}{p_n^2+q_m^2}, \tag{3.2}
$$

$$
p_n = \frac{\pi n}{l_1} \quad , \quad q_m = \frac{\pi m}{l_2}
$$

is the Green function for this boundary value problem.

Method which allows to receive closed form of Green's function instead of infinite series (3.2) is explained briefly below.

In addition to the well-known form of Green's function as double series (3.2, it exists other form of this function [2]:

$$
\begin{aligned}
G(\xi_1,\xi_2,\omega_1,\omega_2) &= \frac{2}{l_1}\sum_{n=1}^{\infty}\frac{\sin(p_n\xi_1)\sin(p_n\omega_1)}{p_n \operatorname{sh}(p_n l_2)} H_n(\xi_2,\omega_2) \\
&= \frac{2}{l_2}\sum_{m=1}^{\infty}\frac{\sin(q_m\xi_2)\sin(q_m\omega_2)}{q_m \operatorname{sh}(q_m l_1)} Q_m(\xi_1,\omega_1)
\end{aligned}
\tag{3.3}
$$

where $p_n = \frac{\pi n}{l_1}$,

$$
H_n(\xi_2,\omega_2) = \begin{cases} \operatorname{sh}(p_n\omega_2)\operatorname{sh}[p_n(l_2-\xi_2)] & l_2 \geq \xi_2 > \omega_2 \geq 0 \\ \operatorname{sh}(p_n\xi_2)\operatorname{sh}[p_n(l_2-\omega_2)] & l_2 \geq \omega_2 > \xi_2 \geq 0 \end{cases}
$$

Expression for $Q_m(\xi_1,\omega_1)$has the similar form.

Let us use formula (3.3) for obtaining of finite form of Green's function.

Results [1] allow to use them for derivation of the following expression:

$$\sum_{n=1}^{\infty}\frac{\mathrm{sh}(n\beta_1)}{\mathrm{sh}(n\beta_2)}\sin(n\beta_3)\sin(n\beta_4)$$
$$(3.4)\qquad =\frac{1-\exp(\beta_1-\beta_2)\cos(\beta_3-\beta_4)}{1-2\exp(\beta_1-\beta_2)\cos(\beta_3-\beta_4)+2\left(\exp(\beta_1-\beta_2)\right)^2}-\frac{1-\exp(\beta_1-\beta_2)\cos(\beta_3+\beta_4)}{1-2\exp(\beta_1-\beta_2)\cos(\beta_3+\beta_4)+2\left(\exp(\beta_1-\beta_2)\right)^2}$$

With integration of (3.4) by β_1 it is possible to get the following series:

$$\sum_{n=1}^{\infty}\frac{\mathrm{ch}(n\beta_1)}{n\,\mathrm{sh}(n\beta_2)}\sin(n\beta_3)\sin(n\beta_4)$$
$$(3.5)\qquad =\frac{1}{2}\ln\left[\frac{\exp(2\beta_2)+2\exp(\beta_1+\beta_2)\cos(\beta_3+\beta_4)-\exp(2\beta_1)}{\exp(2\beta_2)+2\exp(\beta_1+\beta_2)\cos(\beta_3-\beta_4)-\exp(2\beta_1)}\right]+const(\beta_1)$$

Its last term does not depend on β_1 but may depend on all other variables.

Expression (3.3) may be rewritten with the help of (3.5) in $l_2\geq\xi_2>\omega_2\geq 0$ as:

$$G(\xi_1,\xi_2,\omega_1,\omega_2)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{l_1}\xi_1\right)\sin\left(\frac{n\pi}{l_1}\omega_1\right)}{n\,\mathrm{sh}\left(\frac{n\pi}{l_1}l_2\right)}\left\{\mathrm{ch}\left[\frac{n\pi}{l_1}\left(\xi_2-\omega_2-l_2\right)\right]-\mathrm{ch}\left[\frac{n\pi}{l_1}\left(\xi_2+\omega_2-l_2\right)\right]\right\}$$

The same way may be used for derivation of Green's function inside interval $l_2\geq\omega_2>\xi_2\geq 0$.

Thus, application of sum (3.5) allows deriving closed expression for Green's function inside listed above intervals. Value of $const(\beta_1)$is obtained with the help of Green's function properties. While analyzing of these expressions, it finally becomes possible to derive the general expression for Green's function inside all its range of definition:

$$(3.6)\qquad G(\xi_1,\xi_2,\omega_1,\omega_2)=\frac{1}{4\pi}\ln\frac{R_1R_2}{R_3R_4}$$

$$R_1=f\left(\left|\xi_2+\omega_2\right|,\left|\xi_1-\omega_1\right|\right);\qquad R_2=f\left(\left|\xi_2-\omega_2\right|,\left|\xi_1+\omega_1\right|\right)$$

$$R_3=f\left(\left|\xi_2+\omega_2\right|,\left|\xi_1+\omega_1\right|\right);\quad R_4=f\left(\left|\xi_2-\omega_2\right|,\left|\xi_1-\omega_1\right|\right);$$

$$f(a,b)=\left[\mathrm{ch}\left(\frac{\pi}{l_1}(l_2-a)\right)-\cos\left(\frac{\pi}{l_1}b\right)\right]\left[\mathrm{ch}\left(\frac{\pi}{l_2}(l_1-b)\right)-\cos\left(\frac{\pi}{l_2}a\right)\right]$$

The similar approach in 3D case leads to the following expression:

$$G(\xi_1, \xi_2, \xi_3, \omega_1, \omega_2, \omega_3) = \frac{1}{4\pi} \ln \left(\frac{R_1 R_2}{R_3 R_4} \frac{\tilde{R}_1 \tilde{R}_2}{\tilde{R}_3 \tilde{R}_4} \right) \tag{3.7}$$

$$R_1 = f\left(|\xi_3 - \omega_3|, |\xi_1 + \omega_1|\right) \quad \tilde{R}_1 = \tilde{f}\left(|\xi_3 - \omega_3|, |\xi_2 + \omega_2|\right)$$

$$R_2 = f\left(|\xi_3 + \omega_3|, |\xi_1 - \omega_1|\right) \quad \tilde{R}_2 = \tilde{f}\left(|\xi_3 + \omega_3|, |\xi_2 - \omega_2|\right)$$

$$R_3 = f\left(|\xi_3 - \omega_3|, |\xi_1 - \omega_1|\right) \quad \tilde{R}_3 = \tilde{f}\left(|\xi_3 - \omega_3|, |\xi_2 - \omega_2|\right)$$

$$R_4 = f\left(|\xi_3 + \omega_3|, |\xi_1 + \omega_1|\right) \quad \tilde{R}_4 = \tilde{f}\left(|\xi_3 + \omega_3|, |\xi_2 + \omega_2|\right)$$

$$f(a,b) = \left[\text{ch}\left(\frac{\pi}{l_1}(l_3 - a)\right) - \cos\left(\frac{\pi}{l_1} b\right) \right] \left[\text{ch}\left(\frac{\pi}{l_3}(l_1 - b)\right) - \cos\left(\frac{\pi}{l_3} a\right) \right]$$

$$\tilde{f}(a,b) = \left[\text{ch}\left(\frac{\pi}{l_2}(l_3 - a)\right) - \cos\left(\frac{\pi}{l_2} b\right) \right] \left[\text{ch}\left(\frac{\pi}{l_3}(l_2 - b)\right) - \cos\left(\frac{\pi}{l_3} a\right) \right]$$

Thus, equation (3.1) rewritten in closed form with the help of (3.6) or (3.7) may be used for description of distribution of nodes of some grid.

4. Examples. Let us consider some example of application of the obtained solution for generation of grids. On all figures below, the following labels are used: $x_1 = X, x_2 = Y$. It has to be emphasized that shape of 2D domain is the same as in [1].

On Fig. 1 in [1], one can see the grid generated with the help of Laplace's equation in a flat polygon with non-monotone boundaries is shown. It is visible there, that the grid lines are smooth, and the influence of angular points decreases with moving away from boundary. It is very difficult to achieve this effect in interpolation methods. This picture is used as start point for further demonstration.

Illustrations on Fig. 3.1 describe the possibility of above derived results for grid control inside computational domain. The density of node's distribution is varied there proportionally to source term in (2.2). The law of its variation has to be taken in accordance with requirements to grid from end-user (f. ex., according to properties of numerical solution). On other hand, this term must guarantee non-degeneration of MF generated by (2.2). Closed form of generated solution allows fulfilling some a'priori estimations and satisfies to this requirement.

So, the simple tool for generation of everywhere non-degenerated grids satisfying to wide range of requirements along boundaries and inside computational domain may be developed on the base of described above solution.

REFERENCES

[1] Fedosenko, N. and Sokolov, E., *Application of exact solution of some elliptic equations for generation of two- and multi-dimensional analytical grids.* Proc. of ALGORITMY 2002 Conference on Scientific Computing, pp 253-259. http://www.iam.fmph.uniba.sk/amuc/_contributed/algo2002/

[2] Polyanin, A. D., *Handbook on linear equations of mathematical physics.* Fizmatlit, 2001 (in Russian).

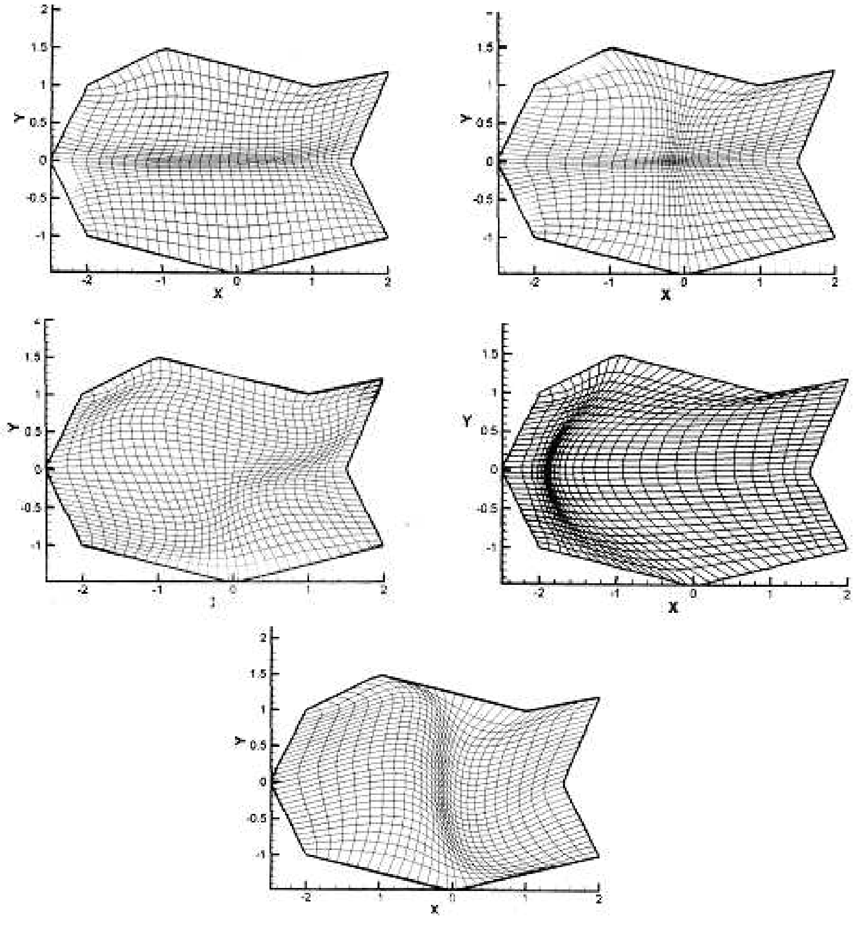

Fig. 3.1.