Proceedings of ALGORITMY 2009
pp. 294–303

DISCONTINUOUS GALERKIN METHOD FOR NONSTATIONARY NONLINEAR CONVECTION–DIFFUSION PROBLEMS: A PRIORI ERROR ESTIMATES*

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Abstract. We deal with a numerical solution of a scalar nonstationary convection-diffusion equation with nonlinear convective as well as diffusive terms which represents a model problem for the solution of the system of the compressible Navier-Stokes equations describing a motion of viscous compressible fluids. We present a discretization of this model equation by the interior penalty discontinuous Galerkin methods. Moreover, under some assumptions on the nonlinear terms, domain partitions and the regularity of the exact solution, we introduce a priori error estimates in the $L^\infty(0,T;L^2(\Omega))$-norm and in the $L^2(0,T;H^1(\Omega))$-semi-norm. A sketch of the proof and numerical verifications are presented.

Key words. discontinuous Galerkin method, convection-diffusion problem, a priori error estimates

AMS subject classifications. 65M60, 65M15, 65M12, 65M20

1. Introduction. Our goal is to develop a sufficiently robust, accurate and efficient numerical method for the solution of the system of the compressible Navier-Stokes equations describing a motion of viscous compressible fluids. Due to the lack of the theory concerning with an existence of the solution of the Navier-Stokes equations we consider the model problem represented by a nonstationary two-dimensional convection–diffusion equation with nonlinear convection as well as diffusion.

Among a wide class of numerical methods, the *discontinuous Galerkin finite element method* (DGFEM) seems to be a promising technique for the solution of convection-diffusion problems. DGFEM is based on a piecewise polynomial but discontinuous approximation, for a survey, see, e.g., [4], [5]. Within this paper we deal with the space semidiscretization of the model problem with the aid three variants of DGFEM, namely nonsymmetric (NIPG), symmetric (SIPG) and incomplete interior penalty Galerkin (IIPG) techniques, see [1].

This article represents a generalization of research papers [7], [8], [9], [10], where the linear diffusion term was considered. Moreover, let us cite works [6], [11], [12], where simpler forms of nonlinear diffusion were analysed.

2. Problem formulation. We consider the following unsteady nonlinear convection-diffusion problem: Let $\Omega \subset I\!\!R^2$ be a bounded polygonal domain, $T > 0$, we seek a function $u : Q_T = \Omega \times (0,T) \to I\!\!R$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} = \operatorname{div}(I\!\!K(u)\,\nabla u) + g \quad \text{in } Q_T, \tag{1}$$

$$u\big|_{\partial\Omega\times(0,T)} = u_D, \tag{2}$$

*This work is a part of the research project MSM 0021620839 financed by the Ministry of Education of the Czech Republic and it was partly supported by the Grant No. 316/2006/B-MAT/MFF of the Grant Agency of the Charles University Prague.

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$$u(x,0) = u^0(x), \quad x \in \Omega, \tag{3}$$

where $g : Q_T \to I\!R$, $u_D : \partial\Omega \times (0,T) \to I\!R$, $u^0 : \Omega \to I\!R$ are given functions, $f_1, f_2 \in C^1(I\!R)$ represent convective terms and the regular matrix $I\!K(u) \in I\!R^{2,2}$ plays a role of nonlinear anisotropic diffusive coefficients. If $I\!K(u) = \varepsilon I\!I$, where ε is a positive constant and $I\!I \in I\!R^{2,2}$ the unit matrix, then problem (1) – (3) reduces to the equation considered in [7], [8], [9], [10]. For simplicity we prescribe the Dirichlet condition on the whole boundary but it is also possible to consider mixed Dirichlet–Neumann boundary conditions.

3. Discretization. Let $\mathcal{T}_h$ $(h > 0)$ be a family of the partitions of the domain $\Omega \subset I\!R^2$ into triangular elements. We do not require the conformity of the mesh, i.e., the so-called hanging nodes are allowed. However, more general elements (even non-convex) can be considered within the frame of DGFEM, see [9]. By $\mathcal{F}_h$ we denote the smallest possible set of all edges of all elements $K \in \mathcal{T}_h$. Furthermore, let $\mathcal{F}_h^I$ and $\mathcal{F}_h^D$ represent the *interior* and *boundary* edges of $\mathcal{T}_h$, respectively. Obviously $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D$. Finally, for each $\Gamma \in \mathcal{F}_h$, we define a unit normal vector $\vec{n}_\Gamma$. We assume that $\vec{n}_\Gamma$, $\Gamma \subset \partial\Omega$ has the same orientation as the outer normal of $\partial\Omega$. For $\vec{n}_\Gamma$, $\Gamma \in \mathcal{F}_I$ the orientation is arbitrary but fixed for each edge.

Over a triangulation $\mathcal{T}_h$ we define the so-called *broken Sobolev space*

$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K) \ \forall K \in \mathcal{T}_h\} \tag{4}$$

with the seminorm $|v|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} |v|^2_{H^k(K)}\right)^{1/2}$, where $|\cdot|_{H^k(K)}$ denotes the standard seminorm on the Sobolev space $H^k(K)$, $K \in \mathcal{T}_h$. Moreover, the approximate solution is sought in a space of piecewise polynomial but discontinuous functions

$$S_{hp} \equiv S_{hp}(\Omega, \mathcal{T}_h) = \{v; v|_K \in P_p(K) \ \forall K \in \mathcal{T}_h\}, \tag{5}$$

where $P_p(K)$ denotes the space of all polynomials on K of degree $\le p$, $K \in \mathcal{T}_h$.

For each $\Gamma \in \mathcal{F}_h^I$ there exist two elements $K_L, K_R \in \mathcal{T}_h$ such that $\Gamma \subset K_L \cap K_R$. We use a convention that K_R lies in the direction of $\vec{n}_\Gamma$ and K_L in the opposite direction of $\vec{n}_\Gamma$. For $v \in S_{hp}$, by $v|_\Gamma^{(L)} =$ trace of $v|_{K_L}$ on Γ, $v|_\Gamma^{(R)} =$ trace of $v|_{K_R}$ on Γ we denote the *traces* of v on edge Γ, which are different in general. Additionally,

$$[v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}, \quad \langle v\rangle_\Gamma = \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\right), \tag{6}$$

denotes the *jump* and the *mean value* of function v over the edge Γ, respectively. For $\Gamma \in \partial\Omega$ there exists an element $K_L \in \mathcal{T}_h$ such that $\Gamma \subset K_L \cap \partial\Omega$. Then for $v \in S_{hp}$, we put: $v|_\Gamma^{(L)} =$ trace of $v|_{K_L}$ on Γ, $\langle v\rangle_\Gamma = [v]_\Gamma = v|_\Gamma^{(L)}$. In case that $[\cdot]_\Gamma$ and $\langle\cdot\rangle_\Gamma$ are arguments of $\int_\Gamma \ldots \,\mathrm{d}S$, $\Gamma \in \mathcal{F}_h$ we omit the subscript Γ and write simply $[\cdot]$ and $\langle\cdot\rangle$, respectively.

Similarly as in [7], it is possible to derive the space semi-discretization of (1) – (3). A particular attention should be paid to the nonlinear diffusive term. In order to replace the interelement continuity, we add some stabilization and penalty terms into formulation of the discrete problem. The convective term is approximated with the aid of a numerical flux $H(\cdot,\cdot,\cdot)$, known from the finite volume method.

Therefore, we say that $u_h \in C^1(0,T; S_{hp})$ is the *semi-discrete solution* of (1) – (3) if $(u_h(0), v_h) = (u^0, v_h) \ \forall v_h \in S_{hp}$ and

$$\begin{aligned}&\left(\frac{\partial u_h(t)}{\partial t}, v_h\right) + b_h(u_h(t), v_h) + a_h^\Theta(u_h(t), v_h) + \alpha J_h^\sigma(u_h(t), v_h)\\ &= \ell_h^\Theta(u_h(t), v_h)\,(t) \qquad \forall v_h \in S_{hp}, \ \forall t \in (0,T),\end{aligned} \tag{7}$$

where

$$a_h^{\Theta}(u,v) = \sum_{K\in\mathcal{T}_h}\int_K \mathbb{K}(u)\,\nabla u\cdot\nabla v\,\mathrm{d}x - \sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma \langle \mathbb{K}(u)\,\nabla u\cdot\vec{n}\rangle[v]\,\mathrm{d}S$$

$$+\,\Theta\sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma \langle \mathbb{K}(u)\,\nabla v\cdot\vec{n}\rangle[u]\,\mathrm{d}S, \tag{8}$$

$$b_h(u,v) = -\sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^{2} f_s(u)\frac{\partial v}{\partial x_s}\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \vec{n}_\Gamma)\,[v]\mathrm{d}S, \tag{9}$$

$$J_h^{\sigma}(u,v) = \sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma \sigma[u]\,[v]\,\mathrm{d}S, \tag{10}$$

$$\ell_h^{\Theta}(u,v)(t) = \int_\Omega g(t)\,v\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \Big(\Theta\,\mathbb{K}(u)\,\nabla v\cdot\vec{n}\,u_D(t) + \sigma\,u_D(t)\,v\Big)\,\mathrm{d}S, \tag{11}$$

and α is a positive constant whose specification follows from assumption (13, b). Nonlinear forms $a_h^{\Theta}(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are the discrete variants of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, respectively. According to value of parameter Θ, we speak of SIPG ($\Theta=-1$), IIPG ($\Theta=0$) or NIPG ($\Theta=1$) variants of DGFEM. Penalty terms are represented by J_h^{σ} and the penalty parameter function σ in (10) is defined by $\sigma|_\Gamma = C_W\mathrm{diam}(\Gamma)^{-1}, \Gamma\in\mathcal{F}_h$, where $C_W>0$ is a suitable constant depending on the used variant of scheme and on the degree of polynomial approximation, the value of multiplicative constant α before the penalty form J_h^{σ} will be specified in Section 4, assumption (13).

The problem (7) exhibits a system of ordinary differential equations for $u_h(t)$ which has to be discretized by a suitable ODE method.

We shall assume that the numerical flux H is Lipschitz continuous (i.e., $|H(u,v,\vec{n}) - H(u^*,v^*,\vec{n})| \le C(|u-u^*|+|v-v^*|)\ \forall u,u^*,v,v^*\in\mathbb{R}\ \forall\vec{n}=(n_1,n_2)$), consistent with the convective fluxes $f_1,\ f_2$ (i.e., $H(u,u,\vec{n}) = f_1(u)n_1+f_2(u)n_2\ \forall u\in\mathbb{R}\ \forall\vec{n}=(n_1,n_2)$) and conservative (i.e., $H(u,v,\vec{n}) = -H(v,u,-\vec{n})\ \forall u,v\in\mathbb{R}\ \forall\vec{n}=(n_1,n_2)$). Then we find that the sufficiently regular solution u of (1) – (3) satisfies

$$\left(\frac{\partial u(t)}{\partial t}, v_h\right) + b_h(u(t),v_h) + a_h^{\Theta}(u(t),v_h) + \alpha J_h^{\sigma}(u(t),v_h) = \ell_h^{\Theta}(u(t),v_h)\,(t) \qquad \forall\, v_h\in S_{hp}\ \forall\, t\in(0,T), \tag{12}$$

4. Error analysis. To carry out the error analysis we need to specify the additional assumptions on mesh, nonlinear diffusion term and regularity of the solution u of the continuous problem. Therefore, we assume that

(A1) The matrix $\mathbb{K}(v) = \{k_{ij}(v)\}_{i,j=1}^2,\ k_{ij}(v):\mathbb{R}\to\mathbb{R}$, appearing in the diffusion terms satisfies

$$\begin{aligned} &(a) && \|\mathbb{K}(v)\|_\infty \le C_U < \infty\ \forall\, v\in\mathbb{R},\\ &(b) && \|\mathbb{K}(v_1)-\mathbb{K}(v_2)\|_\infty \le C_L|v_1-v_2|\ \forall\, v_1,v_2\in\mathbb{R},\\ &(c) && \xi^T\mathbb{K}(v)\xi \ge \alpha\|\xi\|^2,\ \alpha>0,\ \forall\, v\in\mathbb{R},\ \forall\,\xi\in\mathbb{R}^2, \end{aligned} \tag{13}$$

where $\|\cdot\|_\infty$ represents the l^∞-matrix norm, i.e., $\|\mathbb{K}\|_\infty = \max_{1\le i\le n}\sum_{j=1}^n |k_{ij}|$.

(A2) The weak solution u is sufficiently regular, namely

$$\begin{aligned} &(a) && u\in L^2(0,T;H^s(\Omega)),\quad \frac{\partial u}{\partial t}\in L^2(0,T;H^s(\Omega)),\ s\ge 1\\ &(b) && \|\nabla u(t)\|_{L^\infty(\Omega)} \le C_D \quad \text{for a.a. } t\in(0,T), \end{aligned} \tag{14}$$

where $s > 0$ is a given number.

(A3) The triangulations $\mathcal{T}_h,\ h \in (0, h_0)$ are *locally quasi-uniform* and *shape-regular* (for detailed definitions see [6]).

Further, we shall consider the new norm $|||\cdot||| := \sum_{K\in\mathcal{T}_h} |\cdot|^2_{H^1(K)} + J_h^\sigma(\cdot,\cdot)$. Now, we are ready to formulate the main result of this paper.

THEOREM 4.1. *Let assumptions (A1) be satisfied, let u be the exact solution of the continuous problem satisfying (A2). Let $\mathcal{T}_h,\ h \in (0, h_0)$ be a family of triangulations satisfying (A3) and let the numerical flux H from (9) be consistent, conservative and Lipschitz continuous. Let $u_h \in S_{hp}$ be the solution of the discrete problem given by (7). Then the discretization error $e_h = u_h - u$ satisfies*

$$\max_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^T |||e_h(\vartheta)|||^2 \mathrm{d}\vartheta \le Ch^{2(\mu-1)}, \tag{15}$$

where $\mu = \min(p + 1, s)$ and $C > 0$ is a constant independent of h.

Proof. Let $u \in H^s(\Omega)$ be the solution of the continuous problem. For $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the L^2-projection of v on S_{hp}. We set $\xi(t) = u_h(t) - \Pi_h u(t) \in S_{hp}$, $\eta(t) = \Pi_h u(t) - u(t)$, $e_h(t) = u_h(t) - u(t) = \xi(t) + \eta(t)$ for a.a. $t \in (0, T)$. We subtract (12) from (7), set $v_h := \xi$ and add terms $-a_h^\Theta(\Pi_h u, \xi) + \ell_h^\Theta(\Pi_h u, \xi)$ on both sides of this identity. Then we obtain

$$\begin{aligned}
&\left(\frac{\partial\xi}{\partial t}, \xi\right) + \underbrace{a_h^\Theta(u_h(t), \xi) - a_h^\Theta(\Pi_h u, \xi) + \ell_h^\Theta(\Pi_h u, \xi) - \ell_h^\Theta(u_h, \xi) + \alpha J_h^\sigma(\xi, \xi)}_{=:\chi_1} \\
&\quad = \underbrace{-\left(\frac{\partial\eta}{\partial t}, \xi\right) + b_h(u, \xi) - b_h(u_h, \xi) - \alpha J_h^\sigma(\eta, \xi)}_{=:\chi_2} \\
&\qquad + \underbrace{a_h^\Theta(u, \xi) - a_h^\Theta(\Pi_h u, \xi) + \ell_h^\Theta(\Pi_h u, \xi) - \ell_h^\Theta(u, \xi)}_{=:\chi_3}
\end{aligned} \tag{16}$$

With the aid of the *multiplicative trace inequality*, *inverse inequality* and *approximation properties of the space S_{hp}*, (see [9, Lemmas 4.2–4.4]), we estimate terms χ_1, χ_2 and χ_3. At first, we start with treatment of the term χ_3. From (8), (11) and the fact that $[u]_\Gamma = 0\ \forall\Gamma \in \mathcal{F}_h^I$, we have

$$\begin{aligned}
\chi_3 = &\sum_{K\in\mathcal{T}_h} \int_K \Big(I\!\!K(u)\,\nabla u - I\!\!K(\Pi_h u)\,\nabla\Pi_h u\Big)\cdot\nabla\xi\,\mathrm{d}x \\
&- \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big(\langle I\!\!K(u)\,\nabla u\cdot\vec{n}\rangle - \langle I\!\!K(\Pi_h u)\,\nabla\Pi_h u\cdot\vec{n}\rangle\Big)[\xi]\,\mathrm{d}S \\
&+ \Theta \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big(\langle I\!\!K(u)\,\nabla\xi\cdot\vec{n}\rangle[u] - \langle I\!\!K(\Pi_h u)\,\nabla\xi\cdot\vec{n}\rangle[\Pi_h u]\Big)\,\mathrm{d}S \\
&+ \Theta \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \Big(I\!\!K(\Pi_h u)\,\nabla\xi\cdot\vec{n}\,u_D + \sigma\,u_D\,\xi - I\!\!K(u)\,\nabla\xi\cdot\vec{n}\,u_D - \sigma\,u_D\,\xi\Big)\,\mathrm{d}S \\
= &\underbrace{\sum_{K\in\mathcal{T}_h} \int_K I\!\!K(u)\,(\nabla u - \nabla\Pi_h u)\cdot\nabla\xi\,\mathrm{d}x}_{=:\vartheta_1}
\end{aligned} \tag{17}$$

$$
\begin{aligned}
&+ \underbrace{\sum_{K\in\mathcal{T}_h} \int_K \Big(I\!\!K(u) - I\!\!K(\Pi_h u)\Big)\, \nabla \Pi_h u \cdot \nabla \xi \,\mathrm{d}x}_{=:\vartheta_2} \\
&+ \underbrace{\sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big\langle I\!\!K(u)\,(\nabla \Pi_h u - \nabla u)\cdot \vec{n}\Big\rangle [\xi]\,\mathrm{d}S}_{=:\vartheta_3} \\
&+ \underbrace{\sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big\langle (I\!\!K(\Pi_h u) - I\!\!K(u))\, \nabla \Pi_h u \cdot \vec{n}\Big\rangle [\xi]\,\mathrm{d}S}_{=:\vartheta_4} \\
&+ \underbrace{\Theta \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \langle I\!\!K(u)\, \nabla\xi \cdot \vec{n}\rangle [u - \Pi_h u]\,\mathrm{d}S}_{=:\vartheta_5} \\
&+ \underbrace{\Theta \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big\langle (I\!\!K(u) - I\!\!K(\Pi_h u))\, \nabla\xi \cdot \vec{n}\Big\rangle [\Pi_h u - u]\,\mathrm{d}S}_{=:\vartheta_6} \\
&- \Theta \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \Big(I\!\!K(u) - I\!\!K(\Pi_h u)\Big)\, \nabla\xi \cdot \vec{n}\,(u_D - u)\,\mathrm{d}S = \sum_{i=1}^{6} \vartheta_i,
\end{aligned}
$$

where we used the fact that $u = u_D$ on $\Gamma \in \mathcal{F}_h^D$. Analogously as in [11] we derive

$$|\vartheta_1| \le C \cdot C_U h^{\mu-1} |u|_{H^\mu(\Omega)} |\xi|_{H^1(\Omega,\mathcal{T}_h)}, \tag{18}$$

$$|\vartheta_2| \le C\Big(C_U h^{\mu-1} |u|_{H^\mu(\Omega)} + C_D C_L h^{\mu} |u|_{H^\mu(\Omega)}\Big) |\xi|_{H^1(\Omega,\mathcal{T}_h)}, \tag{19}$$

$$|\vartheta_3| \le C \cdot C_U h^{\mu-1} |u|_{H^\mu(\Omega)} J_h^\sigma(\xi,\xi)^{1/2}, \tag{20}$$

$$|\vartheta_4| \le C\Big(C_U h^{\mu-1} |u|_{H^\mu(\Omega)} + C_D C_L h^{\mu} |u|_{H^\mu(\Omega)}\Big) J_h^\sigma(\xi,\xi)^{1/2}, \tag{21}$$

$$|\vartheta_5| \le C \cdot C_U h^{\mu-1} |u|_{H^\mu(\Omega)} |\xi|_{H^1(\Omega,\mathcal{T}_h)}, \tag{22}$$

$$|\vartheta_6| \le C \cdot C_U h^{\mu-1} |u|_{H^\mu(\Omega)} |\xi|_{H^1(\Omega,\mathcal{T}_h)}. \tag{23}$$

where C_U, C_L and C_D are constants from (13a), (13b) and (14b), respectively. Finally, since estimates (18)–(23), we can write

$$|\chi_3| \le C\Big(C_U h^{\mu-1} |u|_{H^\mu(\Omega)} + C_D C_L h^{\mu} |u|_{H^\mu(\Omega)}\Big) |\!|\!|\xi|\!|\!|. \tag{24}$$

The estimate of the second term χ_2 follows from [6, Lemmas 14–15]

$$|\chi_2| \le C\left\{ h^\mu \left|\frac{\partial u}{\partial t}\right|_{H^\mu(\Omega)} \|\xi\|_{L^2(\Omega)} + |\!|\!|\xi|\!|\!| (h^\mu |u|_{H^\mu(\Omega)} + \alpha h^{\mu-1} |u|_{H^\mu(\Omega)} + \|\xi\|_{L^2(\Omega)}) \right\}. \tag{25}$$

Similarly as in (17), we treat the term χ_1.

$$
\begin{aligned}
\chi_1 = &\sum_{K\in\mathcal{T}_h} \int_K \Big(I\!\!K(u_h)\, \nabla u_h - I\!\!K(\Pi_h u)\, \nabla \Pi_h u\Big) \cdot \nabla\xi\,\mathrm{d}x \\
&- \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big(\langle I\!\!K(u_h)\, \nabla u_h \cdot \vec{n}\rangle - \langle I\!\!K(\Pi_h u)\, \nabla \Pi_h u \cdot \vec{n}\rangle\Big) [\xi]\,\mathrm{d}S
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&+ \Theta \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big(\langle \mathbb{K}(u_h)\,\nabla\xi\cdot\vec{n}\rangle[u_h] - \langle \mathbb{K}(\Pi_h u)\,\nabla\xi\cdot\vec{n}\rangle[\Pi_h u] \Big)\,\mathrm{d}S \\
&+ \Theta \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \Big(\mathbb{K}(\Pi_h u)\,\nabla\xi\cdot\vec{n}\,u_D - \mathbb{K}(u_h)\,\nabla\xi\cdot\vec{n}\,u_D \Big)\,\mathrm{d}S + \alpha J_h^\sigma(\xi,\xi) \\
&= \underbrace{\sum_{K\in\mathcal{T}_h} \int_K \mathbb{K}(u_h)\,(\nabla u_h - \nabla\Pi_h u)\cdot\nabla\xi\,\mathrm{d}x}_{=:\vartheta_7} \\
&+ \underbrace{\sum_{K\in\mathcal{T}_h} \int_K \Big(\mathbb{K}(u_h) - \mathbb{K}(\Pi_h u)\Big)\,\nabla\Pi_h u\cdot\nabla\xi\,\mathrm{d}x}_{=:\vartheta_8} \\
&- \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big(\langle \mathbb{K}(u_h)\,\nabla u\cdot\vec{n}\rangle - \langle \mathbb{K}(u_h)\,\nabla\Pi_h u\cdot\vec{n}\rangle \Big)[\xi]\,\mathrm{d}S \\
&- \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big(\langle \mathbb{K}(u_h)\,\nabla\Pi_h u\cdot\vec{n}\rangle - \langle \mathbb{K}(\Pi_h u)\,\nabla\Pi_h u\cdot\vec{n}\rangle \Big)[\xi]\,\mathrm{d}S \\
&+ \Theta \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \langle \mathbb{K}(u_h)\,\nabla\xi\cdot\vec{n}\rangle[u_h - \Pi_h u]\,\mathrm{d}S \\
&+ \Theta \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big\langle (\mathbb{K}(u_h) - \mathbb{K}(\Pi_h u))\,\nabla\xi\cdot\vec{n}\Big\rangle[\Pi_h u]\,\mathrm{d}S \\
&+ \Theta \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \Big(\mathbb{K}(\Pi_h u) - \mathbb{K}(u_h) \Big)\,\nabla\xi\cdot\vec{n}\,u_D\,\mathrm{d}S + \alpha J_h^\sigma(\xi,\xi) \\
&= \vartheta_7 + \vartheta_8 + (\Theta - 1) \underbrace{\sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big\langle \mathbb{K}(u_h)\,(\nabla u_h - \nabla\Pi_h u)\cdot\vec{n}\Big\rangle[u_h - \Pi_h u]\,\mathrm{d}S}_{=:\vartheta_9} \\
&+ \underbrace{\sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big\langle (\mathbb{K}(\Pi_h u) - \mathbb{K}(u_h))\,\nabla\Pi_h u\cdot\vec{n}\Big\rangle[\xi]\,\mathrm{d}S}_{=:\vartheta_{10}} \\
&+ \underbrace{\Theta \sum_{\Gamma\in\mathcal{F}_h} \int_\Gamma \Big\langle (\mathbb{K}(u_h) - \mathbb{K}(\Pi_h u))\,\nabla\xi\cdot\vec{n}\Big\rangle[\eta]\,\mathrm{d}S}_{=:\vartheta_{11}} + \alpha J_h^\sigma(\xi,\xi)
\end{aligned}
$$

For all three mentioned IPG variants according to (26), the term χ_1 satisfies

$$\chi_1 \geq \vartheta_7 - |\vartheta_8| + (\Theta - 1)\vartheta_9 - |\vartheta_{10}| - |\vartheta_{11}| + \alpha J_h^\sigma(\xi,\xi). \tag{27}$$

Analogously as in [11] we derive

$$\vartheta_7 \geq \alpha |\xi|^2_{H^1(\Omega,\mathcal{T}_h)}, \tag{28}$$

$$|\vartheta_8| \leq C\Big(C_U h^{\mu-1} |u|_{H^\mu(\Omega)} + C_D C_L \|\xi\|_{L^2(\Omega)} \Big) |\xi|_{H^1(\Omega,\mathcal{T}_h)}, \tag{29}$$

$$|\vartheta_{10}| \leq C\Big(C_U h^{\mu-1} |u|_{H^\mu(\Omega)} + C_D C_L |\xi|_{L^2(\Omega)} \Big) J_h^\sigma(\xi,\xi)^{1/2}, \tag{30}$$

$$|\vartheta_{11}| \leq C \cdot C_U h^{\mu-1} |u|_{H^\mu(\Omega)} |\xi|_{H^1(\Omega,\mathcal{T}_h)}, \tag{31}$$

where α is constant from (13c).

The special attention is paid to the estimation of term ϑ_9. For NIPG variant ($\Theta = 1$) this term disappears in (27). On the other hand, for IIPG and SIPG variants by a particular choice of the constant C_W we obtain the following inequalities

$$(32)\quad -\vartheta_9 \geq -\frac{\alpha}{2}\Big(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} - J_h^\sigma(\xi,\xi)\Big),\ \text{for } C_W \geq \frac{C_U^2 C_M(1+C_I)}{\alpha^2}\ \text{(IIPG)}$$
$$-2\vartheta_9 \geq -\frac{\alpha}{2}\Big(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} - J_h^\sigma(\xi,\xi)\Big),\ \text{for } C_W \geq \frac{4C_U^2 C_M(1+C_I)}{\alpha^2}\ \text{(SIPG)},$$

where C_M and C_I are constants from [7, Lemmas 1–2], respectively.

Finally we can write

$$(33)\qquad \chi_1 \geq \frac{\alpha}{2}|||\xi|||^2 - C\Big(C_U h^{\mu-1}|u|_{H^\mu(\Omega)} + C_D C_L\|\xi\|_{L^2(\Omega)})\Big)|||\xi|||.$$

From (16), (24), (25), (33) and the identity $\left(\frac{\partial\xi}{\partial t},\xi\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)}$ we get

$$(34)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \frac{\alpha}{2}|||\xi|||^2 - C\Big(C_U h^{\mu-1}|u|_{H^\mu(\Omega)} + C_D C_L\|\xi\|_{L^2(\Omega)})\Big)|||\xi|||$$
$$\leq C\Big\{h^\mu|\partial u/\partial t|_{H^\mu(\Omega)}\|\xi\|_{L^2(\Omega)} + |||\xi|||(h^\mu|u|_{H^\mu(\Omega)} + \alpha h^{\mu-1}|u|_{H^\mu(\Omega)} + \|\xi\|_{L^2(\Omega)})\Big\}$$
$$+ C\Big(C_U h^{\mu-1}|u|_{H^\mu(\Omega)} + C_D C_L h^\mu|u|_{H^\mu(\Omega)}\Big)|||\xi|||.$$

We rewrite the above inequality (34) in the following form

$$(35)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \frac{\alpha}{2}|||\xi|||^2 \leq C(1+C_D C_L)|||\xi|||\ \|\xi\|_{L^2(\Omega)}$$
$$+ C\|\xi\|_{L^2(\Omega)}h^\mu|\partial u/\partial t|_{H^\mu(\Omega)} + C\Big(C_U + h(1+C_D C_L) + \alpha\Big)|||\xi|||h^{\mu-1}|u|_{H^\mu(\Omega)}.$$

Applying Young's inequality to the right-hand-side of (35) gives us

$$(36)\ \frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \alpha|||\xi|||^2 \leq \frac{\alpha}{4}|||\xi|||^2 + \frac{4C^2(1+C_D C_L)^2}{\alpha}\|\xi\|^2_{L^2(\Omega)} + \frac{\alpha}{4}\|\xi\|^2_{L^2(\Omega)}$$
$$+ \frac{4C^2}{\alpha}h^{2\mu}|\partial u/\partial t|^2_{H^\mu(\Omega)} + \frac{\alpha}{4}|||\xi|||^2 + \frac{4C^2(C_U + h(1+C_D C_L)+\alpha)^2}{\alpha}h^{2\mu-2}|u|^2_{H^\mu(\Omega)}.$$

Subsequently, we set $\widetilde{C} = \max(1, (C_U + h(1+C_D C_L)+\alpha)^2)$ and put generic constant $C = 4C^2$, hence

$$(37)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}|||\xi(t)|||^2 \leq \left(\frac{\alpha}{4} + \frac{C(1+C_D C_L)^2}{\alpha}\right)\|\xi(t)\|^2_{L^2(\Omega)}$$
$$+ C\frac{\widetilde{C}}{\alpha}h^{2\mu-2}\Big(|\partial u(t)/\partial t|^2_{H^\mu(\Omega)} + |u(t)|^2_{H^\mu(\Omega)}\Big).$$

The integration of (37) from 0 to $t \in [0,T]$ and the relation $\xi(0) = u_h^0 - \Pi_h u^0 = 0$ yield

$$(38)\quad \|\xi(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^t |||\xi(\vartheta)|||^2\mathrm{d}\vartheta \leq \left(\frac{\alpha}{4} + \frac{C(1+C_D C_L)^2}{\alpha}\right)\int_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\mathrm{d}\vartheta$$
$$+ C\frac{\widetilde{C}}{\alpha}h^{2\mu-2}\int_0^t \Big(|\partial u(\vartheta)/\partial t|^2_{H^\mu(\Omega)} + |u(\vartheta)|^2_{H^\mu(\Omega)}\Big)\mathrm{d}\vartheta.$$

Now the application of Gronwall's lemma (see, e.g. [7, Lemma 10]), implies that

$$\begin{aligned}(39)\qquad & \|\xi(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^t |||\xi(\vartheta)|||^2\,\mathrm{d}\vartheta \\ & \le Ch^{2\mu-2}\frac{\widetilde{C}}{\alpha}\Big(\|u\|^2_{L^2(0,T;H^\mu(\Omega))} + \|\partial u/\partial t\|^2_{L^2(0,T;H^\mu(\Omega))}\Big) \\ & \times \exp\left(\left(\frac{4+C(1+C_DC_L)^2}{4\alpha}\right)t\right), \quad t\in[0,T].\end{aligned}$$

Finally, relation $e_h = \xi + \eta$, the triangle inequality, estimate (39) and estimates from [7, Lemma 6] yield the sought result (15). □

We observe that estimate (15) is suboptimal in the $L^\infty(0,T,L^2(\Omega))$-norm, namely $O(h^{\mu-1})$, but optimal in the $L^2(0,T,H^1(\Omega))$-seminorm, namely $O(h^{\mu-1})$. Moreover, for not sufficiently regular solution $s \le p+1$ the order of convergence is given by the regularity of u (the value s).

5. Numerical example. In this section we verify the a priori error estimates (15). We consider the 2D viscous Burgers equation

$$(40)\qquad \frac{\partial u}{\partial t} + \sum_{s=1}^{2} u\frac{\partial u}{\partial x_s} = \mathrm{div}(\mathbb{K}(u)\,\nabla u) + g \quad \text{in}\ \ \Omega\times(0,T),$$

where matrix $K(w)$ is chosen in the form $K(w) = \varepsilon(2+\arctan(w))\mathbb{I}$. Obviously, $f_s(u) = u^2/2$ for $s = 1,2$. It is possible to show that the problem (40) satisfies assumptions (A1) with $C_U = \varepsilon(2+\frac{\pi}{2})$, $C_L = 1.0$ and $\alpha = \varepsilon(2-\frac{\pi}{2})$. We set $\varepsilon = 0.002$, $\Omega = (0,1)^2$, $T = 10$ and define the function g and the initial and boundary conditions in such a way that the exact solution has the form

$$(41)\qquad u(x_1,x_2,t) = \left(1-\mathrm{e}^{-10t}\right)\hat{u}(x_1,x_2),$$

$$(42)\qquad \hat{u}(x_1,x_2) = 2r^\alpha x_1x_2(1-x_1)(1-x_2) = r^{\alpha+2}\sin(2\varphi)(1-x_1)(1-x_2),$$

where (r,φ) $(r = (x_1^2+x_2^2)^{1/2})$ are the polar coordinates and $\alpha\in\mathbb{R}$ is a constant. For $t = T = 10$ the solution u differs very little from the "steady state" solution $\hat{u}$. The function $\hat{u}$ is equal to zero on $\partial\Omega$ and its regularity depends on the value of α, namely (cf. [3]) $\hat{u}\in H^\beta(\Omega)\quad \forall\beta\in(0,\alpha+3)$, where $H^\beta(\Omega)$ denotes (in general) the Sobolev-Slobodetskii space of functions with "noninteger derivatives".

In the presented numerical tests we use the values $\alpha = 2$ and $\alpha = -3/2$. The value $\alpha = 2$ gives function $\hat{u}$ sufficiently regular ($\in H^\beta(\Omega)$ for $\beta < 5$), whereas the value $\alpha = -3/2$ gives $\hat{u}\in H^\beta(\Omega)$, $\beta < 3/2$. Numerical experiments are carried out with the use of P^1, P^2, P^3 and P^4 polynomial approximations on 6 triangular meshes having 128, 288, 512, 1152, 2048 and 4608 elements for SIPG, NIPG and IIPG methods.

Figures 1 - 4 show computational errors in the $L^2(\Omega)$-norm and in the H^1-seminorm at time $t = T = 10$ and indicate the corresponding experimental orders of convergence (EOC) for $\alpha = 2$ and $\alpha = -3/2$ using SIPG, NIPG and IIPG methods. We observe that

- **regular exact solution (case** $\alpha = 2$**)** Whereas SIPG method gives optimal order of convergence $O(h^{p+1})$ for $p = 1,2,3,4$, the NIPG and IIPG methods give optimal order of convergence $O(h^{p+1})$ for $p = 1$ and $p = 3$ (even degrees)

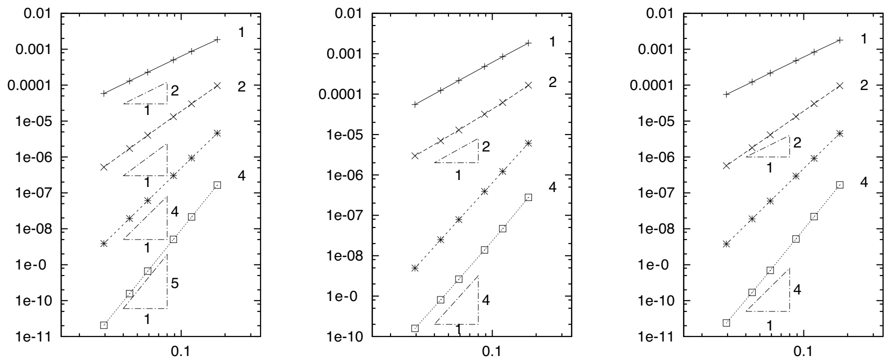

FIG. 1. *Computational error and EOC in the L^2-norm for the SIPG (left), NIPG (middle), IIPG (right) method, the regular solution ($\alpha = 2$)*

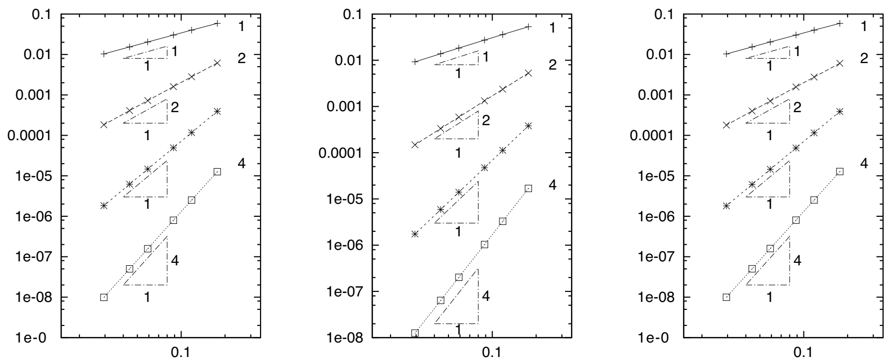

FIG. 2. *Computational error and EOC in the H^1-semi-norm for the SIPG (left), NIPG (middle), IIPG (right) method, the regular solution ($\alpha = 2$)*

and suboptimal $O(h^p)$ for $p = 2$ and $p = 4$ (odd degrees). Moreover, all IPG techniques produce optimal order of convergence in the H^1-seminorm ($O(h^p)$).

- **singular exact solution (case** $\alpha = -3/2$**)** The experimental order of convergence in the L^2-norm is equal to 3/2 and it is equal to 1/2 in the H^1-seminorm. This is in agreement with theoretical results.

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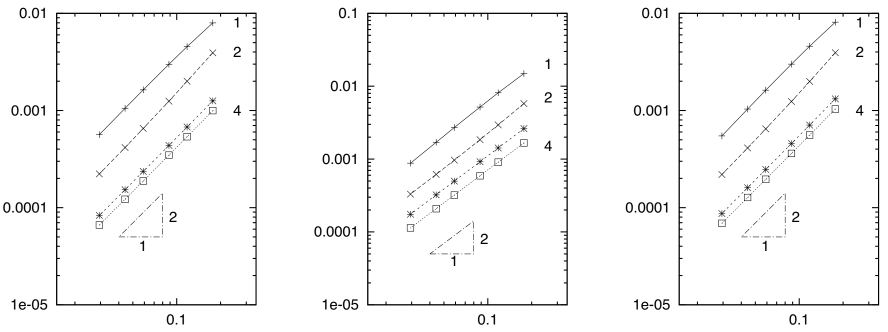

Fig. 3. *Computational error and EOC in the L^2-norm for the SIPG (left), NIPG (middle), IIPG (right) method, the solution having a singularity ($\alpha = -3/2$)*

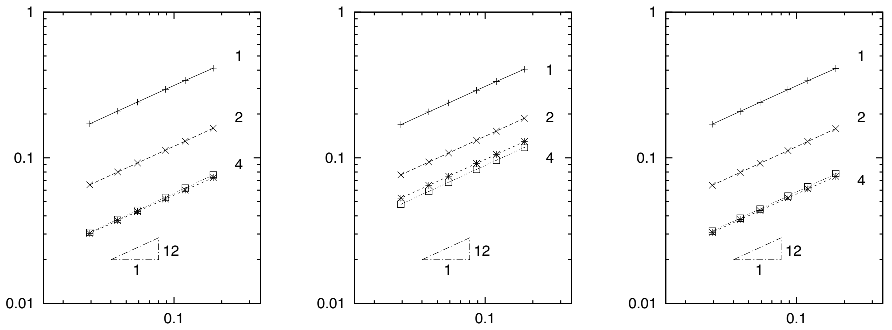

Fig. 4. *Computational error and EOC in the H^1-semi-norm for the SIPG (left), NIPG (middle), IIPG (right) method, the solution having a singularity ($\alpha = -3/2$)*

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