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STABILIZED DDFV SCHEMES FOR STOKES PROBLEM

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Abstract. “Discrete Duality Finite Volume” schemes (DDFV for short) on general meshes are studied here for the linear Stokes problem with Dirichlet boundary conditions. The aim of this work is to analyze the wellposedness of different DDFV schemes and their convergence properties. We first consider the natural extension of the DDVF scheme classically used for the Laplace problem. Unfortunately, its wellposedness is still an open problem on general meshes. To overcome this difficulty, we propose to stabilize the mass conservation equation. We present that stabilized schemes are wellposed for general meshes and we derive some error estimates. Finally, the different schemes (stabilized and unstabilized) are compared on numerical test cases.

Key words. Finite-volume methods, Stokes problem, DDFV methods.

AMS subject classifications. 35Q30, 65N15, 76M12

1. Introduction. The DDFV schemes have been first introduced and studied in [H 00, DO 05] to approximate the Laplace equation on a large class of 2D meshes including non-conformal and distorted meshes. Such schemes require unknowns on both vertices and “centers” of primal control volumes. This allows to reconstitute two-dimensional discrete gradient and divergence operators which are in duality in a discrete sense. The discrete gradient denoted by $\nabla^{\mathfrak{D}}$ is defined to be constant on each diamond cell which is a quadrangle whose vertices are two vertices of an edge and the two control volume centers corresponding to this edge. This framework is recalled in section 2.

The DDFV schemes have then been applied for several linear and non-linear problems. In the case of the linear anisotropic Laplace equation, the DDFV schemes have been studied with mixed boundary conditions [BHK] on general grids. The div curl problems have been discretized with DDFV schemes in [DDO 07]. Also in the case of non-linear diffusion equations for Leray-Lions operators, they have been successfully extended in [ABH 07, BH 08]. In the benchmark [HH 08] of the FVCA5 conference, we see that the DDFV method is a good way to approach the gradient of the solution compared to other finite volume schemes.

For the sake of simplicity, we restrict the presentation to the Stokes equation with Dirichlet boundary conditions and a regular source term. The aim is to find $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $p : \Omega \to \mathbb{R}$ such that:

$$\begin{aligned} -\Delta \mathbf{u} + \boldsymbol{\nabla} p &= \mathbf{f}, \quad \text{in } \Omega, \\ \operatorname{div}(\mathbf{u}) &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}, \quad \text{on } \partial\Omega, \qquad \int_\Omega p(x)\mathrm{d}x = 0. \end{aligned} \tag{1.1}$$

where Ω is a polygonal open bounded connected subset of $\mathbb{R}^2$, $\mathbf{f}$ is a function in $(L^2(\Omega))^2$ and $\mathbf{g}$ is a function in $(L^2(\partial\Omega))^2$ which verifies the compatibility condition:

$$\int_{\partial\Omega} \mathbf{g}(s) \cdot \vec{\mathbf{n}} \mathrm{d}s = 0. \tag{1.2}$$

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For the Stokes problem, it seems to be natural to approximate the velocity on both vertices and centers of primal control volumes and the pressure on the diamond cells. This scheme is known to be wellposed [D 07] only for particular classes of meshes. Indeed, the wellposedness of the scheme relies on an uniform discrete inf-sup condition, which is still an open problem for general meshes. Under usual regularity assumptions about the exact solution, we state here that the convergence rate of the velocity gradient $||\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}||_2$, in L^2 norm, is equal to $\frac{1}{2}$, when this scheme is wellposed.

Two strategies can be considered to overcome the difficulty of the wellposedness: approximate the velocity on the diamond cells and the pressure on both vertices and centers of primal control volumes (see [DDO 07]) or stabilize the mass conservation equation as for finite elements schemes. The stabilized schemes are proven to be wellposed for general meshes. Error estimates are studied for several kind of stabilizations. In particular, we propose a first order convergent scheme in the L^2 norm for the velocity $||\mathbf{u} - \mathbf{u}^{\mathcal{T}}||_2$, for its gradient $||\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}||_2$ and for the pressure $||p - p^{\mathfrak{D}}||_2$ provided that the exact solution satisfies usual regularity assumptions. Finally, in section 4, theoretical error estimates are illustrated with numerical results. The complete proofs of all the results in this paper and further numerical experiments are presented in [K].

2. The DDFV framework. The meshes: we recall here the main notations and definitions taken from [ABH 07]. A DDFV mesh $\mathcal{T}$ is constituted by a primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ and a dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ (Figure 2.1).

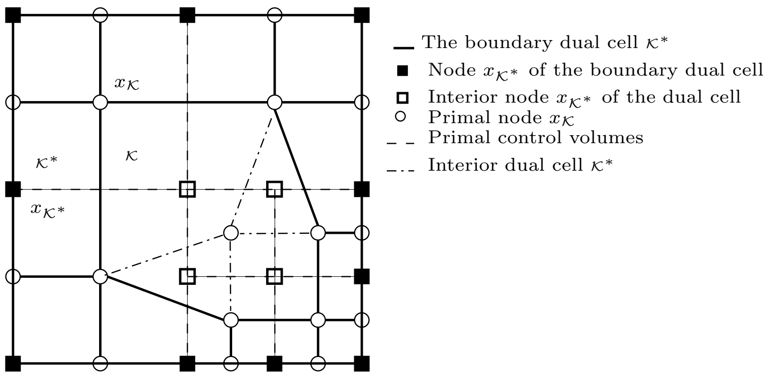


FIG. 2.1. *The mesh $\mathcal{T}$*

The primal mesh $\mathfrak{M}$ is a set of disjoint polygonal control volumes $\mathcal{K} \subset \Omega$ such that $\cup\overline{\mathcal{K}} = \overline{\Omega}$. We denote by $\partial\mathfrak{M}$ the set of edges of the control volumes in $\mathfrak{M}$ included in $\partial\Omega$. Edges in $\partial\mathfrak{M}$ are considered as degenerate control volumes. Then, to each control volume and degenerate control volume $\mathcal{K} \in \mathfrak{M} \cup \partial\mathfrak{M}$, we associate a point $x_{\mathcal{K}} \in \mathcal{K}$. This family of points is denoted by $X = \{x_{\mathcal{K}},\ \mathcal{K} \in \mathfrak{M} \cup \partial\mathfrak{M}\}$.

Let X^* be the set of the vertices of the primal control volumes in $\mathfrak{M}$. It can be split into $X^* = X^*_{int} \cup X^*_{ext}$ where $X^*_{int} \cap \partial\Omega = \emptyset$ and $X^*_{ext} \subset \partial\Omega$. To any point $x_{\mathcal{K}^*} \in X^*_{int}$ (resp. $x_{\mathcal{K}^*} \in X^*_{ext}$), we associate the polygon $\mathcal{K}^*$ whose vertices are $\{x_{\mathcal{K}} \in X,$ such that $x_{\mathcal{K}^*} \in \overline{\mathcal{K}},\ \mathcal{K} \in \mathfrak{M}\}$ (resp. $\{x_{\mathcal{K}^*}\} \cup \{x_{\mathcal{K}} \in X,$ such that $x_{\mathcal{K}^*} \in \overline{\mathcal{K}},\ \mathcal{K} \in (\mathfrak{M} \cup \partial\mathfrak{M})\}$) sorted with respect to the clockwise order of the corresponding control volumes. This defines the set $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ of dual control volumes.

We assume that if $(\mathcal{K}^*, \mathcal{L}^*) \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ such that $\mathcal{K}^* \neq \mathcal{L}^*$, we have $\overset{o}{\mathcal{K}^*} \cap \overset{o}{\mathcal{L}^*} = \emptyset$.

For all neighbour control volumes $\mathcal{K}$ and $\mathcal{L}$, we assume that $\partial\mathcal{K} \cap \partial\mathcal{L}$ is an edge of the primal mesh denoted by $\sigma = \mathcal{K}|\mathcal{L}$. We note $\mathcal{E}$ the set of such edges. We also note $\sigma^* = \mathcal{K}^*|\mathcal{L}^*$ and $\mathcal{E}^*$ for the corresponding dual definitions. Given the primal and dual control volumes, we define the diamond cells $\mathcal{D}_{\sigma,\sigma^*}$ as the quadrangles whose diagonals are a primal edge $\sigma = \mathcal{K}|\mathcal{L} = (x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$ and the corresponding dual edge $\sigma^* = \mathcal{K}^*|\mathcal{L}^* = (x_{\mathcal{K}}, x_{\mathcal{L}})$, (see Figure 2.2). Note that the diamond cells are not necessarily convex. If $\sigma \in \mathcal{E} \cap \partial\overline{\Omega}$, the quadrangle $\mathcal{D}_{\sigma,\sigma^*}$ degenerate into a triangle. The set of the diamond cells is denoted by $\mathfrak{D}$ and we have $\overline{\Omega} = \underset{\mathcal{D}\in\mathfrak{D}}{\cup} \overline{\mathcal{D}}$.

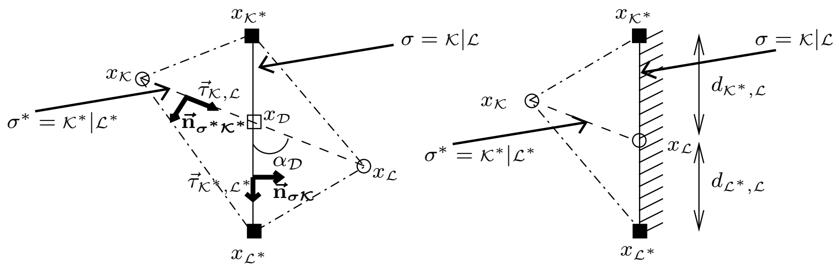


FIG. 2.2. *Notations in the diamond cells. (Left) Interior cell. (Right) Boundary cell.*

For any primal control volume $\mathcal{K} \in \mathfrak{M} \cap \partial\mathfrak{M}$, let $m_{\mathcal{K}}$ be its Lebesgue measure, $\mathcal{E}_{\mathcal{K}}$ the set of its edges if $\mathcal{K} \in \mathfrak{M}$, or the one-element set $\{\mathcal{K}\}$ if $\mathcal{K} \in \partial\mathfrak{M}$. We note by $\mathfrak{D}_{\mathcal{K}} = \{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D},\ \sigma \in \mathcal{E}_{\mathcal{K}}\}$, and $\vec{\mathbf{n}}_{\mathcal{K}}$ the outward unit normal vector to $\mathcal{K}$. We introduce the open ball $B_{\mathcal{K}} := B(x_{\mathcal{K}}, \rho_{\mathcal{K}}) \cap \Omega$ for $\mathcal{K} \in \mathfrak{M}$ and $B_{\mathcal{K}} := B(x_{\mathcal{K}}, \rho_{\mathcal{K}}) \cap \partial\Omega \subset \overline{\mathcal{K}}$ for $\mathcal{K} \in \partial\mathfrak{M}$. The value $\rho_{\mathcal{K}}$ is chosen such that the inclusion is verified. We will also use the corresponding dual notations: $m_{\mathcal{K}^*}$, $\mathcal{E}_{\mathcal{K}^*}$, $\mathfrak{D}_{\mathcal{K}^*}$, $\vec{\mathbf{n}}_{\mathcal{K}^*}$, $B_{\mathcal{K}^*}$ and $\rho_{\mathcal{K}^*}$.

For a diamond cell $\mathcal{D} = \mathcal{D}_{\sigma,\sigma^*}$ whose vertices are $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$, we note by $m_{\mathcal{D}}$ its measure, $x_{\mathcal{D}}$ the center of the diamond cell $\mathcal{D}$, that is the intersection of the primal edge σ and the dual edge σ^*, m_σ the length of the primal edge σ, m_{σ^*} the length of the dual edge σ^*, $d_{\mathcal{D}}$ its diameter, $\mathfrak{s}$ its edges (for example $\mathfrak{s} = [x_{\mathcal{K}}, x_{\mathcal{K}^*}]$), $m_{\mathfrak{s}}$ the length of a diamond edge $\mathfrak{s}$, $m_{\mathfrak{s}^*}$ the length between $x_{\mathcal{D}}$ and $x_{\mathcal{D}'}$ if $\mathfrak{s} = \mathcal{D}|\mathcal{D}'$, $\vec{\mathbf{n}}_{\sigma\mathcal{K}}$ the unit vector normal to σ oriented from $x_{\mathcal{K}}$ to $x_{\mathcal{L}}$, $\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$ the unit vector normal to σ^* oriented from $x_{\mathcal{K}^*}$ to $x_{\mathcal{L}^*}$, $\vec{\boldsymbol{\tau}}_{\mathcal{K},\mathcal{L}}$ the unit vector parallel to σ^* (oriented from $x_{\mathcal{K}}$ to $x_{\mathcal{L}}$), $\vec{\boldsymbol{\tau}}_{\mathcal{K}^*,\mathcal{L}^*}$ the unit vector parallel to σ (oriented from $\mathcal{K}^*$ to $\mathcal{L}^*$), $\alpha_{\mathcal{D}}$ the angle between $\vec{\boldsymbol{\tau}}_{\mathcal{K},\mathcal{L}}$ and $\vec{\boldsymbol{\tau}}_{\mathcal{K}^*,\mathcal{L}^*}$, $d_{\mathcal{K}^*,\mathcal{L}}$ the length between $x_{\mathcal{K}^*}$ and $x_{\mathcal{L}}$ for any boundary degenerate diamond cell and $\rho_{\mathcal{D}}$ is the diameter of the greatest open ball $B_{\mathcal{D}}$ which contains $\mathcal{D}$ and its neighbours. Finally, we define the outward unit normal field vector $\vec{\mathbf{n}}^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ such that $\vec{\mathbf{n}}_{\mathcal{D}} = \vec{\mathbf{n}}_{\sigma\mathcal{K}}$ for $\mathcal{D} = \mathcal{D}_{\sigma,\sigma^*}$.

The unknowns: the DDFV method associates to all primal control volumes $\mathcal{K} \in \mathfrak{M} \cup \partial\mathfrak{M}$ an unknown value $\mathbf{u}_{\mathcal{K}} \in \mathbb{R}^2$ for the velocity, to all dual control volumes $\mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ an unknown value $\mathbf{u}_{\mathcal{K}^*} \in \mathbb{R}^2$ for the velocity and to all diamond cells $\mathcal{D} \in \mathfrak{D}$ an unknown value $p^{\mathcal{D}} \in \mathbb{R}$ for the pressure. We denote the approximate solution on the mesh $(\mathcal{T}, \mathfrak{D})$ by $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$:

$$\mathbf{u}^{\mathcal{T}} = \Big((\mathbf{u}_{\mathcal{K}})_{\mathcal{K}\in(\mathfrak{M}\cup\partial\mathfrak{M})}, (\mathbf{u}_{\mathcal{K}^*})_{\mathcal{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)} \Big), \qquad p^{\mathfrak{D}} = \big((p^{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}} \big).$$

Whenever it is convenient, we associate to the discrete function $\mathbf{u}^{\mathcal{T}}$ the piecewise constant function

$$\mathbf{u}^{\mathcal{T}} \sim \frac{1}{2}(\mathbf{u}^{\mathfrak{M}} + \mathbf{u}^{\mathfrak{M}^*}),$$

where $\mathbf{u}^{\mathfrak{M}} = \sum_{\mathcal{K}\in\mathfrak{M}} \mathbf{u}_{\boldsymbol{\mathcal{K}}} \mathbf{1}_{\mathcal{K}}$ and $\mathbf{u}^{\mathfrak{M}^*} = \sum_{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \mathbf{u}_{\boldsymbol{\mathcal{K}}^*} \mathbf{1}_{\mathcal{K}^*}$. As a consequence, one can define the L^2 norm of $\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*}, \mathbf{u}^{\boldsymbol{\mathcal{T}}}$.

Discrete gradient: we define a consistent approximation of the gradient operator denoted by $\nabla^{\mathfrak{D}} : \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto \left(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, as follows:

$$\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} (\nabla^{\mathcal{D}} u_1^{\mathcal{T}})^t \\ (\nabla^{\mathcal{D}} u_2^{\mathcal{T}})^t \end{pmatrix}, \quad \forall \mathcal{D} \in \mathfrak{D}.$$

and for $i = 1, 2$

$$\nabla^{\mathcal{D}} u_i^{\mathcal{T}} = \frac{1}{2m_{\mathcal{D}}} \left[(u_{i,\mathcal{L}} - u_{i,\mathcal{K}}) m_\sigma \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} + (u_{i,\mathcal{L}^*} - u_{i,\mathcal{K}^*}) m_{\sigma^*} \vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}} \right].$$

By using the value of the diamond cell measure $2m_{\mathcal{D}} = \sin\alpha_{\mathcal{D}} m_\sigma m_{\sigma^*}$, it can be written as follows:

$$\nabla^{\mathcal{D}} u_i^{\mathcal{T}} = \frac{1}{\sin\alpha_{\mathcal{D}}} \left[\frac{u_{i,\mathcal{L}} - u_{i,\mathcal{K}}}{m_{\sigma^*}} \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} + \frac{u_{i,\mathcal{L}^*} - u_{i,\mathcal{K}^*}}{m_\sigma} \vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}} \right].$$

Discrete divergence: we define a consistent approximation of the divergence operator applied to discrete tensor fields denoted by $\mathbf{div}^{\boldsymbol{\mathcal{T}}} : \xi = (\xi^{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \mathbf{div}^{\boldsymbol{\mathcal{T}}} \xi \in \left(\mathbb{R}^2\right)^{\mathcal{T}}$, as follows:

$$\mathbf{div}^{\boldsymbol{\mathcal{K}}} \xi = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma\in\partial\mathcal{K}} m_\sigma \xi^{\mathcal{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}, \ \forall \mathcal{K} \in \mathfrak{M}, \quad \text{and} \quad \mathbf{div}^{\boldsymbol{\mathcal{K}}} \xi = 0, \ \forall \mathcal{K} \in \partial\mathfrak{M},$$

$$\mathbf{div}^{\boldsymbol{\mathcal{K}}^*} \xi = \frac{1}{m_{\mathcal{K}^*}} \sum_{\sigma^*\in\partial\mathcal{K}^*} m_{\sigma^*} \xi^{\mathcal{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}}, \ \forall \mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*.$$

These two operators are in *discrete duality* (giving its name to the scheme) since they can be linked by a discrete Stokes formula (see [DO 05, DDO 07, ABH 07]). In order to write a such formula, we define trace operators and inner products.

Trace operators: we define two trace operators. The first one is denoted by $\boldsymbol{\gamma}^{\boldsymbol{\mathcal{T}}} : \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto \boldsymbol{\gamma}^{\boldsymbol{\mathcal{T}}}(\mathbf{u}^{\boldsymbol{\mathcal{T}}}) \in (\mathbb{R}^2)^{\partial\mathfrak{M}}$, as follows:

$$\boldsymbol{\gamma}_{\boldsymbol{\sigma}}(\mathbf{u}^{\boldsymbol{\mathcal{T}}}) = \frac{d_{\mathcal{K}^*,\mathcal{L}}(\mathbf{u}_{\boldsymbol{\mathcal{K}}^*} + \mathbf{u}_{\boldsymbol{\mathcal{L}}}) + d_{\mathcal{L}^*,\mathcal{L}}(\mathbf{u}_{\boldsymbol{\mathcal{L}}^*} + \mathbf{u}_{\boldsymbol{\mathcal{L}}})}{2m_\sigma}, \qquad \forall\, \sigma \in \partial\mathfrak{M}.$$

This trace operator will impose the Dirichlet boundary conditions in a weak way. The second one is denoted by $\boldsymbol{\gamma}^{\mathfrak{D}} : \boldsymbol{\phi}^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto (\boldsymbol{\phi}^{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}_{ext}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}$.

Inner products: we define the three following inner products

$$[\![\mathbf{v}^{\boldsymbol{\mathcal{T}}}, \mathbf{u}^{\boldsymbol{\mathcal{T}}}]\!]_{\mathcal{T}} = \frac{1}{2} \left(\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} \mathbf{u}_{\boldsymbol{\mathcal{K}}} \cdot \mathbf{v}_{\boldsymbol{\mathcal{K}}} + \sum_{\mathcal{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)} m_{\mathcal{K}^*} \mathbf{u}_{\boldsymbol{\mathcal{K}}^*} \cdot \mathbf{v}_{\boldsymbol{\mathcal{K}}^*} \right), \quad \forall \mathbf{u}^{\boldsymbol{\mathcal{T}}}, \mathbf{v}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}},$$

$$(\boldsymbol{\gamma}^{\mathfrak{D}}(\boldsymbol{\phi}^{\mathfrak{D}}), \boldsymbol{\gamma}^{\boldsymbol{\mathcal{T}}}(\mathbf{u}^{\boldsymbol{\mathcal{T}}}))_{\partial\Omega} = \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \boldsymbol{\phi}^{\mathcal{D}} \cdot \boldsymbol{\gamma}_{\boldsymbol{\sigma}}(\mathbf{u}^{\boldsymbol{\mathcal{T}}}), \quad \forall\, \boldsymbol{\phi}^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}, \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}},$$

$$(\xi^{\mathfrak{D}}, \eta^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} (\xi^{\mathcal{D}} : \eta^{\mathcal{D}}), \quad \forall \xi^{\mathfrak{D}}, \eta^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},$$

where $(\xi^{\mathcal{D}} : \eta^{\mathcal{D}}) = \sum_{1\le i,j\le 2} \xi^{\mathcal{D}}_{i,j}\eta^{\mathcal{D}}_{i,j}$.

THEOREM 2.1 (Discrete Stokes formula). *For all* $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}}$:

$$[\![\boldsymbol{div}^{\boldsymbol{\mathcal{T}}}\xi^{\mathfrak{D}}, \mathbf{u}^{\boldsymbol{\mathcal{T}}}]\!]_{\mathcal{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})_{\mathfrak{D}} + (\boldsymbol{\gamma}^{\mathfrak{D}}(\xi^{\mathfrak{D}}\vec{\boldsymbol{n}}^{\mathfrak{D}}), \boldsymbol{\gamma}^{\boldsymbol{\mathcal{T}}}(\mathbf{u}^{\boldsymbol{\mathcal{T}}}))_{\partial\Omega} \in \mathbb{R}.$$

We also need a discrete divergence of a vector field of $\left(\mathbb{R}^2\right)^{\mathcal{T}}$, which is defined by using the discrete gradient: $\mathrm{div}^{\mathfrak{D}} : \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto (\mathrm{div}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})_{\mathcal{D}\in\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, such that $\mathrm{div}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \mathrm{Trace}(\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})$.

One of the stabilization term will be a non-consistent discrete form of Δp, denoted by $\Delta^{\mathfrak{D}} : p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \mapsto \Delta^{\mathfrak{D}}p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, and defined as follows:

$$\Delta^{\mathcal{D}}p^{\mathfrak{D}} = \frac{1}{m_{\mathcal{D}}} \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\partial\mathcal{D}} \frac{d_{\mathcal{D}}^2 + d_{\mathcal{D}'}^2}{d_{\mathcal{D}}^2}(p^{\mathcal{D}'} - p^{\mathcal{D}}), \qquad \forall\, \mathcal{D} \in \mathfrak{D}.$$

Note that we do not need a consistent discrete form of a Laplacian.

For simplicity, we consider a family of meshes with convex diamond cells. We note $\mathrm{size}(\mathcal{T})$ the maximum of the diameters of the diamond cells in $\mathfrak{D}$. To measure how flat the diamond cells are, we note $\alpha_{\mathcal{T}}$ the unique real in $]0, \frac{\pi}{2}]$ such that $\sin\alpha_{\mathcal{T}} := \min_{\mathcal{D}\in\mathfrak{D}} |\sin\alpha_{\mathcal{D}}|$. We introduce a positive number $\mathrm{reg}(\mathcal{T})$ that quantifies the regularity of a given mesh and is useful to perform the convergence analysis of finite volume schemes like in [ABH 07] and [BH 08].

$$\mathrm{reg}(\mathcal{T}) := \max\Bigg(\frac{1}{\alpha_{\mathcal{T}}}, \max_{\mathcal{D}\in\mathfrak{D}} \max_{\mathfrak{s}\in\mathcal{E}_{\mathcal{D}}} \frac{d_{\mathcal{D}}}{\min(m_{\mathfrak{s}}, m_{\mathfrak{s}^*})}, \max_{\substack{\mathcal{K}\in\mathfrak{M}\\ \mathcal{D}\in\mathfrak{D}_{\mathcal{K}}}} \frac{d_{\mathcal{K}}}{d_{\mathcal{D}}}, \max_{\substack{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*\\ \mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}}} \frac{d_{\mathcal{K}^*}}{d_{\mathcal{D}}}$$
$$\max_{\mathcal{K}\in\mathfrak{M}} \frac{d_{\mathcal{K}}}{\rho_{\mathcal{K}}} + \frac{\rho_{\mathcal{K}}}{d_{\mathcal{K}}}, \max_{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \frac{d_{\mathcal{K}^*}}{\rho_{\mathcal{K}^*}} + \frac{\rho_{\mathcal{K}^*}}{d_{\mathcal{K}^*}}, \max_{\mathcal{D}\in\mathfrak{D}} \frac{d_{\mathcal{D}}}{\rho_{\mathcal{D}}} + \frac{\rho_{\mathcal{D}}}{d_{\mathcal{D}}}\Bigg).$$

For instance, this constant $\mathrm{reg}(\mathcal{T})$ is involved in the following geometrical result: there exists two constants C_1 and C_2 depending on $\mathrm{reg}(\mathcal{T})$ such that for any $\mathcal{K} \in \mathfrak{M}$, $\mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{D} \cap \mathcal{K} \neq \emptyset$ and $\mathcal{D} \cap \mathcal{K}^* \neq \emptyset$, we have

$$C_1 m_{\mathcal{K}} \le m_{\mathcal{D}} \le C_2 m_{\mathcal{K}}, \qquad C_1 m_{\mathcal{K}^*} \le m_{\mathcal{D}} \le C_2 m_{\mathcal{K}^*}.$$

3. DDFV schemes for the Stokes equations. We denote by $\mathbf{f}_{\boldsymbol{\mathcal{K}}}$ (resp. $\mathbf{f}_{\boldsymbol{\mathcal{K}}^*}$) the mean-value of the source term $\mathbf{f}$ on $\mathcal{K} \in \mathfrak{M}$ (resp. on $\mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$) and by $\mathbf{g}_{\boldsymbol{\sigma}}$ the mean-value of the Dirichlet boundary conditions $\mathbf{g}$ on $\sigma \in \partial\mathfrak{M}$. With these choices and the compatibility condition (1.2), we have the two following equalities:

$$\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}\mathbf{f}_{\boldsymbol{\mathcal{K}}} = \sum_{\mathcal{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)} m_{\mathcal{K}^*}\mathbf{f}_{\boldsymbol{\mathcal{K}}^*}, \qquad \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_{\sigma}\mathbf{g}_{\boldsymbol{\sigma}} \cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} = 0.$$

The scheme for the problem (1.1) is written as follows :

$$
\begin{array}{ll}
 & \text{Find } \mathbf{u}^{\mathcal{T}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \text{ and } p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that,} \\
\forall\, \mathcal{K} \in \mathfrak{M}, & \mathbf{div}^{\mathcal{K}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}_{\mathcal{K}}, \\
\forall\, \mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*, & \mathbf{div}^{\mathcal{K}^*}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}_{\mathcal{K}^*}, \\
\forall\, \mathcal{D} \in \mathfrak{D}, & \mathrm{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) + \lambda\ \mathrm{size}(\mathcal{T})p^{\mathcal{D}} - \mu\ \mathrm{size}(\mathcal{T})^2\Delta^{\mathcal{D}}p^{\mathcal{D}} = 0, \\
\forall\, \sigma \in \partial\mathfrak{M}, & \gamma_{\boldsymbol{\sigma}}(\mathbf{u}^{\mathcal{T}}) = \mathbf{g}_{\boldsymbol{\sigma}}, \\
 & \displaystyle\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}\mathbf{u}_{\mathcal{K}} - \sum_{\mathcal{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)} m_{\mathcal{K}^*}\mathbf{u}_{\mathcal{K}^*} = 0, \\
 & \displaystyle\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}p^{\mathfrak{D}} = 0.
\end{array}
\tag{3.1}
$$

with an appropriate choice of $\lambda \geq 0$ and $\mu \geq 0$.

We will study three schemes depending on the values of $\lambda \geq 0$ and $\mu \geq 0$:

- $\lambda = 0,\ \mu = 0$, (US) scheme: it seems to be the more natural way to approximate the system. This unstabilized scheme is studied in [D 07].
- $\lambda = 0,\ \mu > 0$, (BPS) scheme: this is a stabilized scheme inspired by the well known Brezzi-Pitkäranta scheme [BP 84] in the finite element framework.
- $\lambda > 0,\ \mu = 0$, (PS) scheme: it corresponds to a stabilized scheme with the pressure [BEH 05].

3.1. Wellposedness of the scheme.

Unstabilized scheme: the (US) scheme ($\lambda = \mu = 0$) is proved in [D 07] to be wellposed only for some particular classes of meshes described in the following condition. We will say that a given mesh $\mathcal{T}$ satisfies the condition $(\mathcal{H}_{\mathcal{M}})$:

$$
\begin{cases}
\text{If } \mathcal{T} \text{ is a conformal triangular mesh whose angles are less than } \dfrac{\pi}{2}, \\
\text{or if } \mathcal{T} \text{ a non-conformal rectangular mesh.}
\end{cases}
\tag{$\mathcal{H}_{\mathcal{M}}$}
$$

Stabilized scheme: the stabilized scheme (3.1) with $\lambda + \mu > 0$ is wellposed for all meshes.

Theorem 3.1. *If $\lambda = \mu = 0$, we assume that $\mathcal{T}$ verified the condition $(\mathcal{H}_{\mathcal{M}})$, otherwise we just assume that $\mathcal{T}$ is a mesh as described in section 2, the finite volume scheme* (3.1) *admits a unique solution* $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$.

3.2. Error estimates.

3.2.1. First estimates. Under regularity assumptions of the solution of the problem (1.1), one can derive error estimates for the scheme (3.1):

Theorem 3.2. *We assume that the solution $(\boldsymbol{u}, p)$ of the Stokes problem* (1.1) *belongs to $(H^2(\Omega))^2 \times H^1(\Omega)$. For any values of $(\lambda \geq 0, \mu \geq 0)$, under the existence assumptions of the discrete solution $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}})$ of the scheme* (3.1) *(see Theorem 3.1), there exists a constant $C > 0$ depending only on reg($\mathcal{T}$), on the norms of the functions* $\mathbf{f}$*,* $\mathbf{g}$*,* $\boldsymbol{u}$ *and* p*, such that:*

$$
||\boldsymbol{u} - \boldsymbol{u}^{\mathfrak{M}}||_2 + ||\boldsymbol{u} - \boldsymbol{u}^{\mathfrak{M}^*}||_2 + ||\nabla\boldsymbol{u} - \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}||_2 \quad \leq \quad C\ size(\mathcal{T})^{\frac{1}{2}}.
$$

Remark 3.3. *We do not have any error estimate on the pressure, we can just prove that* $||p^{\mathfrak{D}}||_2 \leq C$.

REMARK 3.4. *The boundedness of reg($\mathcal{T}$) imposes only local restriction on the mesh. It is easy to construct a family of locally refined mesh such that reg($\mathcal{T}$) is bounded independently on the level of the refinement.*

3.2.2. Higher order error estimates for (BPS) scheme ($\mu > 0$). In this section, we assume that $\mu > 0$. We improve the error estimates for the (BPS) scheme, and gain the pressure error estimate. These error estimates rely on stability results of the scheme, that we emphasized in Theorem 3.5. The complete proof tooks inspiration from the work of [EHL 06].

THEOREM 3.5 (Stability of the scheme). *Let $\mathbf{u}^{\mathcal{T}}, \widetilde{\mathbf{u}}^{\mathcal{T}}$ and $p^{\mathfrak{D}}, \widetilde{p}^{\mathfrak{D}}$ be two elements of respectively $\left(\mathbb{R}^2\right)^{\mathcal{T}}$ and $\mathbb{R}^{\mathfrak{D}}$, we note:*

$$B(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}) = [\![\mathbf{div}^{\mathcal{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} Id), \widetilde{\mathbf{u}}^{\mathcal{T}}]\!] + (div^{\mathfrak{D}}(\mathbf{u}^{\mathcal{T}}) - \mu size(\mathcal{T})^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}}, \widetilde{p}^{\mathfrak{D}})_{\mathfrak{D}}$$

with $\mu > 0$. Then there exists $C_1 > 0$ and $C_2 > 0$ such that for each pair $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$ such that $\boldsymbol{\gamma}^{\mathcal{T}}(\mathbf{u}^{\mathcal{T}}) = 0$, there exists $\widetilde{\mathbf{u}}^{\mathcal{T}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}}$ such that $\boldsymbol{\gamma}^{\mathcal{T}}(\widetilde{\mathbf{u}}^{\mathcal{T}}) = 0$, $\widetilde{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$:

$$||\nabla^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathcal{T}}||_2 + ||\widetilde{p}^{\mathfrak{D}}||_2 \leq C_1 \left(||\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}||_2 + ||p^{\mathfrak{D}}||_2 \right),$$

and

$$||\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}||_2^2 + ||p^{\mathfrak{D}}||_2^2 \leq C_2 B(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}).$$

We apply the stability Theorem to the difference between some projection of $\mathbf{u}$ and $\mathbf{u}^{\mathcal{T}}$ and to the difference between some projection of p and $p^{\mathfrak{D}}$ where $(\mathbf{u}, p)$ is the solution of the Stokes problem (1.1) and $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$ is the solution of the scheme (3.1) with $\mu > 0$.

THEOREM 3.6. *We assume that the solution $(\boldsymbol{u}, p)$ of the Stokes problem* (1.1) *belongs to $(H^2(\Omega))^2 \times H^1(\Omega)$. Let $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$ be the solution of the scheme* (3.1) *with $\mu > 0$. There exists a constant $C > 0$ depending only on reg($\mathcal{T}$), on the norms of the functions* $\mathbf{f}$, $\mathbf{g}$, $\boldsymbol{u}$ *and* p, *such that:*

$$||\boldsymbol{u} - \boldsymbol{u}^{\mathfrak{M}}||_2 + ||\boldsymbol{u} - \boldsymbol{u}^{\mathfrak{M}^*}||_2 + ||\nabla \boldsymbol{u} - \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}||_2 \quad \leq \quad C \; size(\mathcal{T}),$$

and

$$||p - p^{\mathfrak{D}}||_2 \quad \leq \quad C \; size(\mathcal{T}).$$

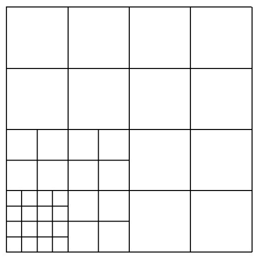
4. Numerical results. We show here some numerical results obtained on a rectangular domain $\Omega =]0,1[^2$. We compare the (US), (BPS) and (PS) schemes on two different tests. The first one is the Green-Taylor vortex on a non-conformal rectangular mesh (Figure 4.1). In the second one, the exact solution is a polynomial function on a general quadrangle mesh (Figure 4.2).

The exact solution $(\mathbf{u}, p)$ being chosen, we define the source term $\mathbf{f}$ and the boundary data $\mathbf{g}$ in such a way that (1.1) is satisfied.

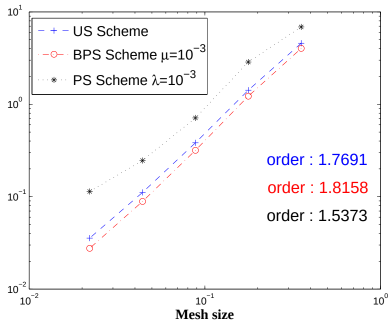
4.1. Test 1 (Green-Taylor vortex). Let us consider the following exact solution:

$$\mathbf{u}(x,y) = \begin{pmatrix} \frac{1}{2}\sin(2\pi x)\cos(2\pi y) \\ -\frac{1}{2}\cos(2\pi x)\sin(2\pi y) \end{pmatrix}, \qquad p(x,y) = \frac{1}{8}\cos(4\pi x)\sin(4\pi y).$$

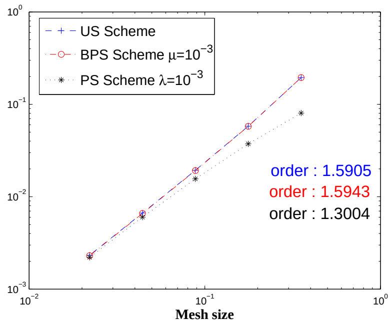
The mesh considered is a non-conformal rectangular mesh (see Figure 4.1(a)). Recall that the (US) scheme is proved to be wellposed on such a rectangular mesh.



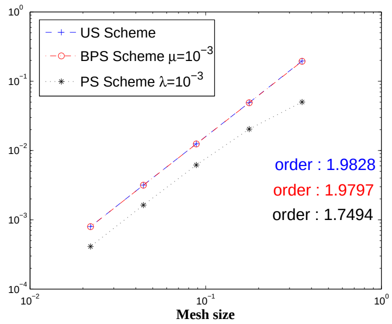
(a) Locally refined rectangular mesh.



(b) $||p - p^{\mathfrak{D}}||_{L^2}$



(c) $||\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}||_{L^2}$



(d) $||\mathbf{u} - \mathbf{u}^{\mathcal{T}}||_{L^2}$

FIG. 4.1. *Green-Taylor vortex, on a rectangular mesh.*

In Figures 4.1(b), 4.1(c) and 4.1(d), we compare the L^2 norm of the error obtained with the three (US), (BPS) and (PS) schemes, for the pressure $||p - p^{\mathfrak{D}}||_{L^2}$, for the velocity gradient $||\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}||_{L^2}$ and for the velocity $||\mathbf{u} - \mathbf{u}^{\mathcal{T}}||_{L^2}$ respectively.

Note that the convergence rates obtained in this numerical test are greater than the theoretical one given in the Theorem 3.2 and 3.6. For the velocity, its gradient and the pressure, we numerically obtain convergence rates equal to 1.5, 2 and 1.7 respectively for the three schemes.

For the (PS) scheme, the convergence rate in L^2 norm for the velocity and the gradient of the velocity is smaller than for the other two schemes even if the (PS) scheme seems to be more precise for the velocity. Nevertheless, for the pressure, the (PS) scheme is less precise than the other two schemes.

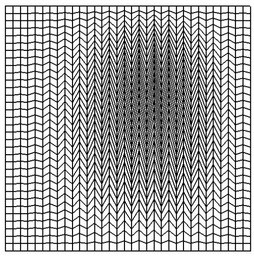
The results obtained with the (BPS) and (US) schemes are essentially the same as far as the velocity is concerned. However, for the pressure, the (BPS) scheme is more precise than the (US) scheme.

As a result, the stabilization induces a precision gain on the velocity for the (PS) scheme, or on the pressure for the (BPS) scheme. Because of the damaging of the error on the pressure, the (PS) scheme does not seem to give important improvements. That is the reason why for the second test, we only compare the (US) scheme and the (BPS) scheme.

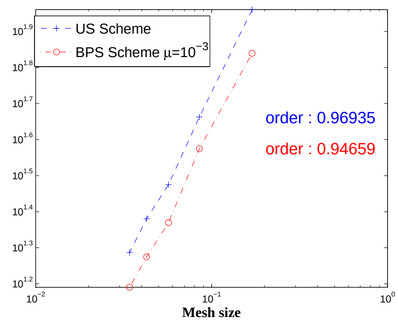
4.2. Test 2. The exact solution on the second test is the following polynomial function:

$$\mathbf{u}(x,y) = \begin{pmatrix} 1000x^2(1-x)^2 2y(1-y)(1-2y) \\ -1000y^2(1-y)^2 2x(1-x)(1-2x) \end{pmatrix}, \qquad p(x,y) = x^2 + y^2 - \frac{2}{3}.$$

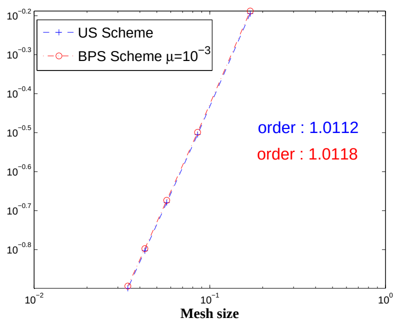
We use the distorted quadrangle mesh, shown on Figure 4.2(a). With such a mesh, we are not able to prove the existence and uniqueness of the solution of the (US) scheme. Nevertheless, numerically the scheme seems to be wellposed and convergent.



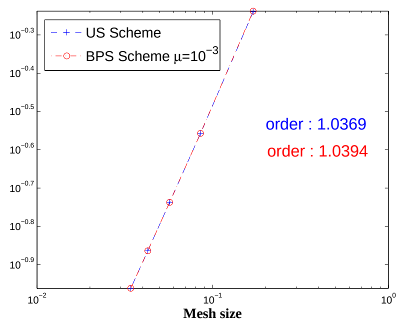
(a) Distorted quadrangle mesh.



(b) $||p - p^{\mathfrak{D}}||_{L^2}$



(c) $||\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}||_{L^2}$



(d) $||\mathbf{u} - \mathbf{u}^{\mathcal{T}}||_{L^2}$

FIG. 4.2. *Polynomial function, on a distorted quadrangle mesh.*

In Figures 4.2(b), 4.2(c) and 4.2(d), we compare the L^2 norm of the error obtained with the two (US) and (BPS) schemes, for the pressure $||p - p^{\mathfrak{D}}||_{L^2}$, for the velocity gradient $||\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}||_{L^2}$ and for the velocity $||\mathbf{u} - \mathbf{u}^{\mathcal{T}}||_{L^2}$ respectively.

As in the first test case, the two schemes give essentially the same results as far as the velocity is concerned and the (BPS) scheme is more precise for the pressure.

As predicted by the Theorem 3.6, we observe a first order convergence for the (BPS) scheme, which seems to be optimal.

5. Conclusion. In this paper, we propose stabilized DDFV finite volume schemes with Dirichlet boundary conditions for the Stokes problem. One can prove a first order convergence of the (BPS) scheme in the L^2 norm for the velocity gradient

$||\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}||_2$, for the velocity and for the pressure. We compare the scheme for different sets of stabilization parameters (λ, μ) on numerical tests. We bring to light some differences. The (PS) scheme gives some precision on the velocity, nevertheless it damages the approximation of the pressure. The (BPS) scheme is as well precise on the velocity as the (US) scheme but its improves the error of the pressure. Therefore, the stabilization induces a precision gain on the velocity for the (PS) scheme, or on the pressure for the (BPS) scheme.

Such an approach will be extended in [K] for the general multifluid Stokes problem:

$$\begin{aligned} -\mathrm{div}\left(\eta(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)\right) + \boldsymbol{\nabla} p &= \mathbf{f}, \quad \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}, \quad \text{on } \partial\Omega, \qquad \int_\Omega p(x)\mathrm{d}x = 0. \end{aligned}$$

with $\eta \in L^\infty(\Omega)$, $\inf \eta > 0$, which can be discontinuous. The discontinuities of η bring us to use the definition of a new discrete gradient like in [BH 08]. We build a new discrete operator corresponding to $\nabla \mathbf{u} + (\nabla \mathbf{u})^t$ and we prove in [K] a discrete Korn inequality. Finally, the corresponding stabilized DDFV scheme is proved to be wellposed and first order convergent on general meshes, even for discontinuous viscosity.

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