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ON THE APPROXIMATION OF THE NULL-CONTROLLABILITY PROBLEM FOR PARABOLIC EQUATIONS*

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Abstract. In this paper we are interested in the study of semi-discrete and full discrete approximations of the null-controllability problem for parabolic equations. We restrict ourselves to the monodimensional case and to finite difference approximations in space. We first show that the semi-discretisation in space of such a problem can be proved to be uniformly controllable with respect to the mesh size if we only try to reach an exponentially small target and not the null target. Then, we extend this result to full discrete problems by using a classical Implicit Euler scheme or a θ-scheme for the time discretization of the problem.

The proofs, not given here, are essentially based on the proof of a partial discrete Lebeau-Robbiano inequality which is itself obtained by proving a global Carleman estimate for a semi-discrete elliptic operator. Attractive features of our approach is that it applies to variable coefficient problems and are not restricted to uniform meshes.

Key words. Null-controllability problem; Second order parabolic equation; Finite difference methods; Uniform observability inequality.

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1. Introduction. Let $\Omega = (a, b)$, and ω be non-empty bounded intervals of $\mathbb{R}$ with $\omega \Subset \Omega$. We consider the following parabolic problem in $(0, T) \times \Omega$, with $T > 0$,

$$\partial_t y - \partial_x(\gamma \partial_x y) = \mathbf{1}_\omega v \text{ on } (0, T) \times \Omega, \quad y|_{\partial\Omega} = 0, \quad \text{and} \quad y|_{t=0} = y_0, \tag{1.1}$$

where the diffusion coefficient γ satisfies

$$0 < \gamma_{\min} \leq \gamma \leq \gamma_{\max} < \infty, \quad \|\partial_x \gamma\|_\infty < \infty. \tag{1.2}$$

G. Lebeau and L. Robbiano prove in [LR95] the null controllability of system (1.1), *i.e.*, for all $y_0 \in L^2(\Omega)$, there exists $v \in L^2((0, T) \times \Omega)$, such that $y(T) = 0$ and $\|v\|_{L^2((0,T)\times\Omega)} \leq C|y_0|_{L^2(\Omega)}$, where $C > 0$ only depends on Ω, ω, γ and T. They in fact constructed the control function v semi-explicitly. This construction is based on the following spectral inequality.

THEOREM 1.1 ([LR95]). *Let $(\phi_k)_{k\in\mathbb{N}^*}$ be a set of $L^2(\Omega)$-orthonormal eigenfunctions of the operator $\mathcal{A} := -\partial_x(\gamma\partial_x)$ with homogeneous Dirichlet boundary conditions, and $(\mu_k)_{k\in\mathbb{N}^*}$ be the set of the associated eigenvalues (with finite multiplicities) sorted in a non-decreasing sequence. There exists $C > 0$ such that for all $\mu \geq 0$*

$$\sum_{\mu_k \leq \mu} |\alpha_k|^2 = \int_\Omega \Big|\sum_{\mu_k\leq\mu} \alpha_k \phi_k\Big|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \Big|\sum_{\mu_k\leq\mu} \alpha_k\phi_k\Big|^2, \qquad \forall (\alpha_k)_{k\in\mathbb{N}^*} \subset \mathbb{R}.$$

The proof of this result relied on local Carleman estimates for the augmented elliptic operator $-\partial_t^2 + \mathcal{A}$ in $(0, T_*) \times \Omega$, for some $T_* > 0$, where t is an additional variable.

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We proved in [BHLa] a discrete in space counter-part of this inequality when the diffusion operator is replaced by a finite-difference operator and then we deduced a corresponding controllability result for the semi-discrete in space parabolic problem. We recall the notation and the precise statement of these results in Section 2.

The proofs of these results, rely on a global discrete Carleman estimate for a related semi-discrete bidimensional elliptic operator.

The aim of the present paper is to present similar controllabity results for full discrete systems, that is to say when one couples the finite-difference in space scheme with an implicit Euler time discretization, or more generally with a θ-scheme time discretization. To this end we first give in Section 3.1 the time discrete counter-part of the well-known observability inequalities and related controllability results in a general setting. Then, in section 3.2, we prove the announced uniform controllability results. These results will hold under a condition between the time step and the mesh size which looks like a CFL condition. In section 3.3, we finally deduce from the above results that some kind of uniform approximate observability inequality holds with a super-algebraically small additional term. We describe the associated minimisation procedure that leads to the practictal computation of a suitable full discrete control function for our problem.

Notice that most of the techniques used in this paper do not depend on the space dimension. We will give in [BHLb] the corresponding detailed results in higher dimension.

2. Uniform controllability for semi-discrete in space approximations.

2.1. Notation and discrete settings. Let us consider the elliptic operator on $\Omega = (a, b)$ given by $\mathcal{A} = -\partial_x(\gamma\partial_x)$ with homogeneous Dirichlet boundary conditions with γ satisfying (1.2).

We introduce finite-difference approximations of the operator $\mathcal{A}$ on weakly-varying meshes. Let $a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$. We refer to this discretization as to the primal mesh $\mathfrak{M} := \{x_i;\ i = 1, \ldots, N\}$. We set $|\mathfrak{M}| := N$. We set $h_{i+\frac{1}{2}} = x_{i+1} - x_i$ and $x_{i+\frac{1}{2}} = (x_{i+1} + x_i)/2$, $i = 0, \ldots, N$, and $h = \max_{0\leq i\leq N} h_{i+\frac{1}{2}}$. We call $\overline{\mathfrak{M}} := \{x_{i+\frac{1}{2}};\ i = 0, \ldots, N\}$ the dual mesh and we set $h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = (h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}})/2$, $i = 1, \ldots, N$.

In the present article, we shall only consider families of meshes obtained as the image of uniform meshes through smooth increasing maps. More precisely, for any smooth function $\psi : [0, 1] \mapsto \overline{\Omega} = [a, b]$ satisfying

$$\psi([0, 1]) = \overline{\Omega}, \quad \text{and} \quad \inf_{[0,1]} \psi' > 0, \tag{2.1}$$

and any N, we shall consider the mesh $\mathfrak{M}$ of Ω defined by

$$x_i = \psi\left(\frac{i}{N+1}\right), \quad \forall i \in \{0, \ldots, N+1\}. \tag{2.2}$$

The dual mesh $\overline{\mathfrak{M}}$ is then defined as described above.

Such a family of meshes is quasi-uniform in the usual sense, that is to say that there exists C_ψ depending only on ψ such that

$$\frac{1}{C_\psi} \leq \frac{h_{i+\frac{1}{2}}}{h} \leq C_\psi, \forall i \in \{0, \ldots, N\},$$

$$\frac{1}{C_\psi} \leq \frac{h_i}{h} \leq C_\psi, \forall i \in \{1, \dots, N\}.$$

We denote by $\mathbb{R}^{\mathfrak{M}}$ and $\mathbb{R}^{\overline{\mathfrak{M}}}$ the sets of discrete functions defined on $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ respectively. If $u \in \mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}}$), we denote by u_i (resp. $u_{i+\frac{1}{2}}$) its value corresponding to x_i (resp. $x_{i+\frac{1}{2}}$). For $u \in \mathbb{R}^{\mathfrak{M}}$ we define

$$u^{\mathfrak{M}} = \sum_{i=1}^{N} \mathbf{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]} u_i \in L^\infty(\Omega).$$

Since no confusion is possible, by abuse of notation, we shall often write u in place of $u^{\mathfrak{M}}$. For $u \in \mathbb{R}^{\mathfrak{M}}$ we define $\int_\Omega u := \int_\Omega u^{\mathfrak{M}}(x)dx = \sum_{i=1}^N h_i u_i$, and similarly, we define the following L^2 inner product and norms on $\mathbb{R}^{\mathfrak{M}}$

$$(u, v)_{L^2} = \int_\Omega u^{\mathfrak{M}}(x) v^{\mathfrak{M}}(x)\, dx, \quad \text{and } |u|_{L^2(\Omega)} = (u, u)_{L^2}^{\frac{1}{2}}.$$

For some $u \in \mathbb{R}^{\mathfrak{M}}$, we shall need to associate boundary conditions that we denote by $u^{\partial\mathfrak{M}} = \{u_0, u_{N+1}\}$. The set of such extended discrete functions is denoted by $\mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$. Homogeneous Dirichlet boundary conditions then consist in the choice $u_0 = u_{N+1} = 0$, in short $u^{\partial\mathfrak{M}} = 0$. We can now define two difference operators $D : \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}} \to \mathbb{R}^{\overline{\mathfrak{M}}}$ and $\bar{D} : \mathbb{R}^{\overline{\mathfrak{M}}} \to \mathbb{R}^{\mathfrak{M}}$, given by

$$(Du)_{i+\frac{1}{2}} := \frac{u_{i+1} - u_{i-1}}{h_{i+\frac{1}{2}}}, \quad \forall u \in \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}, \forall i \in \{0, \dots, N\},$$

$$(\bar{D}u)_i := \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h_i}, \quad \forall v \in \mathbb{R}^{\overline{\mathfrak{M}}}, \forall i \in \{1, \dots, N\}.$$

With the notation introduced above, the consistent finite-difference approximation of $\mathcal{A}u$ with homogeneous boundary conditions that we consider is $\mathcal{A}^{\mathfrak{M}}u = -\bar{D}(\tilde{\gamma}Du)$ for $u \in \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ satisfying $u^{\partial\mathfrak{M}} = 0$, where $\tilde{\gamma} = (\gamma(x_{i+\frac{1}{2}}))_{0\leq i\leq N}$ is the sampling of the function γ on the dual mesh $\overline{\mathfrak{M}}$.

For any subspace $F \subset \mathbb{R}^{\mathfrak{M}}$, we shall denote by $F^\perp$, the L^2-orthogonal complement of F and by Π_F the L^2-orthogonal projection from $\mathbb{R}^{\mathfrak{M}}$ onto F. We shall also note $\mathbf{1}_\omega$ for the characteristic function of ω sampled on the mesh, that is $(\mathbf{1}_\omega)_i = 1$ if and only if $x_i \in \omega$.

2.2. Discrete Lebeau-Robbiano inequality. We note that $\mathcal{A}^{\mathfrak{M}}$ is selfadjoint with respect to the L^2 inner product on $\mathbb{R}^{\mathfrak{M}}$ introduced in (2.1). We denote by $\phi^{\mathfrak{M}}$ a set of discrete L^2 orthonormal eigenfunctions, $\phi_j \in \mathbb{R}^{\mathfrak{M}}$, $1 \leq j \leq |\mathfrak{M}|$, of the operator $\mathcal{A}^{\mathfrak{M}}$ with homogeneous boundary conditions, and by $\mu^{\mathfrak{M}}$ the set of the associated eigenvalues sorted in a non-decreasing sequence, μ_j, $1 \leq j \leq |\mathfrak{M}|$.

The partial Lebeau-Robbiano spectral inequality for the lower part of the spectrum we obtain in [BHLa] reads

Theorem 2.1 (Partial discrete Lebeau-Robbiano inequality). *Let ψ satisfying* (2.1). *There exist $C > 0$, $C_1 > 0$ and $h_0 > 0$ such that, for any mesh $\mathfrak{M}$ obtained from ψ by* (2.2) *and such that $h \leq h_0$, for all $0 < \mu \leq C_1/h^2$, we have*

$$\sum_{\substack{\mu_k\in\mu^{\mathfrak{M}}\\ \mu_k\leq\mu}} |\alpha_k|^2 = \int_\Omega \Big| \sum_{\substack{\mu_k\in\mu^{\mathfrak{M}}\\ \mu_k\leq\mu}} \alpha_k \phi_k \Big|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \Big| \sum_{\substack{\mu_k\in\mu^{\mathfrak{M}}\\ \mu_k\leq\mu}} \alpha_k \phi_k \Big|^2, \qquad \forall (\alpha_k)_{1\leq k\leq|\mathfrak{M}|} \subset \mathbb{R}.$$

Note that this inequality is optimal in some sense: for dimension reasons this inequality cannot be true for the whole spectrum of the operator $\mathcal{A}^{\mathfrak{M}}$. In particular, it is reasonable to think that C_1 should behave like $|\omega|/|\Omega|$.

2.3. Controllability results and observability inequalities. From Theorem 2.1, we deduced in [BHLa] the following uniform controllability result for the semi-discrete problem.

We introduce the following finite dimensional spaces

$$E_j = \operatorname{Span}\{\phi_k;\ 1 \leq \mu_k \leq 2^{2j}\} \subset \mathbb{R}^{\mathfrak{M}}, \quad j \in \mathbb{N}. \tag{2.3}$$

THEOREM 2.2. *Let $T > 0$ and ψ satisfying* (2.1). *There exist $h_0 > 0$, $C_T > 0$ and C_1, C_2, $C_3 > 0$ such that for any mesh $\mathfrak{M}$ obtained from ψ by* (2.2) *such that $h \leq h_0$, and all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a semi-discrete control function $v \in L^2(]0,T[,\mathbb{R}^{\mathfrak{M}})$ such that the solution to*

$$\partial_t y + \mathcal{A}^{\mathfrak{M}} y = \partial_t y - \bar{D}(\tilde{\gamma} D y) = \mathbf{1}_\omega v(t), \quad y^{\partial\mathfrak{M}} = 0, \quad y|_{t=0} = y_0. \tag{2.4}$$

satisfies $\Pi_{E_{j^{\mathfrak{M}}}} y(T) = 0$, and

$$\int_0^T |v(t)|^2_{L^2(\Omega)}\, dt \leq C_T^2 |y_0|^2_{L^2(\Omega)}, \ \textit{and}\ |y(T)|_{L^2(\Omega)} \leq C_2 e^{-C_3/h^2} |y_0|_{L^2(\Omega)},$$

with

$$j^{\mathfrak{M}} = \max\{j;\ 2^{2j} \leq C_1/h^2\}. \tag{2.5}$$

Thanks to this result, we deduced the following h-uniform approximate observability inequality for the semi-discrete problem under-study.

THEOREM 2.3. *Let $T > 0$ and ψ satisfying* (2.1). *There exist $h_0 > 0$, $C_{obs} > 0$ and $C > 0$ depending on Ω, ω and T, and ψ, such that: for any mesh $\mathfrak{M}$ obtained from ψ by* (2.2) *such that $h < h_0$, the semi-discrete solution q in $\mathscr{C}^\infty([0,T], L^2(\Omega))$ to*

$$\begin{cases} -\partial_t q + \mathcal{A}^{\mathfrak{M}} q = 0 & \textit{in } (0,T)\times\Omega, \\ q = 0 & \textit{on } (0,T)\times\partial\Omega, \\ q(T) = q_T \in \mathbb{R}^{\mathfrak{M}}, \end{cases}$$

satisfies

$$|q(0)|^2_{L^2(\Omega)} \leq C_{obs}^2 \int_0^T\!\!\int_\omega |q(t)|^2\ dt + C e^{-C/h^2} |q_T|^2_{L^2(\Omega)}.$$

Using this observability inequality, we can now provide some constructive way to compute a suitable semi-discrete control function. To this end, let $h \mapsto \phi(h) \in \mathbb{R}^+$ be a function which tends to zero when h goes to 0 and such that $e^{-C/h^2}/\phi(h) \to 0$. We have the following result.

THEOREM 2.4. *We consider the same assumptions and notation as in Theorem 2.3.*

For any mesh $\mathfrak{M}$ *obained from* ψ *by* (2.2) *such that* $h \leq h_0$, *and any* $y_0 \in \mathbb{R}^{\mathfrak{M}}$, *we consider the functional* $q_T \in \mathbb{R}^{\mathfrak{M}} \mapsto J^{\mathfrak{M}}(q_T)$ *defined by*

$$J^{\mathfrak{M}}(q_T) = \frac{1}{2}\int_0^T |q(t)|^2_{L^2(\omega)}\,dt + \frac{\phi(h)}{2}|q_T|^2_{L^2(\Omega)} + (y_0, q(0))_{L^2(\Omega)},$$

where $t \mapsto q(t)$ *is the solution to the adjoint problem* $-\partial_t q(t) + \mathcal{A}^{\mathfrak{M}} q(t) = 0$ *with final data* $q(T) = q_T$.

This functional $J^{\mathfrak{M}}$ *has a unique mimiser denoted by* $q_{F,opt} \in \mathbb{R}^{\mathfrak{M}}$. *This minimiser produces a solution* q *of the adjoint problem such that, if we define the control function* $v(t) = \mathbf{1}_\omega q(t)$ *then we have:*

- *The cost of the control is bounded as follows*

$$\int_0^T |v(t)|^2_{L^2(\omega)}\,dt \leq C^2_{obs}|y_0|^2_{L^2(\Omega)}.$$

- *The controlled solution* y *to* (2.4) *is such that*

$$|y(T)|_{L^2(\Omega)} \leq \sqrt{\phi(h)}C_{obs}|y_0|_{L^2(\Omega)}.$$

3. On the controllability and observability of fully discrete systems.

3.1. General framework. We consider in this section the problem of controlling fully discrete approximations of system (2.4). More precisely, for $M > 0$ and $\delta t = T/M$, We shall consider two time-discretization schemes:

- The implicit Euler scheme:

$$\begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} y^{n+1} = \mathbf{1}_\omega v^{n+1}, \quad \forall n \in \{0, \ldots, M-1\}, \end{cases} \tag{3.1}$$

- The θ-scheme, with $\theta \in [1/2, 1)$:

$$\begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}}(\theta y^{n+1} + (1-\theta)y^n) = \mathbf{1}_\omega v^{n+1}, \quad \forall n \in \{0, \ldots, M-1\}, \end{cases} \tag{3.2}$$

where, in both cases, $(v^n)_{1\leq n\leq M} \in (\mathbb{R}^{\mathfrak{M}})^M$ is a fully discrete control.

Naturally, (3.1) is nothing but (3.2) when $\theta = 1$. We present the two schemes separately, even though most of the following results are similar for both schemes. In fact, the only particular case we will encounter is the one of the Crank-Nicholson scheme, that is when $\theta = 1/2$, which is just at the stability limit.

Let us first state a relationship between a partial controllability result and a suitable observability inequality. For this result we treat in the same way the Implicit Euler scheme and the θ-scheme.

THEOREM 3.1. *Let* E *be a subspace of* $\mathbb{R}^{\mathfrak{M}}$ *such that* $\mathcal{A}^{\mathfrak{M}} E \subset E$. *Let* $\theta \in [0, 1]$, *and* $F = \ker(I - \delta t(1-\theta)\mathcal{A}^{\mathfrak{M}})$. *For a given* $C_{obs} > 0$, *the following statements are equivalent.*

1. *For any* $y_0 \in \mathbb{R}^{\mathfrak{M}}$, *there exists* $v = (v^n)_{1\leq n\leq M} \in (\mathbb{R}^{\mathfrak{M}})^M$ *such that*

$$\sum_{n=1}^M \delta t|v^n|^2_{L^2(\Omega)} \leq C^2_{obs}|y_0|^2_{L^2(\Omega)}, \tag{3.3}$$

and such that the solution to

$$\begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}}(\theta y^{n+1} + (1-\theta)y^n) = \mathbf{1}_\omega v^{n+1}, \quad \forall n \in \{0, \dots, M-1\}, \end{cases}$$

satisfies $\Pi_{E\cap F^\perp} y^M = 0$.

2. *Any solution* $q = (q^n)_{1\leq n\leq M+1}$ *of the adjoint problem in* $E \cap F^\perp$*:*

$$\begin{cases} q^{M+1} \in E \cap F^\perp, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}^{\mathfrak{M}} q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}}(\theta q^n + (1-\theta)q^{n+1}) = 0, \quad \forall n \in \{M-1, \dots, 1\}, \end{cases} \tag{3.4}$$

satisfies

$$|q^1 - \delta t(1-\theta)\mathcal{A}^{\mathfrak{M}} q^1|^2_{L^2(\Omega)} \leq C^2_{obs} \sum_{n=1}^{M} \delta t |q^n|^2_{L^2(\omega)}. \tag{3.5}$$

Notice, when $\theta < 1$, the particular form of the first iterate of the adjoint problem and of the left-hand side member of the observability inequality (3.5). In many cases the space F is trivial (in particular for the implicit Euler scheme). In the cases where $F \neq \{0\}$, we remark that one single iteration of the forward scheme (without control) satisfy the property

$$\Pi_{E\cap F^\perp} y^M = 0 \;\Rightarrow\; y^{M+1} = (I + \delta t\theta \mathcal{A}^{\mathfrak{M}})^{-1}(I - \delta t(1-\theta)\mathcal{A}^{\mathfrak{M}})y^M, \text{is such that } \Pi_E y^{M+1}.$$

This is the key ingredient to get rid of this non trivial kernel F in the proofs of the following results.

3.2. Partial observability inequalities and uniform controllability results. We shall now use the discrete Lebeau-Robbiano inequality recalled in Section 2 in order to prove actually that, under suitable assumptions the above observability inequality is satisfied for the spaces E_j defined in (2.3) and for $j \leq j^{\mathfrak{M}}$.

THEOREM 3.2. *Let* $T > 0$, ψ *satisfying* (2.1) *and consider* h_0, C_T, C_1, C_2, C_3 *given in Theorem 2.2. We recall that* $j^{\mathfrak{M}} \in \mathbb{N}$ *is given by* (2.5) *and we define* $\mu^{\mathfrak{M}}_{max} = \frac{C_1}{h^2}$.

Let now θ *be given in* $[1/2, 1]$. *There exists* $C > 0$ *(which does not depend on* T*) such that the following observability inequality holds true for any mesh* $\mathfrak{M}$ *obtained from* ψ *by* (2.2) *such that* $h \leq h_0$

$$|q^1 - \delta t(1-\theta)\mathcal{A}^{\mathfrak{M}} q^1|^2_{L^2(\Omega)} \leq C(1+T^2)\frac{e^{C2^j}}{T} \sum_{n=1}^{M} \delta t |q^n|^2_{L^2(\omega)},$$

for any solution to the adjoint problem (3.4) *with* $E = E_j$*, for any* $j \leq j^{\mathfrak{M}}$.

Notice that this result holds without any restriction on the time step δt. We also remark that the observability inequality is true for any final data in E_j and in particular for any final data in $E_j \cap F^\perp$, so that we will be able to apply Theorem 3.1.

With this result at hand we can conclude to the following uniform controllability results for the schemes under study.

THEOREM 3.3 (Implicit Euler scheme and θ-scheme with $\theta > 1/2$). *Let $T > 0$, ψ satisfying (2.1) and $\theta \in]1/2, 1]$.*

Let $0 < \beta \leq 1$, $\alpha > 0$ and $0 < \beta' < \beta$. There exists $C_T > 0$, $C > 0$, and $h_0 > 0$ such that for any mesh $\mathfrak{M}$ obtained from ψ by (2.2) such that $h \leq h_0$, and any $M \in \mathbb{N}^*$ such that $\delta t = T/M \leq \alpha h^{1+\beta}$ we have:

For any $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a full discrete control $v = (v^n)_{1\leq n\leq M}$ such that:

- *The solution $(y^n)_{0\leq n\leq M}$ to (3.1) satisfies*

$$\Pi_{E_j^{\mathfrak{M}}} y^M = 0, \quad \text{and } |y^M|_{L^2(\Omega)} \leq Ce^{-C/h^{1+\beta'}} |y_0|_{L^2(\Omega)}.$$

- *The control v satifies*

$$\sum_{n=1}^{M} \delta t |v^n|^2_{L^2(\omega)} \leq C_T^2 |y_0|^2_{L^2(\Omega)}.$$

In the case $\theta = 1/2$, the result we obtain is weaker since a more restrictive condition on δt is imposed.

THEOREM 3.4 (Crank-Nicholson scheme). *Let $T > 0$, ψ satisfying (2.1) and assume $\theta = 1/2$. Let $\alpha > 0$ and $0 < \beta' < 1$. There exists $C_T > 0$, $C > 0$ and $h_0 > 0$ such that for any mesh $\mathfrak{M}$ obtained from ψ by (2.2) such that $h \leq h_0$, and any $M \in \mathbb{N}^*$ such that $\delta t = T/M \leq \alpha h^2$ we have:*

For any $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a full discrete control $v = (v^n)_{1\leq n\leq M} \in (\mathbb{R}^{\mathfrak{M}})^M$ such that:

- *The solution $(y^n)_{0\leq n\leq M}$ to (3.2) satisfies*

$$\Pi_{E_j^{\mathfrak{M}}} y^M = 0, \quad \text{and } |y^M|_{L^2(\Omega)} \leq Ce^{-C/h^{1+\beta'}} |y_0|_{L^2(\Omega)}.$$

- *The control v satifies*

$$\sum_{n=1}^{M} \delta t |v^n|^2_{L^2(\omega)} \leq C_T^2 |y_0|^2_{L^2(\Omega)}.$$

3.3. Global uniform approximate observability inequalities and applications. From the two previous results, we can show the following global uniform approximate observability result.

THEOREM 3.5. *Let $T > 0$, ψ satisyfing (2.1) and $\theta \in [1/2, 1]$. Let $0 < \beta \leq 1$, $0 < \beta' < \beta$ and $\alpha > 0$. If $\theta = 1/2$, we further assume that $\beta = 1$.*

There exists $C_{obs} > 0$, $C > 0$ and $h_0 > 0$ such that, for any mesh $\mathfrak{M}$ obtained from ψ by (2.2) such that $h \leq h_0$, and any $M \in \mathbb{N}$ such that $\delta t = T/M \leq \alpha h^{1+\beta}$ we have

$$|q^1 - \delta t(1-\theta)\mathcal{A}^{\mathfrak{M}} q^1|^2_{L^2(\Omega)} \leq C^2_{obs} \sum_{n=1}^{M} \delta t |q^n|^2_{L^2(\omega)} + Ce^{-\frac{C}{h^{1+\beta'}}} |q^{M+1}|^2_{L^2(\Omega)},$$

for $(q^n)_n$ solution of the adjoint problem (3.4) associated to any $q^{M+1} \in \mathbb{R}^{\mathfrak{M}}$.

With this result at hand we can provide, just like for the semi-discrete problem, a constructive way to compute an uniformly bounded control v for which the solution $(y^n)_n$ to the controlled problem has a final state as small as we want (see Theorem 2.4).

Let $h \mapsto \phi(h) \in \mathbb{R}^+$ be a function such that $\phi(h) \to 0$ and $e^{-\frac{C}{h^{1+\beta'}}}/\phi(h) \to 0$, when $h \to 0$. Typically, we can choose for $\phi(h)$ any positive power of h.

Theorem 3.6. *We consider the same assumptions and notation as in Theorem 3.5.*

For any mesh $\mathfrak{M}$ defined by (2.2) with $h \leq h_0$, and any $y_0 \in \mathbb{R}^{\mathfrak{M}}$, we consider the functional $q_F \in \mathbb{R}^{\mathfrak{M}} \mapsto J_{\delta t}^{\mathfrak{M}}(q_F)$ defined by

$$J_{\delta t}^{\mathfrak{M}}(q_F) = \frac{1}{2}\sum_{n=1}^{M} \delta t |q^n|^2_{L^2(\omega)} + \frac{\phi(h)}{2}|q_F|^2_{L^2(\Omega)} + (y_0, q^1 - \delta t(1-\theta)\mathcal{A}^{\mathfrak{M}} q^1)_{L^2(\Omega)},$$

where $(q^n)_n$ is the solution of the adjoint problem (3.4) with final data $q^{M+1} = q_F$.

This functional $J_{\delta t}^{\mathfrak{M}}$ has a unique minimiser $q_{F,opt,\delta t} \in \mathbb{R}^{\mathfrak{M}}$. This minimiser produces a solution $(q^n)_n$ to (3.4), and if one define $v^n = \mathbf{1}_\omega q^n$ for any $1 \leq n \leq M$, then the discrete control v satisfies :

- *The cost of the control is bounded as follows*

$$\sum_{n=1}^{M} \delta t |v^n|^2_{L^2(\omega)} \leq C_{obs}^2 |y_0|^2_{L^2(\Omega)}.$$

- *The controlled solution $(y^n)_n$ associated to v is such that*

$$|y^M|_{L^2(\Omega)} \leq \sqrt{\phi(h)} C_{obs} |y_0|_{L^2(\Omega)}.$$

For instance if one choose $\phi(h) = h^{2p}$ for any given p, one can construct this way a uniformly bounded sequence of controls leading to a final state not larger than $h^p C_{obs}|y_0|_{L^2(\Omega)}$. Notice in particular that the value of C_{obs} does not depend on the particular choice of ϕ we use.

The practical computation of q_{opt} can be performed by a conjugate gradient solver as proposed in [GL94]. Each iteration of this solver consists in solving the solution to a first full discrete adjoint problem then the solution to a full discrete direct problem.

Notice that similar semi-discrete h-uniform approximate observability and controllabiliy results are given in [LT06]. The results in this reference are stronger in some sense since they apply to a wider class of semi-discrete problems under suitable assumptions on the continuous problem under study and the considered discretisation. Nevertheless, in the context we discussed in this paper, our results are much more precise since we obtain exponentially small targets associated to uniformly bounded sequences of controls. In particular, the results in [LT06] implies Theorem 2.4 but only for a function $\phi(h) = h^\beta$ for a suitable exponent $\beta > 0$ which is given by the properties of the problem under study. In particular, for a finite element discretisation of the heat equation, and for a boundary control problem (this a minor difference with the present study), they show that one can take $\beta = 0.45$ so that the computed target has a size of $h^{0.225}$ which is a very low convergence rate.

Notice that our results are in some sense optimal since it is known that, in higher dimensions, there exists high frequencies modes whose associated eigenfunctions are localized in space (see [LZ98b, Zua06]). In particular, if the control domain ω does not intersect the support of these eigenmodes, then it is guaranteed that we can not achieve exactly the null target for any initial data y_0. Nevertheless, since these very particular modes correspond to the highest frequencies in the discrete problem (of order $\sim \frac{C}{h^2}$), we can expect to reach a target of size $e^{\frac{-CT}{h^2}}$. This is exactly the result we proved in the semi-discrete case (Theorem 2.2) and also in the full discrete case (Theorems 3.3 and 3.4) with a slightly lower power of h in the exponential term.

4. Convergence when the time step goes to zero. We conclude this paper by a convergence result of the full discrete controls sequence, when $\delta t \to 0$, towards a semi-discrete control function. More precisely, the following results holds.

THEOREM 4.1. *We consider the assumptions and notation as in Theorems 2.4 and 3.6. In particular, we suppose given a function $h \mapsto \phi(h)$ as described just before these two theorems.*

For any $\delta t = T/M$ small enough, we define the piecewise constant function

$$\tilde{v}_{\delta t}(t) = \sum_{n=1}^{M} \mathbf{1}_{](n-1)\delta t, n\delta t[}(t)\, v^n, \quad \forall t \in]0, T[,$$

where $(v^n)_n$ is the full discrete control obtained by minimizing the functional $J_{\delta t}^{\mathfrak{m}}$ given in Theorem 3.6.

We have the following convergence result

$$\int_0^T |\tilde{v}_{\delta t}(t) - v(t)|^2_{L^2(\omega)}\, dt \xrightarrow[\delta t \to 0]{} 0,$$

where $t \mapsto v(t)$ is the semi-discrete control function obtained by minimizing the functional $J^{\mathfrak{m}}$ given in Theorem 2.4.

Note that this result gives a strong convergence property of the controls. Hence, it also implies the strong convergence in a suitable sense of the full discrete controlled solution $(y^n)_n$ towards the semi-discrete controlled solution y.

5. Conclusions. We provide in this paper uniform observability and approximate null-controllability results for the semi-discretisation in space and the full discretisation of 1D distributed control problem for a linear parabolic equation. The originality of this work is that is provides a way to compute effectively a uniformly bounded (with respect to the discretization parameters) sequence of controls for which the associated controlled solution of the parabolic discrete problem reach a final state which is essentially as small as we want. More precisely, we are able to prove a super-algebraically smallness estimate for this final state.

We also prove the strong convergence of the full discrete control functions towards a corresponding semi-discrete control function.

We shall give in further works some numerical illustrations of the various properties given in this paper.

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