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FUTURES TRADING WITH TRANSACTION COSTS *

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Abstract. We consider an investor, who takes positions in the futures contracts, pays proportional transaction costs, do not consume and is interested in his/her wealth far in the future. We assume that the futures price is an arithmetic Brownian motion and this assumption together with the restriction to utility function with hyperbolic absolute risk aversion (HARA) enable us to evaluate interval investment strategies. It is shown that the optimal interval strategy is also optimal among a wide class of admissible strategies.

Key words. HARA utility function, futures trading, transaction costs

AMS subject classifications. 60H30, 60G44, 91B28

1. Introduction. One of possible approaches to the problem of investment is to maximize the expected value of certain transformation of investors wealth at a certain time in the future. It is reasonable to assume that such a transformation should be strictly increasing and concave and it is referred as a utility function. One of the most desirable such a function is the logarithmic one and it dates at least to Daniel Bernoulli in the eighteen century. It is known as Kelly criterion, see Kelly (1956), whose objective was to maximize the exponential growth rate rather than to use any utility function. Breiman (1961), Algoet and Cover (1988) showed that maximizing logarithmic utility leads to asymptotically maximal growth rate and asymptotically minimal expected time to reach a presigned goal. Bell and Cover (1988) showed that the expected log-optimal portfolio is also game theoretically optimal in a single play or in multiple plays of the stock market for a wide variety of pay off functions. Browne and Whitt (1996) used Bayesian approach in order to derive optimal gambling and investment policies for cases in which the underlying stochastic process has parameter values that are unobserved random variables. For further properties of Kelly criterion see Bell and Cover (1980), Rotando and Thorp (1992), Thorp (1997), Janeček (1999). Although this criterion has a lot of desirable properties, Samuelson (1971) and Thorp (1975) showed that maximizing geometric mean does not mean to end with a higher utility after a long time of investment. It is sufficient to consider other than logarithmic utility function with hyperbolic absolute risk aversion (HARA).

This paper is devoted to the simplest problem of investment in the futures contracts in the presence of proportional transaction costs. Similarly as in [15], we assume that the futures price is as an arithmetic Brownian motion. In contrast to [15], we consider an investor who does not consume, but he/she withdraw from the market at the end of a very large time horizon. Our objective is to maximize the asymptotic exponential growth rate of the certainty equivalent of the wealth process. Note that in case of logarithmic utility function, this criterion agrees with the aim to maximize the asymptotic exponential growth rate of the wealth process itself, compare this aim with Kelly's objective above.

See [2],[3],[8],[9],[14],[15],[18],[20],[21],[23] for further papers on the investment and

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consumption problem mostly in the presence of transaction costs based on Merton approach to the investment.

The motivation of the paper comes from [15], where the so called "Merton problem" for the futures trading is considered in case of constant coefficients and where some asymptotic results are given for small transaction costs besides some qualitative results. On the other hand, our approach is most related to the paper [11], which solves the problem that can be regarded as a limiting case of the Merton problem. The main difference here is that we consider futures trading instead of stock trading and we consider arithmetic Brownian motion for the futures prices instead of geometric Brownian motion for the stock market price. Both models assume that the corresponding coefficients are constant, which enables us to obtain explicit results in both cases and both results look very similar, but the case of futures trading cannot be derived from the case of stock trading. See [10],[12] for additional papers on stock trading with transaction costs following the same approach as it is considered in this paper.

The proofs corresponding to the stated lemmas and theorems in this paper are available at `www.karlin.mff.cuni.cz/~dostal/futures.pdf` or on request.

2. Notation. Let us denote by $\mathcal{W}_t$ the wealth process of the investor who invests in a money market and takes positions in futures contracts on some asset or index. For simplicity we assume that the interest rate r is zero. In contrast to the geometric Brownian motion model for a stock price, we adopt similarly as in [15] an arithmetic Brownian motion model for the futures price,

$$F_t = F_0 + \mu t + \sigma W_t, \tag{2.1}$$

where $F_0, \mu \in R$ and $\sigma > 0$ are constants, and W is a standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. In the case of the stock market price, only the multiplicative changes matter, not the price itself, and similarly in this model, only changes in the futures price matter (not the futures price itself). This is a natural theoretical reason to consider the model of an arithmetic instead of geometric Brownian motion model for the futures price, besides the reasons that fluctuations in the futures price of many underlying processes (e.g., Eurodollar futures) do not have the multiplicative scaling relative to the futures price inherent in a geometric Brownian motion model, see [15]. It is not realistic to assume that the drift coefficient of the futures price is known and constant in the long run, but it is important to consider the case of constant coefficients fist, since it enables us to obtain explicit results as can be seen in this paper. If we consider the case of stock trading, we are able to obtain explicit results in case of constant coefficients, see [11], and we are able to obtain almost optimal strategies when the coefficients of the model are non-constant and when the rate of return is not observable for small transaction taxes, see [12]. One may imagine that we could find almost optimal strategies also in case of futures trading, but one cannot expect explicit results generally.

In addition, the agent may take any long or short position in futures contracts by paying a small transaction cost $\lambda_1 \geq 0$ or $\lambda_2 \geq 0$, respectively, times the size of the trade required to attain the position, where $\lambda := \lambda_1 + \lambda_2 > 0$. See [15] for a short discussion of this assumption with the conclusion that such an assumption is acceptable for large traders. We assume that the investor is interested in the expected utility of his/her wealth in the long run. In order to be able to obtain results in explicit form, we restrict ourselves to utility functions with hyperbolic absolute risk aversion

(HARA) unbounded from below on $(0,\infty)$,

$$\mathcal{U}_\gamma(x) = \frac{1}{\gamma}\, x^\gamma \qquad \text{if} \quad \gamma < 0 \tag{2.2}$$

$$= \ln x \qquad \text{if} \quad \gamma = 0. \tag{2.3}$$

As $\gamma \le 0$, we get that $\mathcal{U}_\gamma(0) := \mathcal{U}_\gamma(0_+) = -\infty$, which corresponds to the property that such an investor never risk ruin. We also need to introduce corresponding exponential utility functions $e_\gamma(x) := \mathcal{U}_\gamma(e^x)$. We denote by $\mathcal{N}_t$ the number of futures contracts owned by an agent at time t and by $\mathcal{R}_t$ the ratio process $\mathcal{N}_t/\mathcal{W}_t$. Note that the agent is able to cancel all the futures contracts with remaining positive wealth if and only if the ratio process $\mathcal{R}_t$ belongs to the interval $(-1/\lambda_1, 1/\lambda_2)$, which will be denoted by A and referred as the set of all admissible ratios. Note that $1/\lambda_1$ and $1/\lambda_2$ will be considered to be equal to $+\infty$ whenever $\lambda_1 = 0$ or $\lambda_2 = 0$, respectively.

3. Zero transaction costs. If the number of the futures contracts is constant or if the there are no transaction costs,

$$\mathrm{d}\mathcal{W}_t = \mathcal{N}_t \,\mathrm{d}F_t = \mathcal{R}_t\mathcal{W}_t[\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t], \tag{3.1}$$

$$\mathrm{d}\ln \mathcal{W}_t = \tfrac{1}{2}\,\sigma^2(2\theta\mathcal{R}_t - \mathcal{R}_t^2)\,\mathrm{d}t + \sigma\mathcal{R}_t \,\mathrm{d}W_t \tag{3.2}$$

holds, where $\theta := \sigma^{-2}\mu$ is the value of the ratio process $\mathcal{R}_t$ maximizing the logarithmic drift coefficient of the wealth process. So, if we restrict ourselves to strategies such that the ratio process is bounded, we get that there is no strategy giving the higher expected logarithmic utility than the strategy keeping the ratio process on θ which will be referred as the log-optimal ratio. Further,

$$\mathcal{W}_t^{-\gamma}\,\mathrm{d}e_\gamma(\ln \mathcal{W}_t) = \tfrac{1-\gamma}{2}\,\sigma^2(2\Theta\mathcal{R}_t - \mathcal{R}_t^2)\,\mathrm{d}t + \sigma\mathcal{R}_t \,\mathrm{d}W_t \tag{3.3}$$

holds with $\Theta := \theta/(1-\gamma)$ and this value will be referred as the optimal ratio. Note that if we restrict ourselves to strategies such that the ratio process is bounded, $\mathcal{U}_\gamma(\mathcal{W}_t) = e_\gamma(\ln \mathcal{W}_t)$ is a product of an exponential martingale $\mathcal{E}_t$ with $\mathcal{E}_t^{-1}\,\mathrm{d}\mathcal{E}_t = \gamma\sigma\mathcal{R}_t \,\mathrm{d}W_t$ starting from 1 and the process

$$e_\gamma(\ln \mathcal{W}_0 + \tfrac{1-\gamma}{2}\,\sigma^2 \textstyle\int_0^t (2\Theta\mathcal{R}_s - \mathcal{R}_s^2)\,\mathrm{d}s),$$

which is maximal if the ratio process $\mathcal{R}_t$ is kept on the value Θ. As the maximal value of this process is deterministic, we get that keeping the ratio process $\mathcal{R}_t$ on the value Θ really gives the maximal expected utility at each time t.

4. Motivation. In the presence of transaction costs, it is not possible to find a strategy giving the maximal expected utility of the wealth process at each time t, but it is possible to find a strategy giving the maximal asymptotic growth rate ν of the certainty equivalent of the wealth process similarly as in [10],[11],[12]. More precisely, we are able to find a function $f \in C^2(A)$ and $\nu \in R$ such that $e_\gamma(U_t)$ is generally an $\mathcal{F}_t$-supermartingale and it is an $\mathcal{F}_t$-martingale in a special case, where $U_t = \ln \mathcal{W}_t - f(\mathcal{R}_t) - \nu t$. Then νt is a non-stationary part and $f(\mathcal{R}_t)$ is a stationary part of the process, which compensates $\ln \mathcal{W}_t$ so that $e_\gamma(U_t)$ is an $\mathcal{F}_t$-martingale in the optimal case. Denoting the special case by hat, we obtain the following inequalities

$$E\mathcal{U}_\gamma(\mathcal{W}_t) \le \mathcal{U}_\gamma\big(e^{\nu t + O(1)}\big) = e_\gamma(\nu t + O(1)), \tag{4.1}$$

$$E\mathcal{U}_\gamma(\hat{\mathcal{W}}_t) = \mathcal{U}_\gamma\big(e^{\nu t + O(1)}\big) = e_\gamma(\nu t + O(1)) \tag{4.2}$$

as $t \to \infty$, i.e., the certainty equivalent of the wealth process is generally less or equal to $e^{\nu t + O(1)}$ and equal to $e^{\nu t+O(1)}$ in the special case. In particular,

$$\limsup_{t\to\infty} \tfrac{1}{t} e_\gamma^{-1} E e_\gamma(\ln \mathcal{W}_t) \le \lim_{t\to\infty} \tfrac{1}{t} e_\gamma^{-1} E e_\gamma(\ln \hat{\mathcal{W}}_t) = \nu \tag{4.3}$$

and therefore we are able to solve the problem of maximizing the left hand side of (4.3) and also the same problem with lim sup replaced by lim inf. Note that $e_\gamma(\ln \mathcal{W}_t) = \mathcal{U}_\gamma(\mathcal{W}_t)$. If $\gamma = 0$, then (4.3) holds almost surely with $e_\gamma^{-1} E e_\gamma$ omitted. For the corresponding theorems see theorem 8.1, lemma 9.1 and corollary 10.6.

5. Basic dynamics. If the agent increase or decrease the number of futures contracts about $|\Delta \mathcal{N}_t|$, he/she pays the transaction costs $\lambda_1 \Delta \mathcal{N}_t$ or $-\lambda_2 \Delta \mathcal{N}_t$, respectively, and therefore the following value remains the same

$$\mathcal{W}_t + \lambda_1 \mathcal{N}_t = \mathcal{W}_t(1 + \lambda_1 \mathcal{R}_t) \quad \text{or} \quad \mathcal{W}_t - \lambda_2 \mathcal{N}_t = \mathcal{W}_t(1 - \lambda_2 \mathcal{R}_t), \tag{5.1}$$

respectively. It follows from (3.1) that $\mathrm{d}\mathcal{R}_t = B(\mathcal{R}_t)\,\mathrm{d}t + S(\mathcal{R}_t)\,\mathrm{d}W_t$ holds if the number of futures contracts does not change, where

$$B(x) = \sigma^2 x^2 (x - \theta), \quad S(x) = -\sigma x^2. \tag{5.2}$$

Let us introduce the control processes $\mathcal{R}_t^\pm$. They are assumed to be non-decreasing left-continuous $\mathcal{F}_t$-adapted with $\mathcal{R}_0^\pm = 0$. They increase or decrease the ratio process $\mathcal{R}_t$ by increasing or decreasing the agent's position in futures contracts, respectively, so that

$$\mathrm{d}\mathcal{R}_t = B(\mathcal{R}_t)\,\mathrm{d}t + S(\mathcal{R}_t)\,\mathrm{d}W_t + \mathrm{d}\mathcal{R}_t^+ - \mathrm{d}\mathcal{R}_t^- \tag{5.3}$$

holds. If the control processes are continuous or if we deal with them as they were continuous, we get from the law of constant values (5.1) that

$$\mathrm{d}\ln \mathcal{W}_t = \tfrac{1}{2}\sigma^2 (2\theta \mathcal{R}_t - \mathcal{R}_t^2)\,\mathrm{d}t + \sigma \mathcal{R}_t\,\mathrm{d}W_t - \zeta_+(\mathcal{R}_t)\,\mathrm{d}\mathcal{R}_t^+ - \zeta_-(\mathcal{R}_t)\,\mathrm{d}\mathcal{R}_t^- \tag{5.4}$$

holds as the $\mathrm{d}\mathcal{R}_t^\pm$-coefficients of logarithm of (5.1) should be equal to zero, where

$$\zeta_+(x) = \tfrac{\lambda_1}{1+\lambda_1 x}, \qquad \zeta_-(x) = \tfrac{\lambda_2}{1-\lambda_2 x}. \tag{5.5}$$

6. Assumptions and restrictions. Let us denote by $\mathcal{A} = \mathcal{A}(\Omega, \mathcal{F}, P)$ the set of all processes with values in a compact subset of A. We regard the ratio process $\mathcal{R}_t$ as admissible if it belongs to $\mathcal{A}$. The strategies not satisfying this condition will not be allowed.

We denote by $\mathcal{L}^- = \mathcal{L}^-(\Omega, \mathcal{F}, P)$ the set of all processes on $R^+ := [0, \infty)$ defined on the probability space $(\Omega, \mathcal{F}, P)$ with finite all negative moments. Only those strategies such that the wealth process $\mathcal{W}_t$ belongs to $\mathcal{L}^-$ will be considered. For simplicity, we assume that $\mathcal{W}_0 = w_0 > 0$ is a deterministic random variable.

We assume that in the presence of transaction costs, $\mathcal{N}_t$ is a process of locally finite variation, left-continuous and $\mathcal{F}_t$-adapted with the minimal decomposition $\mathcal{N}_t = \mathcal{N}_t^+ - \mathcal{N}_t^-$ to non-decreasing $\mathcal{F}_t$-adapted left-continuous processes $\mathcal{N}_t^\pm$ with $\mathcal{N}_0^\pm = 0$. Further, we restrict ourselves to strategies that do not take short and long positions in the futures contracts at the same time. Then the total value of transaction costs on the interval $[0, t)$ is of the form $\lambda_1 \mathcal{N}_t^+ + \lambda_2 \mathcal{N}_t^-$. Similarly as in [12], we consider a special type of integration in the presence in jumps that allows us to compute with stochastic differentials, as every integrator was a continuous process.

Further, we assume that $\theta \neq 0$ and therefore $\Theta \neq 0$. If $\theta = \Theta = 0$, the optimal strategy is to take always zero position in futures contracts in case of zero transaction costs. Since such a strategy does nothing, it is optimal also in case of non-zero proportional transaction costs.

7. Advanced dynamics. First, we introduce lemma that tells us what ODE we have to solve and how the boundary conditions look. Its proof is based just on Itô formula for continuous semimartingales.

LEMMA 7.1. *Let $f \in C^2(A)$. Let $\mathcal{W}_t > 0, \mathcal{R}_t \in A$ hold for every $t \geq 0$. Then the process $U_t := \ln \mathcal{W}_t - f(\mathcal{R}_t) - \nu t$ is an $\mathcal{F}_t$-semimartingale with*

$$e^{-\gamma U_t}\, de_\gamma(U_t) = d^\nu_f(\mathcal{R}_t)\, dt + v_f(\mathcal{R}_t)\, dW_t + \delta^f_+(\mathcal{R}_t)\, d\mathcal{R}^+_t + \delta^f_-(\mathcal{R}_t)\, d\mathcal{R}^-_t, \tag{7.1}$$

where $v_f(x) = \sigma x(1 + xf'(x)), \delta^f_\pm(x) = -\zeta_\pm(x) \mp f'(x)$ and

$$d^\nu_f(x) = \tfrac{1-\gamma}{2}\, \sigma^2(2\Theta x - x^2) - \nu - f'(x)\tilde{B}(x) - \tfrac{1}{2}[f''(x) - \gamma f'(x)^2]S^2(x) \tag{7.2}$$

$$\tilde{B}(x) = B(x) + \gamma\sigma x S(x) = (1-\gamma)\sigma^2 x^2(x - \Theta), \tag{7.3}$$

where $\Theta := \theta/(1-\gamma)$.

We say that an $\mathcal{F}_t$-Itô process has bounded coefficients if it is a sum of $\mathcal{F}_t$-adapted Lipschitz process and a continuous local $\mathcal{F}_t$-martingale with Lipschitz quadratic variation. A process X_t is assumed to be Lipschitz if and only if there is $c \in R$ such that $|X_s - X_t| \leq c|s - t|$ holds for every $s, t \geq 0$.

LEMMA 7.2. *Let $f \in C^2(A), \mathcal{W}_t > 0, \mathcal{R}_t \in \mathcal{A}$. Put $U_t := \ln \mathcal{W}_t - f(\mathcal{R}_t) - \nu t$.*
(i) If U_t is a an $\mathcal{F}_t$-Itô process with bounded coefficients, then $\mathcal{W}_t \in \mathcal{L}^-$.
(ii) If $\mathcal{W}_t \in \mathcal{L}^-$, then

$$\mathcal{V} := e_\gamma(U_0) + \textstyle\int e^{\gamma U_s} v_f(\mathcal{R}_s)\, dW_s \qquad \text{is an } \mathcal{F}_t\text{-martingale.} \tag{7.4}$$

This lemma says that (i) strategies satisfying certain condition are admissible and that (ii) the diffusion part of U_t has increments with mean value zero provided that an admissible strategy is considered.

8. Interval strategies. Let us assume that $\alpha < \beta$ are in A and consider an interval strategy denoted by $[(\alpha, \beta)]$ that keeps the ratio process $\mathcal{R}_t$ within the interval $[\alpha, \beta]$ and does nothing if $\mathcal{R}_t \in (\alpha, \beta)$. More precisely, we say that the strategy $[(\alpha, \beta)]$ is applied if there exist continuous non-decreasing $\mathcal{F}_t$-adapted processes $\mathcal{R}^\pm_t$ such that the ratio process $\mathcal{R}_t$ satisfies (5.3), that $\mathcal{R}_t \in [\alpha, \beta]$ holds for every $t \geq 0$ and that

$$\textstyle\int_0^\infty \mathbf{1}_{[\mathcal{R}_t > \alpha]}\, \mathrm{d}\mathcal{R}^+_t = 0, \qquad \int_0^\infty \mathbf{1}_{[\mathcal{R}_t < \beta]}\, \mathrm{d}\mathcal{R}^-_t = 0. \tag{8.1}$$

Note that given the above mentioned processes, there is an almost surely unique positive continuous $\mathcal{F}_t$-semimartingale $\mathcal{W}_t$ representing the wealth process such that (5.4) holds with the initial condition $\mathcal{W}_0 = w_0 > 0$. See theorem 6.4 in [10] that if $\mathcal{R}_0 = r_0 \in [\alpha, \beta]$, such processes $\mathcal{R}_t, \mathcal{R}^\pm_t$ exist and therefore we can (say that we) apply the strategy $[(\alpha, \beta)]$. The corresponding portfolio market price will be denoted by $\mathcal{W}^{\alpha,\beta}_t$ and the corresponding ratio process by $\mathcal{R}^{\alpha,\beta}_t$. If the initial ratio, further denoted by $\mathcal{R}_{0_-}$, does not belong to $[\alpha, \beta]$, we take positions in futures contracts in order to achieve that $\mathcal{R}_0 = \alpha$ if $\mathcal{R}_{0-} < \alpha$ and $\mathcal{R}_0 = \beta$ if $\mathcal{R}_{0-} > \beta$ at time $t = 0$. This may cause that $\mathcal{R}_t, \mathcal{R}_t^{\pm}$ are not left-continuous at $t = 0$, but we are interested in these processes on R^+.

In order to better understand the meaning of the initial value $\mathcal{R}_{0-}$, the reader can imagine that all processes are constant on $(-\infty, 0)$ with possible jumps at $t = 0$ from left. The rigorous approach is considered only on time interval $[0, \infty)$, which corresponds to the assumption that there are no initial contracts or that the initial values of the processes are adjusted. Note that this adjustment is negligible thanks to our criterion based on the asymptotics of the certainty equivalent of the wealth process.

The following theorem tells us that the right evaluation of the interval strategy is ν provided that we are able to find a solution to certain ODE with certain boundary conditions. The theorem follows immediately from lemmas 7.1 and 7.2.

THEOREM 8.1. *Let $f \in C^2(A)$ and $\nu \in R$, and $\alpha, \beta \in A$ be such that $\alpha < \beta$,*

$$d_f^\nu(x) = 0 \quad \text{holds for every } x \in [\alpha, \beta], \quad \delta_+^f(\alpha) = 0 = \delta_-^f(\beta). \tag{8.2}$$

Then $e_\gamma(U_t)$ is an $\mathcal{F}_t$-martingale and

$$\nu(\alpha, \beta) := \lim_{t\to\infty} \frac{1}{t}\, e_\gamma^{-1} E e_\gamma(\ln \mathcal{W}_t) = \nu \tag{8.3}$$

and $\mathcal{W}_t \in \mathcal{L}^-$, when applying the strategy $[(\alpha, \beta)]$.

REMARK 8.2. See corollary 10.6 that for every $\alpha < \beta$ in A with the same sign there exists $f \in C^2(A)$ and $\nu \in R$ such that the assumptions of theorem 8.1 are satisfied, and we get that the strategy $[(\alpha, \beta)]$ is admissible.

9. Comparison of strategies. In this section, we assume that the policies $\alpha < \beta$ from A are as in lemma 11.1 and that $g \in C^2(A)$ is as it introduced in section 11 so that

$$d_g^\nu(x) = 0 \text{ on } [\alpha, \beta], \quad \delta_+^g(x) = 0 \text{ on } (-1/\lambda_1, \alpha], \quad \delta_-^g(x) = 0 \text{ on } [\beta, 1/\lambda_2)$$

hold with $\nu = \frac{\sigma^2}{2}\, u(\alpha, \beta)$ and that $d_g^\nu(x), \delta_\pm^g(x) \le 0$ hold on A.

Further, we will use the following notation

$$U_t = \ln \mathcal{W}_t - g(\mathcal{R}_t) - \nu t, \qquad U_t^{\alpha,\beta} = \ln \mathcal{W}_t^{\alpha,\beta} - g(\mathcal{R}_t^{\alpha,\beta}) - \nu t.$$

If V_t, Z_t are random processes, we write $V_t = o_{as}(Z_t)$ if $V_t = o(Z_t)$ holds as $t \to \infty$ for almost every element of Ω.

LEMMA 9.1. *Let $\mathcal{W}_t \in \mathcal{L}^-, \mathcal{R}_t \in \mathcal{A}$. (i) Then $e_\gamma(U_t)$ is an $\mathcal{F}_t$-supermartingale, and $e_\gamma(U_t^{\alpha,\beta})$ is an $\mathcal{F}_t$-martingale. In particular,*

$$E\mathcal{U}_\gamma(\mathcal{W}_t^{\alpha,\beta}) = \mathcal{U}_\gamma\big(e^{\nu t + O(1)}\big), \quad E\mathcal{U}_\gamma(\mathcal{W}_t) \le \mathcal{U}_\gamma\big(e^{\nu t + O(1)}\big) \quad \text{as} \quad t \to \infty. \tag{9.1}$$

(ii) Let $\gamma = 0$. Then $\mathcal{W}_t \le \exp\{\nu t + o_{as}(t)\}$, and $\mathcal{W}_t^{\alpha,\beta} = \exp\{\nu t + o_{as}(t)\}$.
In particular if $\gamma = 0$, outside of a P-null set we have that

$$\nu = \lim_{t\to\infty} \tfrac{1}{t} \ln \mathcal{W}_t^{\alpha,\beta} \ge \limsup_{t\to\infty} \tfrac{1}{t} \ln \mathcal{W}_t.$$

10. Solution to (8.2). In order to simplify the form of the results, we introduce the following transformations

$$\xi_+(x) = \xi_{\lambda_1}(x) = \tfrac{x}{1+\lambda_1 x}, \qquad \xi_-(x) = \xi_{-\lambda_2}(x) = \tfrac{x}{1-\lambda_2 x}, \tag{10.1}$$

where $\xi_a(x) = \frac{x}{1+ax}$. Further, note that $\xi_0(x) = x$ and that $\xi_a \circ \xi_b = \xi_{a+b}$, which gives that $\xi_a^{-1} = \xi_{-a}$.

REMARK 10.1. Let us consider the equation (8.2), see (7.2) and (5.2), in the form

$$\tfrac{x^4}{1-\gamma}[f''(x) - f'(x)^2] + (x[1 + xf'(x)] - \Theta)^2 = \Theta^2 - \tfrac{2\nu\sigma^{-2}}{1-\gamma} \tag{10.2}$$

with the boundary conditions $\delta_+^f(\alpha) = 0 = \delta_-^f(\beta)$ in the form

$$f'(\alpha) = -\zeta_+(\alpha), \quad f'(\beta) = \zeta_-(\beta). \tag{10.3}$$

As $\mp\zeta'_\pm(x) = \zeta_\pm(x)^2$, the principle of the smooth fit at the boundary conditions is of the form

$$f''(\alpha) = \zeta_+^2(\alpha) = f'(\alpha)^2, \quad f''(\beta) = \zeta_-^2(\beta) = f'(\beta)^2. \tag{10.4}$$

Considering $x \in \{\alpha, \beta\}$, we obtain that the right-hand side of (10.2) is equal to ω^2 for some $\omega \geq 0$. Using again the boundary conditions (10.3) and (10.4), we obtain the following requirements

$$(\xi_+(\alpha) - \Theta)^2 = (\xi_-(\beta) - \Theta)^2 = \omega^2, \quad \nu = \tfrac{1-\gamma}{2}\,\sigma^2(\Theta^2 - \omega^2). \tag{10.5}$$

Let us consider a substitution

$$f'(x) = y(\tfrac{1}{x})\tfrac{1}{x^2} - \tfrac{1}{x}, \quad \text{i.e.} \quad y(\tfrac{1}{x}) = x(1 + xf'(x)). \tag{10.6}$$

Then the condition $d_f^\nu(x) = 0$ is of the form

$$y'(u) + F(y(u)) = 0, \qquad \text{with} \qquad F(y) = \gamma y^2 + 2\theta y - 2\nu\sigma^{-2} \tag{10.7}$$

and the boundary conditions (10.3) are of the form

$$y(\tfrac{1}{\alpha}) = \xi_+(\alpha), \quad y(\tfrac{1}{\beta}) = \xi_-(\beta). \tag{10.8}$$

Further, note that

$$d_f^\nu(x) = -\tfrac{1}{2}\,\sigma^2 x^4[f''(x) - f'(x)^2] - \tfrac{1-\gamma}{2}\,\sigma^2[(y(\tfrac{1}{x}) - \Theta)^2 - \omega^2] \tag{10.9}$$

holds provided that ν is as in (10.5) and $y(\frac{1}{x})$ as in (10.6) and also note that

$$f'(x) = \mp\zeta_\pm(x) \quad \Rightarrow \quad y(\tfrac{1}{x}) = x(1 + xf'(x)) = \xi_\pm(x). \tag{10.10}$$

REMARK 10.2. Let $u_0, y_0 \in R$ be such that $F(y_0) \neq 0$. Then $y(u) := G^{-1}(u)$ is the unique maximal solution to (10.7) with the boundary condition $y(u_0) = y_0$, where $G(y) = u_0 - \int_{y_0}^y F(x)^{-1}\,\mathrm{d}x$. Further, G' attains values in $R\backslash\{0\}$, and we get that the functions G and y are strictly monotone.

Proof. As $F(y_0) \neq 0$ and as F is a polynomial, the function G is defined correctly just on the maximal open interval containing y_0 of y such that $F(y) \neq 0$. Obviously,

$G'(y) = -F(y)^{-1} \neq 0$. Further, $G(y_0) = u_0$ and $y'(u) = G'(y(u))^{-1} = -F(y(u))$. As F is a locally Lipschitz function, we have the uniqueness. □

LEMMA 10.3. *Let $\nu \in R$ and $u_1, u_2, y_1, y_2 \in R$ with $u_1 < u_2$ be such that*

$$u_2 - u_1 = \textstyle\int_{y_2}^{y_1} F(y)^{-1}\, dy. \tag{10.11}$$

Then there exists $y \in C^1[u_1, u_2]$ satisfying $y' + F(y) = 0$ on $[u_1, u_2]$ with $y(u_1) = y_1$ and $y(u_2) = y_2$.

Proof. As the right-hand side of (10.11) has to converge and F is a polynomial, we obtain that $F(y) \neq 0$ holds on $[y_1 \wedge y_2, y_1 \vee y_2]$. By remark 10.2, the inversion of $G(y) := u_1 - \int_{y_1}^{y} F(x)^{-1}\,\mathrm{d}x$ is a solution to $y' + F(y) = 0$ on $[u_1, u_2]$ and it is obviously a C^1-function as F is a polynomial and as G' has values in $R\backslash\{0\}$. Further, $u_2 = u_1 + \int_{y_2}^{y_1} F(y)^{-1}\,\mathrm{d}y = G(y_2)$, which is nothing else but $y(u_2) = y_2$. □

REMARK 11.4. Let assume that $\alpha < \beta$ have the same non-zero sign and that

$$\frac{1}{\alpha} - \frac{1}{\beta} = \int_{\xi_+(\alpha)}^{\xi_-(\beta)} \frac{dy}{\gamma y^2 + 2\theta y - 2\nu\sigma^{-2}}. \tag{10.12}$$

By lemma 10.3, there exists $y \in C^1[1/\beta, 1/\alpha]$ satisfying (10.7-10.8). Then

$$f(x) = \textstyle\int y\left(\frac{1}{x}\right)\frac{\mathrm{d}x}{x^2} - \ln|x| \in C^2[\alpha, \beta]$$

is a solution to (8.2), see remark 10.1.

LEMMA 10.5. *Let $\alpha < \beta$ be from A with the same sign, then there exists just one $u(\alpha, \beta) \in R$ such that*

$$\frac{1}{\alpha} - \frac{1}{\beta} = \int_{\xi_+(\alpha)}^{\xi_-(\beta)} \frac{dy}{\gamma y^2 + 2\theta y - u(\alpha, \beta)}. \tag{10.13}$$

COROLLARY 10.6. *Let $\alpha < \beta$ be from A with the same sign, then $\nu := \frac{\sigma^2}{2}\, u(\alpha, \beta)$ satisfies (10.12) and therefore there exists $f \in C^2[\alpha, \beta]$ satisfying (8.2) with the above introduced value of ν.*

11. Smooth fit. Let $\alpha < \beta$ be from A with the same sign. By corollary 10.6, there exists $f \in C^2[\alpha, \beta]$ and $\nu = \frac{\sigma^2}{2}\, u(\alpha, \beta)$ satisfying (8.2). Put

$$g(x) = f(x) \quad \text{if} \quad x \in [\alpha, \beta] \tag{11.1}$$

$$g(x) = f(\alpha) + \ln \tfrac{1+\lambda_1 \alpha}{1+\lambda_1 x} \quad \text{if} \quad x \in (-1/\lambda_1, \alpha] \tag{11.2}$$

$$g(x) = f(\beta) + \ln \tfrac{1-\lambda_2 \beta}{1-\lambda_2 x} \quad \text{if} \quad x \in [\beta, 1/\lambda_2). \tag{11.3}$$

Obviously, $g \in C^1(A)$. We are looking for the policies $\alpha, \beta \in A$ such that $g \in C^2(A)$. Note that this requirement corresponds to the stationary condition of function u at the point (α, β) are therefore we can imagine that we are looking for (α, β) maximizing u. Note that $\nu(\alpha, \beta) = \frac{\sigma^2}{2}\, u(\alpha, \beta)$ holds, where $\nu(\alpha, \beta)$ is given by (8.3).

LEMMA 11.1. *Let $\omega \in [0, |\Theta|)$ and $\alpha < \beta$ from A be such that*

$$\xi_+(\alpha) = \Theta - \omega, \quad \xi_-(\beta) = \Theta + \omega, \quad u(\alpha, \beta) = (1 - \gamma)(\Theta^2 - \omega^2). \tag{11.4}$$

Then $g \in C^2(A)$. *Moreover,* $d_g^\nu \le 0$ *and* $\delta_\pm^g \le 0$ *hold on* A.

Further, we will show that there exists $\omega \in (0, |\Theta|)$ such that (11.4) holds. It will immediately follow from the following lemma.

LEMMA 11.2. *The function* $L(\omega)$ *is continuous increasing on* $(-|\Theta|, |\Theta|)$ *with range* R

$$L(\omega) := \frac{1}{\Theta - \omega} - \frac{1}{\Theta + \omega} - \int_{\Theta-\omega}^{\Theta+\omega} \frac{dy}{\gamma y^2 + 2\theta y - (1-\gamma)(\Theta^2 - \omega^2)}. \tag{11.5}$$

Further, $L(0) = 0$ *and* $L(\omega) = q\omega^3 + o(\omega^3)$ *holds as* $\omega \to 0^+$, *where* $q = \frac{4}{3}(1-\gamma)\Theta^{-4}$.

LEMMA 11.3. *There exists just one* $\omega_\lambda := \omega \in (0, |\Theta|)$ *such that* $\lambda := \lambda_1 + \lambda_2 = L(\omega)$, *and there are* $\alpha < \beta$ *from* A *such that (11.4) holds.*

REMARK 12.4. By lemma 11.2, $\mathcal{L}(x) := L(\sqrt[3]{x}) = qx + o(x)$ as $x \to 0$, i.e. $\mathcal{L}'(0) = q \neq 0$ and we get from the theorem on explicitly defined function that $(\mathcal{L}^{-1})'(0) = q^{-1}$, i.e. $\mathcal{L}^{-1}(\lambda) = q^{-1}\lambda + o(\lambda)$. Then

$$\omega_\lambda^3 = \tfrac{3}{4}\tfrac{\Theta^4}{1-\gamma}\,\lambda + o(\lambda) \tag{11.6}$$

holds as $\lambda \to 0^+$. Since $\lambda_1, \lambda_2 = O(\lambda)$, we get that

$$\alpha = \xi_{-\lambda_1}(\Theta - \omega_\lambda) = \Theta - \omega_\lambda + O(\lambda), \tag{11.7}$$

$$\beta = \xi_{-\lambda_2}(\Theta + \omega_\lambda) = \Theta + \omega_\lambda + O(\lambda). \tag{11.8}$$

12. Conclusion. Let us assume that we have constant (and deterministic) coefficients $\mu, \sigma \neq 0$. If $\mu = 0$, the optimal strategy would be to invest all the wealth in the money market and not to be interested in futures contracts (and the case $\sigma = 0$ would lead to a deterministic model).

Then we get that $\Theta = \sigma^{-2}\mu/(1-\gamma) \neq 0$ is the optimal ratio of the wealth process that should be "invested" in the futures contracts in case of zero transaction costs. In the presence of transaction costs, it is optimal just to keep the ratio process within the interval $[\alpha, \beta]$, where $\xi_+(\alpha) = \Theta - \omega$ and $\xi_-(\beta) = \Theta + \omega$, where ω is a parameter connected with the width of the no-trade region given by the equation $L(\omega) = \lambda := \lambda_1 + \lambda_2$. See (11.5) for the definition and properties of function $L(\omega)$ and (10.1) for the definition of functions $\xi_\pm(x)$.

Note that the optimal strategy gives the maximal value of $\lim \frac{1}{t} e_\gamma^{-1} E e_\gamma(\ln \mathcal{W}_t)$ as $t \to \infty$, which for $\gamma < 0$ gives that

$$E\mathcal{U}_\gamma(\mathcal{W}_t) \le E\mathcal{U}_\gamma(\hat{\mathcal{W}}_t)\exp\{o(t)\} \tag{12.1}$$

holds as $t \to \infty$, where $\hat{\mathcal{W}}_t$ stands for the wealth process corresponding to the optimal strategy described above. See lemma 9.1 that we can write $O(1)$ instead of $o(t)$ in (12.1).

In case of logarithmic utility, i.e. $\gamma = 0$, we have a better comparison of strategies given by lemma 9.1 for a large family of stopping times τ in the form

$$E \ln \mathcal{W}_\tau \le E \ln \hat{\mathcal{W}}_\tau + K, \tag{12.2}$$

where $K \geq |f(\mathcal{R}_t) - f(\hat{\mathcal{R}}_t)|$ is a constant not depending on τ, but depending on the strategy that is compared with the optimal one. This inequality for stopping times immediately gives that such a strategy is in certain sense optimal also in a time changed model. In order to appreciate (12.1) and (12.2), note that $E\mathcal{U}_\gamma(\mathcal{W}_t)$ and $E \ln \mathcal{W}_t$ has asymptotically exponential and linear growth, respectively, whenever considering any interval strategy for example.

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