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AN EXTENDED-MATRIX PRECONDITIONER FOR NONSELF-ADJOINT NONSEPARABLE ELLIPTIC EQUATIONS*

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Abstract. In a previous paper we proposed a preconditioning technique that generalizes an idea by M. Griebel. In this generalization, although all considerations are presented in a very general algebraic framework, the main idea behind them is to use as preconditioner a rectangular matrix constructed with the transfer operators between successive discretization levels of the initial problem. In this way we get an extended linear system, which is no more invertible, but still consistent such that any of its solutions generates the unique one of the initial system. For an appropriate choice of the preconditioner, this extended system is mesh-independent well conditioned, thus many classes of iterative solvers can be successfuly used. We prove this in the present paper for picewise linear finite element discretization of two types of nonself-adjoint nonseparable elliptic boundary value problems. Numerical experiments on 1D and 2D versions of the considered problems are also presented for CGN and Kaczmarz iterations.

Key words. multilevel discretization, spectrally equivalent matrices, mesh independent preconditioning, positive semidefinite systems, CGN algorithm, Kaczmarz's projection method

AMS subject classifications. 65F10, 65F20, 65N22, 65N55

1. Introduction.

In this paper we shall refer to the following boundary value problem (b.v.p., for short): let $\Omega = (0,1)^d$, (usually $d = 1, 2, 3$), L a second order operator on Ω and the partial differential equation

$$\begin{cases} Lu = f, & \text{on } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases} \tag{1}$$

We shall suppose that a variational formulation is available in the following form: find $u \in U(\Omega)$ such that

$$a(u,v) = \langle f, v\rangle_{L^2}, \forall v \in V(\Omega), \tag{2}$$

where a is bilinear and bounded, $f \in L^2(\Omega)$, $\langle\cdot,\cdot\rangle_{L^2}, \|\cdot\|_{L^2}$ are the $L^2(\Omega)$ scalar product and norm and $U(\Omega), V(\Omega)$ are appropriate Hilbert spaces of real valued functions defined on Ω.

REMARK 1.1 *Concerning the functional* a *we may suppose that it fits into one of the following three general cases: coercive, $(U(\Omega), V(\Omega))$-coercive or weakly coercive which cover many important classes of b.v.p. (see e.g. [11], chapter 7); together with an appropriate choice of the function spaces $U(\Omega), V(\Omega)$ in each of these cases we have existence and unicity for the solution of the variational problem (2) (this property will be supposed for the rest of the paper).*

Let $k \geq 2$ be a given integer, $n_k = (2^k - 1)^d$ and $B_k = \{\varphi_1^{(k)}, \ldots, \varphi_{n_k}^{(k)}\}$ a standard finite element basis (e.g. picewise d-linear). Then the linear system associated to (2)

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is

$$A_k x_k = b_k, \tag{3}$$

where

$$(A_k)_{ij} = a(\varphi_j^{(k)}, \varphi_i^{(k)}),\ (b_k)_i = \langle f, \varphi_i^{(k)} \rangle_{L^2},\ i,j = 1,\ldots,n_k. \tag{4}$$

The matrix A_k is supposed to be invertible, thus the system (3) has a unique solution $x_k \in I\!R^{n_k}$. If the functional a is in addition symmetric and coercive, a multilevel based preconditioning technique was proposed by M. Griebel in [5] (see also [12] as well for some connections with domain decomposition methods). This will be briefly described in Section 2. In more general hypothesis concerning a and based on Griebel's idea and some results from [3], we developed in [10] a general algebraic multilevel preconditioning technique which will be described in Section 3. In Section 4 we first consider two one dimensional nonself-adjoint b.v.p. for which we make a theoretical analysis concerning the fulfilment of the previously mentioned assumptions. Then, numerical experiments on these 1D b.v.p. together with a two dimensional case are also described.

2. The symmetric case - Griebel's original approach. Let us suppose that the variational formulation (2) is made on $U(\Omega) = V(\Omega) = H_0^1(\Omega)$ and the bilinear functional a from (2) is symmetric, bounded and coercive, i.e.

$$|a(u,v)| \leq M \parallel u \parallel_{H^1} \parallel v \parallel_{H^1},\ a(u,u) \geq \mu \parallel u \parallel_{H^1}^2, \forall u, v \in H_0^1(\Omega), \tag{5}$$

where $\parallel v \parallel_{H^1}^2 = \sum_{i=1}^d \parallel \frac{\partial v}{\partial x_i} \parallel_{L^2}^2 + \parallel v \parallel_{L^2}^2$. Then the finite element discretization matrix A_k from (4) will be symmetric and positive definite (SPD, for short), i.e.

$$\langle A_k z, z \rangle > 0, \forall z \in I\!R^{n_k}, z \neq 0 \tag{6}$$

(by $\langle \cdot, \cdot \rangle, \parallel \cdot \parallel$ we shall denote the euclidean scalar product and norm on some space $I\!R^q$). Let

$$V_1 \subset V_2 \subset \ldots \subset V_k \tag{7}$$

be a sequence of spaces of picewise d-linear functions associated to a sequence of uniform, equidistant, nested grids

$$\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_k, \tag{8}$$

$n_q = (2^q - 1)^d$ the dimension of $V_q, q = 1, 2, \ldots, k$ and $B_q = \{\varphi_1^{(q)}, \ldots, \varphi_{n_q}^{(q)}\}$ the finite element basis in V_q. Let also $\hat{B}_k \subset V_k$ and m_k be given by

$$\hat{B}_k = B_1 \bigcup B_2 \bigcup \ldots \bigcup B_k,\ m_k = n_1 + n_2 + \ldots + n_k. \tag{9}$$

The functions from $\hat{B}_k$ are linearly dependent and generate the subspace V_k. Each function $\varphi_j^{(q)} \in V_q \subset V_{q+1}$ has a unique representation as an element of V_{q+1}, of the form

$$\varphi_j^{(q)} = \sum_{i=1}^{n_{q+1}} c_{ij} \varphi_i^{(q+1)}, j = 1, \ldots, n_q. \tag{10}$$

We then consider the $n_{q+1} \times n_q$ matrix I_q^{q+1} given by

$$\left(I_q^{q+1}\right)_{ij} = c_{ij} \tag{11}$$

and, for $q = 1, 2, \ldots, k-1$ we define the $n_k \times n_q$ matrices S_q^k by

$$S_q^k = I_{k-1}^k I_{k-2}^{k-1} \ldots I_q^{q+1} \tag{12}$$

and the $n_k \times m_k$ matrix S_k (in block notation)

$$S_k = \left[\begin{array}{c|c|c|c|cccc} & & & & 1 & & & \\ & & & & & 1 & & \\ S_1^k & S_2^k & \ldots & S_{k-1}^k & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array} \right], \tag{13}$$

in which the last block is the $n_k \times n_k$ unit matrix. We then consider the preconditioned version of (3)

$$\hat{A}_k \hat{x}_k = \hat{b}_k, \tag{14}$$

where

$$\hat{A}_k = S_k^t A_k S_k, \quad \hat{b}_k = S_k^t b_k. \tag{15}$$

It results that the $m_k \times m_k$ matrix $\hat{A}_k$ is symmetric and positive semidefinite. Moreover, because the matrix S_k is full row rank, it can be proved that for any solution $\hat{x}_k \in I\!R^{m_k}$ of (14), $S_k \hat{x}_k \in I\!R^{n_k}$ is the unique solution of (3) (see [5], [10]). If for a square matrix B we shall denote by $\sigma^*(B)$ the set of all its nonzero eigenvalues, in [5] (see also the references therein) Griebel proved that, the generalized spectral condition number of $\hat{A}_k$, defined by

$$cond(\hat{A}_k) = \sqrt{\frac{\max\{\lambda, \lambda \in \sigma^*(\hat{A}_k^t \hat{A}_k)\}}{\min\{\lambda, \lambda \in \sigma^*(\hat{A}_k^t \hat{A}_k)\}}} \tag{16}$$

is mesh-independently bounded, i.e. S_k from (13) is an efficient preconditioner. Moreover, the matrix

$$G_k = S_k S_k^t \tag{17}$$

is spectrally equivalent with the inverse of the standard discretized Laplacian Δ_k (e.g. the 5-point stencil for $d = 2$), i.e. there exist constants α_1, α_2 independent on the mesh size such that (see e.g. [3])

$$\alpha_1 \leq \frac{\langle G_k x, x \rangle}{\langle \Delta_k^{-1} x, x \rangle} \leq \alpha_2, \forall x \in I\!R^{n_k}. \tag{18}$$

3. The nonsymmetric case - a general algebraic approach. We start from (3), considered as an arbitrary algebraic system of equations and define the matrices

$$M_k = \frac{1}{2}(A_k + A_k^t), \quad R_k = \frac{1}{2}(A_k - A_k^t). \tag{19}$$

We suppose that the symmetric part of A_k, M_k is SPD and observe that R_k satisfies

$$R_k^t = -R_k. \tag{20}$$

Let $m_k \geq n_k$, S_k an arbitrary full row rank $n_k \times m_k$ matrix and G_k defined as in (17). We introduce the following assumptions.

Assumption 1. *The matrix G_k is spectrally equivalent with M_k^{-1} (see e.g.* (18)*).*

Assumption 2. *It exists a constant $\beta \geq 0$, independent on n_k and m_k such that*

$$\rho(G_k R_k) \leq \beta. \tag{21}$$

We then define the matrix $\hat{A}_k$ and vector $\hat{b}_k$ as in (15). The following general result was proved in [10].

THEOREM 3.1. *In the above hypothesis and under the Assumptions 1 and 2 we have*

$$cond(\hat{A}_k) \leq \frac{\alpha_2 + \beta}{\alpha_1}, \tag{22}$$

i.e. the preconditioned matrix $\hat{A}_k$ has a mesh independent generalized spectral condition number.

In what follows we shall prove two useful theoretical results related to the previous assumptions. These results will be used in the next section in the construction of the mesh independent preconditioner for two classes of convection-diffusion problems. In order to simplify the presentation and because the results are general we shall cancel the index k.

PROPOSITION 3.2. *Let H be a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_H$, $a : H \times H \longrightarrow I\!R$ bilinear, bounded and coercive (see* (5)*) and for a given $n \geq 2$, $\{\varphi_1, \ldots, \varphi_n\} \subset H$ a linearly independent system. If A, B are the SPD matrices defined by*

$$(A)_{ij} = \frac{1}{2}\left(a(\varphi_j, \varphi_i) + a(\varphi_i, \varphi_j)\right),\ (B)_{ij} = \langle \varphi_j, \varphi_i \rangle_H, \forall i, j = 1, \ldots, n, \tag{23}$$

then A and B are spectrally equivalent.

Proof. Let $B = CC^t$, with C invertible be a decomposition of B, $\hat{A} = C^{-1}AC^{-t}$ and $x = (x_1, \ldots, x_n)^t \in I\!R^n$ an arbitrary fixed vector. From (23) and the definition of $\hat{A}$ we successively obtain

$$\langle \hat{A}C^t x, C^t x \rangle = \langle Ax, x \rangle = a(u_x, u_x), \tag{24}$$

with $u_x = \sum_{i=1}^n x_i \varphi_i$ and, by also using (5)

$$\mu \parallel u_x \parallel_H^2 \ \leq\ \langle \hat{A}C^t x, C^t x \rangle \ \leq\ M \parallel u_x \parallel_H^2 . \tag{25}$$

Because $\parallel u_x \parallel_H^2 = \langle Bx, x \rangle = \parallel C^t x \parallel^2$, from (25), for $z = C^t x$ we get

$$\mu \parallel z \parallel^2 \ \leq\ \langle \hat{A}z, z \rangle \ \leq\ M \parallel z \parallel^2,$$

thus

$$\mu \leq \lambda_{min}(C^{-1}AC^{-t}) \leq \lambda_{max}(C^{-1}AC^{-t}) \leq M, \tag{26}$$

relation which is equivalent with the spectral equivalence of the matrices A and B, independently on the decomposition $B = CC^t$ (see e.g. [13]). □

PROPOSITION 3.3. *Let G, T be two $n \times n$ SPD spectrally equivalent matrices and R as in* (20). *The following are equivalent:*
(i) it exists a constant $\beta > 0$, independent on n such that

$$\rho(TR) \le \beta; \tag{27}$$

(ii) it exists a constant $\gamma > 0$, independent on n such that

$$\rho(GR) \le \gamma. \tag{28}$$

Proof. It suffices to prove only one implication. For this, let $\beta > 0$ be such that (27) holds and

$$G = CC^t, \quad T = \Gamma\Gamma^t \tag{29}$$

two arbitrary decompositions. From (20) it results that the matrix C^tRC is normal, thus we successively have

$$\rho(GR) = \rho(C^tRC) = \max_{x \ne 0} \frac{\langle C^tRCx, x\rangle}{\langle x, x\rangle} = \max_{y = Cx \ne 0} \frac{\langle Ry, y\rangle}{\langle G^{-1}y, y\rangle}. \tag{30}$$

But, because G and T are spectrally equivalent, so will be G^{-1} and T^{-1} (see e.g. [13]), thus there exist constants $\alpha_1, \alpha_2 > 0$, independent on n such that

$$\alpha_1 \le \frac{\langle G^{-1}z, z\rangle}{\langle T^{-1}z, z\rangle} \le \alpha_2, \forall z \in I\!R^n. \tag{31}$$

Then, from (30), (31), (27) and because the matrix $\Gamma^tR\Gamma$ is normal we successively get

$$\rho(GR) = \max_{y \ne 0} \frac{\langle Ry, y\rangle}{\langle G^{-1}y, y\rangle} = \max_{y \ne 0} \frac{\langle Ry, y\rangle}{\langle T^{-1}y, y\rangle} \frac{\langle T^{-1}y, y\rangle}{\langle G^{-1}y, y\rangle} \le \frac{1}{\alpha_1} \max_{y \ne 0} \frac{\langle Ry, y\rangle}{\langle T^{-1}y, y\rangle} = \tag{32}$$

$$\frac{1}{\alpha_1} \max_{z = \Gamma^{-1}y \ne 0} \frac{\langle R\Gamma z, \Gamma z\rangle}{\langle z, z\rangle} = \frac{1}{\alpha_1}\rho(\Gamma^tR\Gamma) = \frac{1}{\alpha_1}\rho(TR) \le \frac{\beta}{\alpha_1},$$

i.e. exactly the inequality (28) with $\gamma = \frac{\beta}{\alpha_1}$ and the proof is complete. □

4. Numerical experiments.

4.1. 1 D convection-diffusion problems. We have first considered the problem (see [3])

$$\begin{cases} -(u_x)_x + cu_x + (cu)_x + eu & = f, \quad x \in (0,1) \\ u(0) = u(1) & = 0 \end{cases} \tag{33}$$

with $c(x) = \gamma x, \; e(x) = \frac{1}{1+x}, \; x \in (0,1)$ and f such that $u_{ex}(x) = x - x^2, x \in (0,1)$ is its unique exact solution. For (33) we have a variational formulation of the type (2) with $\Omega = (0,1)$ and the bilinear functional a given by

$$a(u,v) = \int_0^1 u'v'\mathrm{dx} + 2\int_0^1 cu'v\mathrm{dx} + \int_0^1 (c' + e)uv\mathrm{dx}. \tag{34}$$

The above bilinear functional a is bounded and $(H^1(0,1), H^0(0,1))$-coercive (see [11]). For the discretization of (33)-(34) we considered the approach from Section 2. On the finest level $n_k \geq 4, h = 1/n_k, x_i = ih \in (0,1), i = 1, \ldots, n_k - 1$ we considered picewise linear basic functions $\varphi_1^{(k)}, \ldots, \varphi_{n_k}^{(k)}$. The coefficients of the corresponding matrix A_k and the right hand side b_k from (4) were approximated with Simpson's rule with 2 nodes on each interval $[x_i, x_{i+1}], i = 0, \ldots, n_k - 1$. In this way we got the following expressions

$$(A_k)_{ii} = 2 + \frac{h^2}{6}\left(\frac{1}{(i-\frac{1}{2})h+1} + \frac{4}{ih+1} + \frac{1}{(i+\frac{1}{2})h+1}\right), \tag{35}$$

$$(A_k)_{i,i-1} = -1 - \gamma h^2 (i - \frac{1}{2}) + \frac{h^2}{(6i-3)h+6},$$

$$(A_k)_{i,i+1} = -1 + \gamma h^2 (i + \frac{1}{2}) + \frac{h^2}{(6i+3)h+6},$$

$$(b_k)_i = \frac{h^2}{3}\left(f\left(\left(i - \frac{1}{2}\right)h\right) + f(ih) + f\left(\left(i + \frac{1}{2}\right)h\right)\right).$$

The matrices S_k, G_k were constructed as in (12)-(13) and (17), respectively with I_q^{q+1} from (11) as the standard multilevel interpolation operator, i.e. in stencil notation (see e.g. [2]),

$$\left[I_q^{q+1}\right]_h = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}_h \tag{36}$$

Let Δ_k be the standard three diagonal discrete Laplacian, i.e. $(\Delta_k)_{ii} = 2, (\Delta_k)_{i,i-1} = -1, (\Delta_k)_{i,i+1} = -1$. It is well-known that

$$(\Delta)_k)_{ij} = \int_0^1 (\varphi_j^{(k)})' \; (\varphi_i^{(k)})' dx = \langle \varphi_j^{(k)}, \varphi_i^{(k)} \rangle_{H_0^1}. \tag{37}$$

Verification of Assumption 1. Griebel proved that (see [5] and references therein) the matrix G_k is spectrally equivalent with Δ_k^{-1}. Moreover, the symmetric part of a from (34), $a_s(u,v) = \frac{1}{2}(a(u,v) + a(v,u))$ satisfies the hypothesis from Proposition 3.2 (for $H = H_0^1(0,1)$) and $(M_k)_{ij} = \frac{1}{2}((A_k)_{ij} + (A_k^t)_{ij}) = a_s(\varphi_j^{(k)}, \varphi_i^{(k)})$. Thus, according to this result M_k will be spectrally equivalent with Δ_k. It then results that (see e.g. [13]) G_k is spectrally equivalent with M_k^{-1}.
Verification of Assumption 2. From (19) and (35) it results that in our case the skew-symmetric part of A_k, R_k given by

$$(R_k)_{ii} = 0, \; (R_k)_{i,i-1} = -\gamma h^2 (i - \frac{1}{2}), \; (R_k)_{ii} = \gamma h^2 (i + \frac{1}{2}) \tag{38}$$

conicides with the matrix $\tilde{R}$ from [3]. In this paper, the authors proved that it exists a constant $\beta > 0$, independent on n_k or the spectrum of A_k such that

$$\rho(\Delta_k^{-1}\tilde{R}) = \rho(\Delta_k^{-1} R_k) \leq \beta. \tag{39}$$

As Δ_k^{-1} is spectrally equivalent with G_k, by directly applying Proposition 3.3 we obtain $\rho(G_k R_k) \leq \gamma$, for some γ depending on β and the spectral equivalence constants for Δ_k^{-1} and G_k, but not on n_k. Thus, according to Theorem 3.1 the extended matrix $\hat{A}_k$ from (15) is mesh independent well conditioned. This is also confirmed by the tests from Table 1. There, for different values of γ and n_k we solved the preconditioned system (14)-(15) with the CGN algorithm (CG for normal equation; see [4], [1]) and Kaczmarz's projection method (see e.g. [1]) and the stopping test (see e.g. [3])

$$\| \text{ residual } \| \leq 10^{-6}. \tag{40}$$

REMARK 4.1 *In the papers [5] and [6] the authors used as iterative solver the Gauss-Seidel method; this is no more possible in our case because the system (14) is no more symmetric; this is the reason for which we have used the Kaczmarz's projection algorithm, which for consistent systems as (14) is convergent and equivalent with the Gauss-Seidel iteration applied to $\hat{A}_k \hat{A}_k^t \hat{y}_k = \hat{b}_k$, with $\hat{x}_k = \hat{A}_k^t \hat{y}_k$ (see e.g. [1]).*

The second 1D b.v.p. that we have analysed was the heat convective transfer equation, without internal sources and nonhomogeneous Dirichlet boundary conditions

$$\begin{cases} -u''(x) + 2\alpha u'(x) + u(x) & = 0, \ x \in (0,1) \\ u(0) = 0, u(1) = 1 & \end{cases} \tag{41}$$

For (41) we have a variational formulation of the type (2) with $\Omega = (0,1)$ and the bilinear functional a given by (see [11])

$$a(u,v) = \int_0^1 u'v'\mathrm{dx} + 2\alpha \int_0^1 u'v\mathrm{dx} + \int_0^1 uv\mathrm{dx}. \tag{42}$$

The bilinear functional a from (42) is bounded and $(H^1(0,1), H^0(0,1))$-coercive (see [11]). By using the same discretization as for (33) we obtained the associated linear system, with A_k, b_k given by

$$(A_k)_{ii} = 2 + 2\frac{h^2}{3}, (A_k)_{i,i-1} = -1 + \frac{h^2}{6} - \alpha h, (A_k)_{i,i+1} = -1 + \frac{h^2}{6} + \alpha h, \tag{43}$$

$$(b_k)_1 = \ldots = (b_k)_{n_k - 2} = 0, (b_k)_{n_k - 1} = 1 - \frac{h^2}{6} - \alpha h.$$

Verification of Assumption 1. We follow exactly the same procedure as before for the problem (33).

Verification of Assumption 2. In this case we obtain

$$(R_k)_{ii} = 0, \ (R_k)_{i,i-1} = -\alpha h, \ (R_k)_{i,i+1} = \alpha h. \tag{44}$$

Then we follow exactly the way in [3], pp. 49-50 and obtain the inequality

$$\frac{\langle R_k x, x\rangle}{\langle \Delta_k x, x\rangle} \leq \epsilon, \ \forall x \in \mathbb{R}^{n_k}, \tag{45}$$

for some ϵ independent on n_k. Now, if $\Delta_k = CC^t$ is a decomposition of Δ_k, then the matrix $C^{-1} R_k C^{-t}$ is normal (see (20)), thus we successively have (by also using (45))

$$\rho(\Delta_k^{-1} R_k) = \rho(C^{-1} R_k C^{-t}) = \max_{x \neq 0} \frac{\langle C^{-1} R_k C^{-t} x, x\rangle}{\langle x, x\rangle} =$$

$$\max_{y=C^{-t}x\neq 0} \frac{\langle R_k y, y\rangle}{\langle \Delta_k y, y\rangle} \leq \epsilon. \tag{46}$$

From (46), the fact that G_k and Δ_k^{-1} are spectrally equivalent and by again applying Proposition 3.3 we get (21) for some $\beta > 0$ independent on n_k . The results of the numerical experiments for different values of α and n_k and the same stopping rule (40) are presented in Table 2.

4.2. A 2 D convection-diffusion problem. We considered the two dimensional problem (see [3])

$$\begin{cases} -(cu_x)_x - (bu_y)_y + du_y + (du)_y + gu & = f, \text{ in } \Omega = (0,1)\times(0,1) \\ u & = 0 \text{ on } \mathrm{Fr}(\Omega) \end{cases} \tag{47}$$

with $c(x,y) = \mathrm{e}^{-xy}, b(x,y) = \mathrm{e}^{xy},\ d(x,y) = \gamma(x+y),\ g(x,y) = \frac{1}{1+x+y}$, where $\gamma \in (0,\infty)$ and the right hand side f such that $u(x,y) = xy(1-x)(1-y)$ is the exact solution. For (47) we have a variational formulation of the type (2) with the bilinear functional a given by

$$a(u,v) = \int_\Omega (cu_x v_x + bu_y u_y)\mathrm{dxdy} + \int_\Omega d(u_y v - uv_y)\mathrm{dxdy} + \int_\Omega guv\mathrm{dxdy} \tag{48}$$

The functional a is bounded and $(H^1(\Omega), H^1(\Omega))$-coercive. For the discretization of (47)-(48) we considered a regular triangulation obtained by dividing the domain into squares with sides $h = \frac{1}{n_k}$ and then dividing these into two triangles. The corresponing basis functions are linear on each triangle surrounding a node (see [9], figures 12-14). The corresponding two dimensional integrals over the triangles T are evaluated numerically using 3-point Gauss formulae

$$\int_T F = \frac{h^2}{6}\left(F(P_1) + F(P_2) + F(P_3)\right) \tag{49}$$

where P_1, P_2, P_3 are the midpoints of the triangles edges. The resulting matrix A_k is non-symmetric 7-diagonal. For the construction of the matrix S_k (see (11)-(13)) we used the standard linear multilevel interpolation operators associated to the above described multilevel discretization (see e.g. [2], [7], [8]), i.e. in stencil notation

$$\left[I_q^{q+1}\right]_h = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}_h \tag{50}$$

The preconditioned system is again solved using CGN and Kaczmarz algorithms with the same (40) stopping rule. The results are presented in Table 3. We observe the same good behaviour of the preconditioning, as in the one dimensional cases from the previous section.

4.3. Computational aspects and final comments. 1. We described in this paper a preconditioning technique based on the construction of an extended system matrix. In practice, the system (14)-(15) is not efectively formed but, because the preconditioner S_k from (13) is constructed with the multigrid interpolation operators, for operations of the form $\hat{A}_k z$ requested by an iterative solver (e.g. CGN or Kaczmarz), the computational effort per iteration step will be of order $\mathcal{O}(n_k)$ (see e.g. [8]).

Moreover, if the generalized spectral condition number of $\hat{A}_k$ is mesh independent, we expect a small number of iterations (comparing with the dimension n_k) which allow us to conclude that a final $\mathcal{O}(n_k)$ computational work will be needed. This is an advantage with respect to classical preconditioning techniques based on Cholesky decomposition of the discrete Laplacian or the finite element basis Gram matrix, for which the usual order is $\mathcal{O}(n_k^2)$ (see e.g. [3], [14]).

2. The results in Tables 1, 3 and 2 are comparable with those in [3] and [14], respectively. But, in those papers the preconditioning is made with the Cholesky factors of the discrete Laplacian or finite element basis functions Gram matrix, respectively.

3. The results in Table 1 show a good behaviour for both CGN and Kaczmarz iterations. In Table 2 the values of the Péclet number (Pe=$2\alpha h$ for our problem (41)) are also indicated. We can see a good "equilibration" between this constant (which controles the numerical stability of the discretization, see e.g. [7]) and the behaviour of the two iterative solvers. For the two dimensional problem (47) the behaviour of our solvers is not so good (see Table 3). In this case Assumption 1 still holds, but we don't know if this is true also for Assumption 2. One possible way for overcoming this difficulty would be to consider other choices for the I_q^{q+1} operators in (50) (see also comment 4 from below).

4. In the numerical experiments from the above sections we used the classical interpolation operators (36) or (50). But, our approach based on the assumptions 1 and 2 is quite general, thus other (possible better) choices for the interpolation operators in the preconditioner (13) can be considered, such that the spectral equivalence requested in Assumption 1 is fulfilled (work is in progress in this direction).

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Table 1
Problem (33)

γ	n_k	$cond(A_k)$	$cond(\hat{A}_k)$	Nr. iter CGN	Nr. iter Kaczmarz
0	16	95	8	13	18
	32	380	15	20	23
	64	1520	30	38	28
	128	6083	60	64	34
5	16	71	3	14	15
	32	282	7	23	21
	64	1146	14	39	27
	128	4585	28	70	32
50	16	20	3	13	19
	32	63	3	21	19
	64	241	3	29	20
	128	1002	7	44	25
100	16	22	6	15	35
	32	51	5	25	26
	64	144	4	32	24
	128	576	5	42	24

Table 2
Problem (41)

α	n_k	Pe	$cond(A_k)$	$cond(\hat{A}_k)$	Nr. iter CGN	Nr. iter Kaczmarz
0	16	0	93	8	14	21
	32	0	376	15	25	28
	64	0	1506	30	44	36
	128	0	6028	60	80	44
10	16	1.25	29	1.3	13	17
	32	0.625	115	2.5	19	22
	64	0.31	458	5	32	28
	128	0.15	1834	10	57	36
50	16	6.25	11	4	14	50
	32	3.12	26	2.2	15	38
	64	1.56	102	1.2	15	34
	128	0.78	407	2	19	40
100	16	12.5	12	8	14	89
	32	6.25	22	4.4	26	67
	64	3.12	53	2.2	17	51
	128	1.56	206	1.2	16	44

Table 3
Problem (47)

γ	n_k	$cond(A_k)$	$cond(\hat{A}_k)$	Nr. iter CGN	Nr. iter Kaczmarz
0	8	25.7	6	34	60
	16	115	11	56	106
	32	522	20	83	149
5	8	15	5	31	50
	16	65.8	9.5	54	91
	32	317	18	79	163
50	8	6	8.8	49	58
	16	16	15	105	76
	32	51	23	187	141
100	8	7	14	65	99
	16	16	23	157	133
	32	43	51	171	135