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SHAPE MODELLING OF AIR BELLOWS SPRINGS *

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Abstract. From the shape of the bellows air spring (BAS) we can derive the geometrical characteristic (Volume, Effective area and Index of effective area), as the most significant static parameters of BAS. Knowledge of them is very important for designer of BAS. Designer can change the design of retainers and spring to optimize the properties of it, before manufacturing. In this article we describe development of two models for solution of this problem and discuss the results.

Key words. bellows air spring, static characteristic, Newton's method, Runge-Kutta's method

AMS subject classifications. 34B60, 65D17, 65R20

1. Introduction. In this article we assume that the fabric reinforced elastomer bellows is ideal (by ideal we mean absolute rigidity in tensioning and absolute elasticity in bending). This assumption reduces a spatial problem to a planar problem. The design of the bellows is sketched on the Fig. 1. Force generating axial shift (vertical moving of the upper retainer, denoted z), is given only by air's work. For practical purposes, we used negative direction for extension and positive direction for compression.

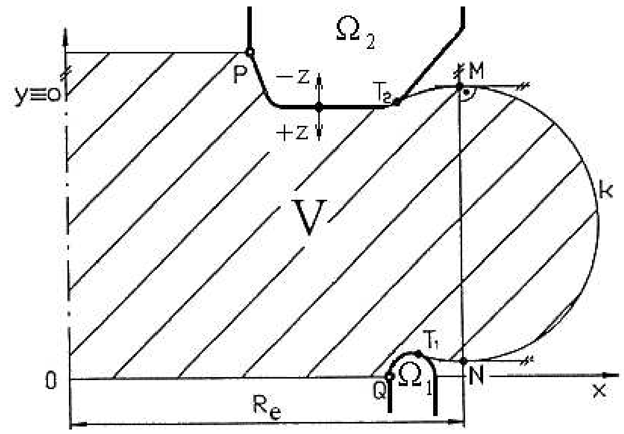

FIG. 1. *Design of the bellows - partial axial section; Ω_1-lower retainer; Ω_2-upper retainer; k-meridian curve of fabric-reinforced elastomer bellows; R_e-radius of the circular effective area; P, Q-terminal points of bellows; T_1, T_2-contact points; M, N-fictive point of release*

We can omit the part of the k curve laying on the right side from line M, N.

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In these points, forces are only in radial direction. The entire axial force of spring is given only by action of air on the residuary area. This area is known under the term effective area [2] and is defined as

$$S(z) = \pi R_e^2(z), \tag{1}$$

where Re(z) is radius of release points M or N. The utilization principle of virtual work gives

$$S(z) = -\frac{dV(z)}{dz}, \tag{2}$$

where V(z) is volume of air spring. Last of major parameters of the BAS is index of effective area,

$$U(z) = \frac{dS(z)}{dz}. \tag{3}$$

2. Model No. 1. This model is based on finding free part of the meridian curve k, in which the body volume V is maximal. Volume V is formed by rotating of area P around the spring axis o. The assumption is, that the meridian length L (sum of length of meridian curve from point P to T_2, length of meridian curve k and length of meridian curve from point T_1 to Q) is constant during the spring axial deformation and internal excess pressure of air. The meridian curve k can be expressed parametrically by equations

$$x = x(s), \qquad y = y(s), \tag{4}$$

where

$$dx^2 + dy^2 = ds^2, \tag{5}$$

and thus

$$(x')^2 + (y')^2 = 1. \tag{6}$$

The BAS internal volume is

$$V = \int_0^L x x' y ds. \tag{7}$$

The meridian end points P and Q have coordinates $P = [x(0), y(0)]$, $Q = [x(L), y(L)]$. We look for the coordinates $x(s)$, $y(s)$ of the meridian curve, which maximize the equation (7) and fulfill the boundary conditions

$$[x(s), y(s)] \notin \Omega_1 \cup \Omega_2, \tag{8}$$

$$[x(s), y(s)] \in \Gamma_1 \Rightarrow [x'(s), y'(s)].\vec{n}_{\Gamma_1} = 0, \tag{9}$$

$$[x(s), y(s)] \in \Gamma_2 \Rightarrow [x'(s), y'(s)].\vec{n}_{\Gamma_2} = 0, \tag{10}$$

where Ω_1 and Ω_2 are domains which define upper and lower retainer, Γ_1 and Γ_2 are boundaries of domains Ω_1 and Ω_2. These equations are especially given for contact points T_1, T_2.

Our problem of finding the maximal V is equal to finding the maximum of the functional

$$\Phi = \int_0^L xx'yds + \int_0^L \lambda[x'(s)^2 + y'(s)^2 - 1]ds, \tag{11}$$

where $x(s)$, $y(s)$ and $\lambda(s)$ are unknown functions of the parameter s. In fact that the variation of the functional (11) must equal zero, following equations have to be fulfilled,

$$x'y - (xy)' - 2(\lambda x')' = 0, \tag{12}$$

$$xx' - 2(\lambda y')' = 0. \tag{13}$$

These equations together with the conditions (6), (8), (9) and (10) represent a system of differential equations for unknown $x(s)$, $y(s)$ and $\lambda(s)$. By eliminating $\lambda(s)$ from equations (12) and (13), the system of differential equations of 2^{nd} order is obtained,

$$x'' = \frac{-2xy'^2}{x^2 - C}, \qquad y'' = \frac{2xx'y}{x^2 - C}. \tag{14}$$

Instead of the optional parameter C, it is suitable to introduce a curvature of meridian curve ρ. The elementary differential geometry gives

$$x''^2 + y''^2 = \rho^2, \tag{15}$$

then from the equations (14) and (15) follows, that

$$\rho = \frac{2xy'^2}{x^2 - C}, \qquad x'' = -\rho y', \qquad y'' = \rho x'. \tag{16}$$

With a simple modification we obtain the system differential equations of 1^{st} order

$$x' = x_1, \qquad y' = y_1, \qquad x_1' = -\rho y_1, \qquad y_1' = \rho x_1, \qquad \rho' = \rho\frac{x_1}{x}. \tag{17}$$

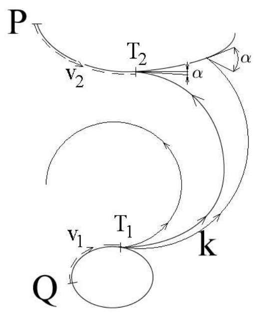

FIG. 2. *Principle of iterations; k-meridian curve of fabric-reinforced elastomer bellows; P, Q-terminal points of bellows; T_1, T_2-contact points; v_1-length of boundary Γ_1 between point Q and T_1; v_2-length of boundary Γ_2 between point P and T_2; α-contact angle*

The system of differential equations (17) is solved by Runge-Kutta four-point method. For $\rho = const$, the first four equations of the system are the differential equations which describe the circle with radius $r = \frac{1}{\rho}$. From the fifth equation of the system (17), it is obvious, that for a great x, i.e. for great distance from the spring axis, $\rho' \approx 0$, thus $\rho = constant$, or for $x \to \infty$ the curve is converging to the circle.

Algorithm of this model is a multi-iterative process. We choose starting point T_1 and initial curvature ρ. Then we start Runge-Kutta method. If the curve does not reach upper retainer, curvature ρ must be changed, until we acquire tangential contact in point T_2, we determine accuracy of contact angle $\alpha < \epsilon$. Then we check the length L and correct start point T_1. We used simple interval bisection method. When the length L is correct, we find result of task for given shift z and we calculate effective area $S(z)$ (using point on curve, where $y' = 0$, see points M, N on Fig. 1. and equation (1)) and volume $V(z)$.

2.1. Convolutions touch problem. For the double convolution bellows can happen contact of bellows in compression interval. We assume, the bellows are the same. Then we solve this task as to be axisymetrical by axis x. We establish new variable g - length of contact of bellows.

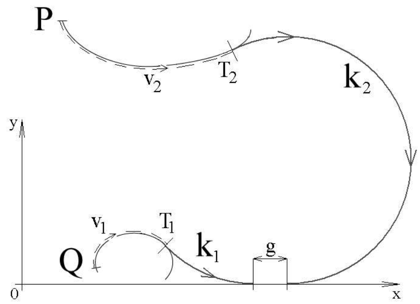

FIG. 3. *Convolutions touch problem - principle of iterations; g - length of line of convolutions contact; k_1, k_2 - meridian curves of fabric-reinforced elastomer bellows; P, Q - terminal points of bellows; T_1, T_2-contact points; v_1-length of boundary Γ_1 between point Q and T_1; v_2-length of boundary Γ_2 between point P and T_2*

For this case, the algorithm is a little more complicated. We have to find two curves, but we have only one relation between them - the volume under them is maximal.

We choose start point T_2 and by modification of curvature ρ_2, we achieve tangential contact with y-axis. Then we choose start point T_1 and by modification of curvature ρ_1, we achieve tangential contact with y-axis. Length of meridian is sum of lengthes of curves v_2, k_2, v_1, k_1 and g, which is the length of contact line, see Fig. 3. We compare it with length L and correct start point T_1. If the length L is correct, we must modificate start point T_2 and start the next iteration step, so long, until we accomplish the maximal volume V. When we find the maximal volume V, we find result of problem for given shift z and we can calculate effective area $S(z)$.

For verification of the model, we compared results of equations (1) and (2). The results are really identical, see Fig. 7. Main limitation of this model is its time costs. To eliminate this, a faster model was derived.

3. Model No. 2. The faster model is based on retrieval of the free part of the meridian k as a part of circle. This case means to solve

$$f_1 \equiv v_1 + v_2 + R\Big((\pi - \varphi_{1(v_1)}) + (\varphi_{2(v_2)} - \pi)\Big) - L = 0, \tag{18}$$

$$\mathbf{f}_{2,3} = \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} = \big(\mathbf{x}_{1(v_1)} + R\,\mathbf{n}_{1(v_1)}\big) - \big(\mathbf{x}_{2(v_2)} + R\,\mathbf{n}_{2(v_2)}\big) = \mathbf{0}, \tag{19}$$

where v_1 is length of meridian part, which is laying on boundary domain Ω_1 between points Q and T_1, v_2 is length of meridian part, which is laying on boundary domain Ω_2 between points P and T_2, R is radius of the circular arc of bellows, φ_1, φ_2 are arguments of vectors $\mathbf{n}_1$, $\mathbf{n}_2$, see Fig. 4.

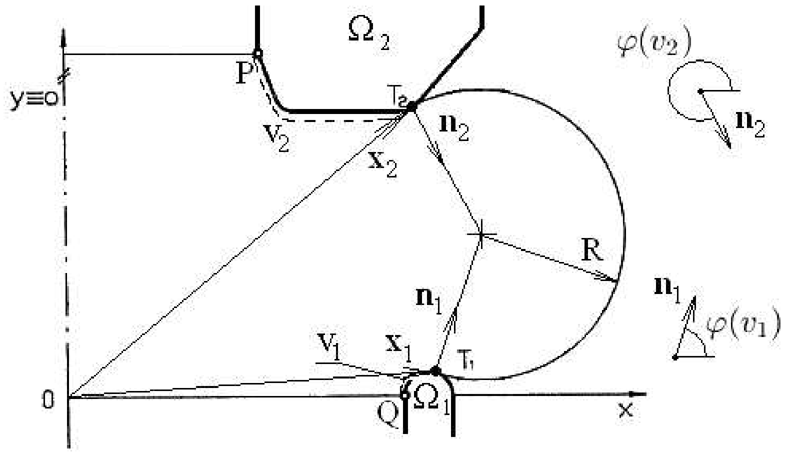

FIG. 4. *Principle of faster model; Ω_1 - lower retainer; Ω_2 - upper retainer; P, Q -terminal points of bellows; T_1, T_2 - contact points; v_1-length of boundary Γ_1 between point Q and T_1; v_2-length of boundary Γ_2 between point P and T_2; x_1, x_2 - vectors of coordinates of contact points T; n_1, n_2 - unit vectors of exterior normal of boundary of domain Ω in contact points T; φ_1, φ_2 - angles of unit vectors n_1 and n_2; R - radius of the circular arc of bellow*

The system of three equations (18) and (19) with three unknown variables v_1, v_2 and R is solved numerically by the generalized Newton's method. After remarking of variable $R = v_3$, an iterative equation can be written as follows

$$^{k+1}\mathbf{v} = {}^{k}\mathbf{v} - \Big(\mathbf{J}(^{k}\mathbf{v})\Big)^{-1}\mathbf{f}(^{k}\mathbf{v}), \tag{20}$$

where

$$^{k+1}\mathbf{v} = \begin{bmatrix} ^{k+1}v_1 \\ ^{k+1}v_2 \\ ^{k+1}v_3 \end{bmatrix}, \qquad ^{k}\mathbf{v} = \begin{bmatrix} ^{k}v_1 \\ ^{k}v_2 \\ ^{k}v_3 \end{bmatrix},$$

$\mathbf{J}(\mathbf{v})$ is Jacobi matrix of functions f_i, where $i = 1, 2, 3$; $\mathbf{f}(\mathbf{v})$ is vector of functions f_i and k is step of iteration.

The iterations are finished when the determinate accuracy

$$\sum_{i=1}^{3}\Big(\,^{k+1}v_i - ^{k}v_i\Big) < \varepsilon$$

is obtained.

Model based on this iteration process converge much faster then model No. 1. In this case, the radius of the effective area is identical with x-coordinate of the center of circular arc and from the equation (1) we calculate the value of effective area. But, there is a big difference compared to model No. 1.

The shapes of solutions (shapes of curves) for both models look very similar, but positions of release points M and N are different (see Fig. 6 for three positions of retainers). The results validate hypothesis written below the set of equations (17). The graph of comparison between effective areas computed by both models is shown on Fig. 7. The effective areas (lines S_c and S_m on the graph) differ. But computed volumes of air spring (lines V_c and V_m) are nearly the same.

Value of effective area obtained by equation (2) is very close to value of exact model No. 1, see Fig. 7.

3.1. Convolutions touch problem. For this model we assume, the radius of inner part of circle arc is identical with outer part of circle arc. For this model and the assumption, we completed equations (18) and (19) by variable g, see Fig. 5.

$$f_1 \equiv v_1 + v_2 + R\Big((\pi - \varphi_{1(v_1)}) + (\varphi_{2(v_2)} - \pi)\Big) + g - L = 0, \tag{21}$$

$$\mathbf{f}_{2,3} = \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} = \big(\mathbf{x}_{1(v_1)} + R\,\mathbf{n}_{1(v_1)}\big) - \big(\mathbf{x}_{2(v_2)} + R\,\mathbf{n}_{2(v_2)}\big) + \begin{bmatrix} g \\ 0 \end{bmatrix} = \mathbf{0}. \tag{22}$$

In this set of equations are four unknowns - v_1, v_2, R and g. But, there deal equation for radius

$$R_{(v_1)} = \frac{y\ sin\varphi_{1(v_1)}}{1 - sin\varphi_{1(v_1)}}, \tag{23}$$

see grey sector on Fig.5.

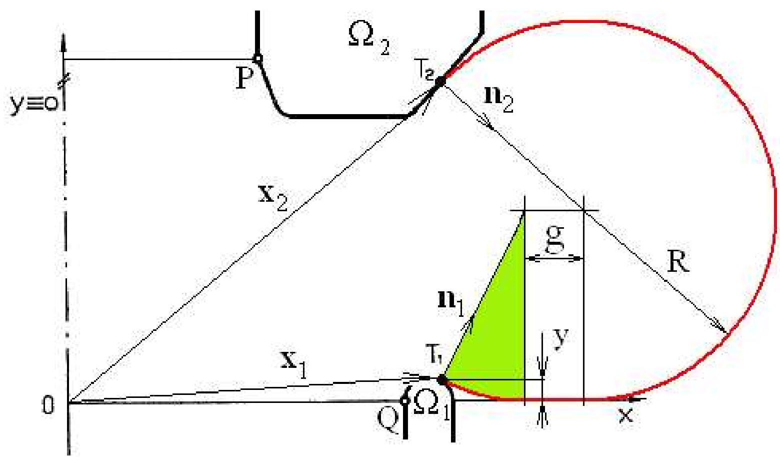

FIG. 5. *Convolutions touch problem; g - length of convolutions contact; Ω_1 - lower retainer; Ω_2 - upper retainer; P, Q - terminal points of bellows; T_1, T_2 - contact points; v_1-length of boundary Γ_1 between point Q and T_1; v_2-length of boundary Γ_2 between point P and T_2; x_1, x_2 - vectors of coordinates of contact points T; n_1, n_2 - unit vectors of exterior normal of boundary of domain Ω in contact points T; R - radius of the circular arc of bellow*

For the equation (21) holds

$$g_{(v_1)} = L - \Big[v_1 + v_2 + R\Big((\pi - \varphi_{1(v_1)}) + (\varphi_{2(v_2)} - \pi)\Big)\Big], \tag{24}$$

where unknowns are v_1 and v_2 and we can solve them with using by the generalized Newton's method, see equation (20).

4. Conclusion. Both of described models are applicable to calculation of geometrical characteristics of BAS. Time cost of model No. 1 is extreme. Usually, we use model No. 2, which gives results in much shorter time. This model is used in user-friendly system for calculation of geometrical characteristics of BAS. But, there are areas for further development.

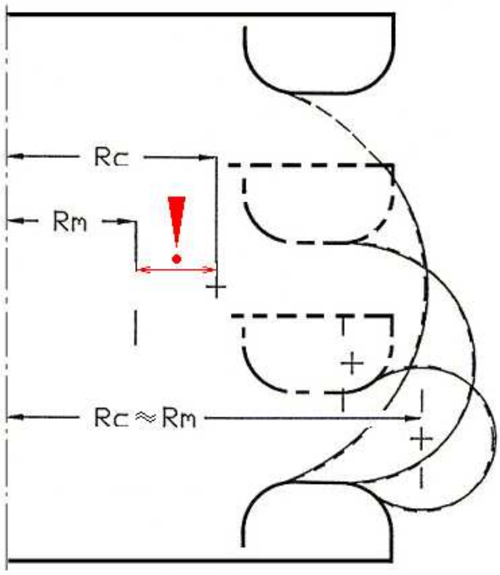

FIG. 6. *Desing of BAS VJ 80–07 (sizes of radii of effective areas); R_c center of the circular arc (free parts of the bellows meridian are plotted by dashed line); R_m radius of fictive points of release M, N for the curve of maximum volume (free parts of the bellows meridian are plotted by continuous line)*

The assumption of constant bellows length is not quite certain. Bellows from fabric-reinforced elastomer change the length in dependence on deformation and on air pressure too. This problem necessitates new formulation of problem, involving the interaction of bellows fabric-reinforced elastomer. This model can provide quite new scope of solving tasks, for example radial loading of retainers (shift in axis x) or inclination of them.

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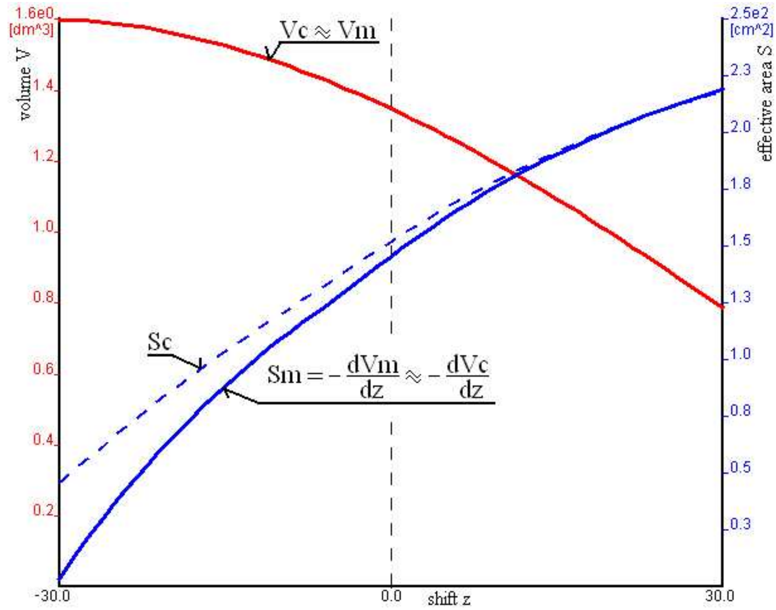

FIG. 7. *Graph of geometrical characteristics (volume V and effective area S) of BAS VJ 80-07; V_m - volume of BAS by model No. 1; V_c - volume of BAS by model No. 2; S_m - effective area of BAS by model No. 1; S_c - effective area of BAS by model No. 2*