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NUMERICAL JUSTIFICATION OF ASYMPTOTIC EDDY–CURRENTS MODEL FOR HETEROGENEOUS MATERIALS*

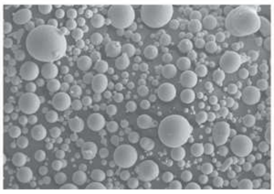
V. MELICHER† AND J.BUŠA JR.†

Abstract. We study electromagnetic properties of a heterogeneous material in low-frequency case. We review briefly the corresponding eddy–currents model, which is multiscale in nature. The homogenized model, obtained in [8], is presented. We justify the asymptotic model numerically on number of examples.

Key words. homogenization, multiscale, heterogeneous materials, magnetic dust, soft magnetic materials

AMS subject classifications. 35B27, 78M40, 80M40

1. Introduction. In the manufacturing of compact electronic devices one tries to lower the voltage and to raise the current loads in order to increase the performace and to reduce the power consumption. Consequently there is still greater need for a low-loss magnetic materials. The most promising are composite materials, which combine small conducting and isolating particles together, known as “magnetic dust” materials. An example of such an material is Fe-based metallic glass dust material called *SENNTIX* developed by NEC [7], see Figure 1. The idea behind the magnetic



a)

Characteristic / Material	Saturated magnetic flux density *Bs* [T]	Relative permeability *μ* 100 kHz	Coercive force *Hc* [A/m]	Electrical resistivity *ρ* [μΩ-cm]
SENNTIX	1.3	6000	2.0	130
Pure iron	2.2	≦200	64	10
6.5% Si-Fe	1.8	1600	20	80
Fe amorphous	1.5	2900	3.0	115

b)

FIG. 1.1. *SENNTIX magnetic material: a) microscopic structure; b) electro–magnetic properties*

dust materials is very simple. A piece from such an material acts as a ferromagnet with added Mass of the ferromagnetic dust particles (in the case of SENNTIX it is ferum). In the same time, the material has a great electrical resistivity. The eddy–currents are induced only locally in the ferromagnetic particles.

An efficient computational model would allow engineers to study such materials lowering the development costs. It would allow to design new materials easily, e.q. to try different compositions. The problem is multiscale in nature. We are interested actually only in macroscopic electro–magnetic properties of the material. However, these are strongly dependent on the microscopic set–up. Different approaches exist to tackle the muliscale problems. Classical multiscale modelling methods such as multigrid, fast multipole method or adaptive mesh refinement try to resolve details of the

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†Department of Mathematical Analysis, Nfam2, Ghent University, 9000 Ghent, Belgium(valdemar.melicher@ugent.be, jan.busa@ugent.be).

solution on the micro scale level and so the cost of the methods is the cost of full micro scale solver.

The purpose of new multiscale methods, is to resolve the macro scale behavior of the multiscale model with a cost that is less than the cost of full micro scale solver. Naturally, to achieve this, we may not require to have all the information on micro-level and we have to use special properties of the micro scale problem such as scale separation. For the overview of modern multiscale methods we refer reader to [5], where the heterogeneous multiscale method (HMM) is considered and references therein. The HMM for magnetic dust materials is analyzed in [3].

In this contribution we will numerically analyze the asymptotic eddy–currents model obtained in [8]. Not numerical concepts (like HMM) but the homogenization theory is used to obtain this asymptotic model. It is a classical approach [4]. The model is derived for periodic materials, however it is well-know in the homogenization theory, that the periodic case can be generalized for e.q. materials with stochastic structure [2, 6].

The article is organized as follows. In Section 2 we introduce the *eddy–currents* model for linear heterogeneous materials, then its homogenized version in Section 3. The Section 4 deals with numerical justification of the asymptotic model.

2. Eddy–currents model. The evolution of low-frequent electromagnetic fields in a heterogeneous linear material is governed by the Maxwell equations

$$\nabla \times \boldsymbol{E} = -i\omega\mu\boldsymbol{H}, \tag{2.1}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_a + \sigma\boldsymbol{E}, \tag{2.2}$$

$$\nabla \cdot (\mu\boldsymbol{H}) = 0, \tag{2.3}$$

$$\nabla \cdot (\epsilon\boldsymbol{E}) = \rho, \tag{2.4}$$

where $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{J}_a$, ρ are the electric field intensity, the magnetic field intensity, the applied current density and the electric charge density, respectively. The matrices σ, μ and ϵ are the conductivity, the permeability and the permittivity, respectively. We omitted the displacement current term. It can be safely neglected, when the frequency ω is sufficiently low. The justification of this so–called *eddy–current* model can be found in [1]. After elimination of $\boldsymbol{H}$ from the Maxwell's equations (2.1)-(2.2) we obtain[1]

$$\nabla \times (\nu\nabla \times \boldsymbol{E}) + i\omega\sigma\boldsymbol{E} = -i\omega\boldsymbol{J}_a \text{ in } \Omega, \tag{2.5}$$

where $\nu = \mu^{-1}$. We suppose that Ω is a bounded domain in $\mathbb{R}^3$. We accompany (2.5) by perfect conducting boundary condition

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \tag{2.6}$$

Γ being the boundary of Ω.

Usual set–up to model is sketched in Figure 2. Eddy–currents are induced in a conducting obstacle due to an external excitation. The external field is generated by an source – coil. We assume that the obstacle is from an periodic heterogeneous material. In whole Ω, the material parameters are dependent on both the micro (in the obstacle) and the macro variable (in air). For simplicity of explanation we assume that the material parameters σ, μ and ϵ are periodic with period ε in whole Ω. The generalization with insulator incorporated can be easily handled.

[1] Gauss's magnetic law (2.3) is a consequence of (2.1).

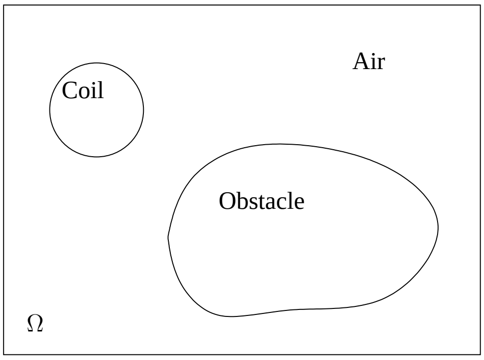


FIG. 2.1. *Domain.*

We assume ϵ, σ, ν are matrices and that there exist constants $0 < \epsilon_{\min}, \epsilon_{\max} < \infty$, $0 < \sigma_{\max} < \infty$, $0 < \nu_{\min}, \nu_{\max} < \infty$ such that

$$\epsilon_{\min}|\boldsymbol{\xi}|^2 \le \boldsymbol{\xi}^* \cdot (\epsilon + \epsilon^*) \cdot \boldsymbol{\xi} \le \epsilon_{\max}|\boldsymbol{\xi}|^2 \tag{2.7}$$
$$0 \le i\omega\boldsymbol{\xi}^* \cdot (\sigma - \sigma^*) \cdot \boldsymbol{\xi} \le \sigma_{\max}|\boldsymbol{\xi}|^2 \tag{2.8}$$
$$\nu_{\min}|\boldsymbol{\xi}|^2 \le \boldsymbol{\xi}^* \cdot (\nu + \nu^*) \cdot \boldsymbol{\xi} \le \nu_{\max}|\boldsymbol{\xi}|^2 \tag{2.9}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^3$. Moreover we assume that ϵ, σ, ν are $Y-$ periodic, where $Y = [0,1]^3$. Thus $\epsilon^\varepsilon := \epsilon(\boldsymbol{x}/\epsilon)$, $\sigma^\varepsilon := \sigma(\boldsymbol{x}/\epsilon)$, $\nu^\varepsilon := \nu(\boldsymbol{x}/\epsilon)$ are periodic with period ϵ.

3. Homogenized model. In [8] we studied the homogenization of (2.5). First, we used an Hodge decomposition to write the electric field as $\boldsymbol{E} = \boldsymbol{A} - \nabla p$, where $\boldsymbol{A}$ is a vector potential and p is a scalar potential. The homogenization was performed componentwise in suitable functional spaces. The homogenization of the scalar potential is classical. We focus on the vector potential, in the case of zero electric charges the solution to

$$\nabla \times (\nu^\varepsilon \nabla \times \boldsymbol{A}^\varepsilon) + i\omega\sigma^\varepsilon \boldsymbol{A}^\varepsilon = -i\omega \boldsymbol{J}_a \quad \text{in } \Omega, \tag{3.1}$$
$$\boldsymbol{A}^\varepsilon \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma. \tag{3.2}$$

This is virtually the same system as (2.5)–(2.6) for electric field, however considered in a different functional space. We moreover have

$$\nabla \cdot (\epsilon^\varepsilon \boldsymbol{A}^\varepsilon) = 0 \quad \text{a.e. in } \Omega. \tag{3.3}$$

We denoted the corresponding unique solution as $\boldsymbol{A}^\varepsilon$ to emphasize the dependence on ε.

For a sufficiently smooth $\boldsymbol{J}_a$ the vector potential $\boldsymbol{A}^\varepsilon$ converges in a weak sense to $\boldsymbol{A}$, the unique solution to the homogenized vector potential formulation

$$\nabla \times (\nu^h \nabla \times \boldsymbol{A}) + i\omega\sigma^h \boldsymbol{A} = -i\omega \boldsymbol{J}_a \quad \text{in } \Omega, \tag{3.4}$$
$$\boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma \tag{3.5}$$

as $\varepsilon \to 0$. We moreover have

$$\nabla \cdot (\epsilon^h \boldsymbol{A}) = 0 \quad \text{a.e. in } \Omega. \tag{3.6}$$

The homogenized ϵ^h, ν^h and σ^h are given by

$$\epsilon^h := \frac{1}{|Y|} \int_Y \epsilon(\boldsymbol{y})[I - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}(\boldsymbol{y})]\, d\boldsymbol{y}, \tag{3.7}$$

$$\sigma^h := \frac{1}{|Y|} \int_Y \sigma(\boldsymbol{y})[I - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}(\boldsymbol{y})]\, d\boldsymbol{y} \tag{3.8}$$

and by

$$\nu^h := \frac{1}{|Y|} \int_Y \nu(\boldsymbol{y})[I - \nabla_{\boldsymbol{y}} \times E(\boldsymbol{y})]\, d\boldsymbol{y}, \tag{3.9}$$

where $\boldsymbol{\chi}$ is a unique solution to

$$\nabla_{\boldsymbol{y}} \cdot \{\epsilon(\boldsymbol{y})\,[I - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}(\boldsymbol{y})]\} = \mathbf{0} \quad \text{a.e. in } Y \tag{3.10}$$

and matrix E is a unique solution to

$$\nabla_{\boldsymbol{y}} \times \{\nu(\boldsymbol{y})\,[I - \nabla_{\boldsymbol{y}} \times E(\boldsymbol{y})]\} = \{0\} \quad \text{a.e. in } Y, \tag{3.11}$$

$$\nabla_{\boldsymbol{y}} \cdot E(\boldsymbol{y}) = \{0\} \quad \text{a.e. in } Y, \tag{3.12}$$

$$\frac{1}{|Y|} \int_Y E(\boldsymbol{y}) = \{0\}. \tag{3.13}$$

4. Numerical analysis. The solution of the homogenized problem goes as follows

1. Solve microproblems (3.10) and (3.11)–(3.13) on the cell Y.
2. Obtain the homogenized coefficients by integration according to (3.7)–(3.9).
3. Solve asymptotic macro problem (3.4)–(3.5).

The purpose of this contribution is to study the homogenized model numerically on number of examples. To take into account the most of phenomena occuring in the eddy–currents modelling, we have to work at two dimensions. We consider vector potentials of the form $\boldsymbol{A} = (\boldsymbol{A}_1, \boldsymbol{A}_2, 0)$. Let us consider the permittivity, the conductivity and the permeability matrices respectively in the following form

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu \end{pmatrix}. \tag{4.1}$$

We do not distinguish between the scalars and the matrices by notation. It will be always clear from the context what we refer to.

Both the micro problems and the macro eddy–currents problem are solved by finite element method. Lagrange elements of the first order are used. The domain Ω is the unite square $[0,1]^2$. At both the micro and the macro level regular meshes are taken. Let us call H the diameter of the macro and h the diameter of the micro mesh. The "exact" solutions are computed on a very fine mesh.

4.1. Example 1. Let us first assume that ν^ε is only one periodic and let us take zero conductivity, i.e.,

$$\epsilon^\varepsilon = 1, \quad \sigma^\varepsilon = 0, \quad \nu^\varepsilon = \sin(2\pi x_1/\varepsilon)(x_1/\varepsilon - [x_1/\varepsilon]) + \sin(2\pi x_2/\varepsilon)(x_2/\varepsilon - [x_2/\varepsilon]) + 3, \tag{4.2}$$

where $\varepsilon = 0.05$ is taken. Since $\epsilon^\varepsilon = 1$ microproblem (3.10) has trivial solution zero. As the source we take $-i\omega \boldsymbol{J}_a =: \boldsymbol{f} = (1, 1, 0)$, where $\omega = 1$. The "exact" solution is computed on a fine mesh with $h = 0.005$. Microproblem (3.11)–(3.13) is solved on a mesh with $h = 0.01$. The resulting homogenized ν^h is approximately 2.57. Merely taking the average of ν^ε through cell gives around 2.68. The asymptotic problem is again solved on a mesh with $H = 0.01$. The relative L^2-errors are 0.0276 for the average ν^ε and 0.0074 for the homogenized ν^h. The results are depicted in Figure 4.1. The real part of the solutions is rendered. Color depicts amplitude.

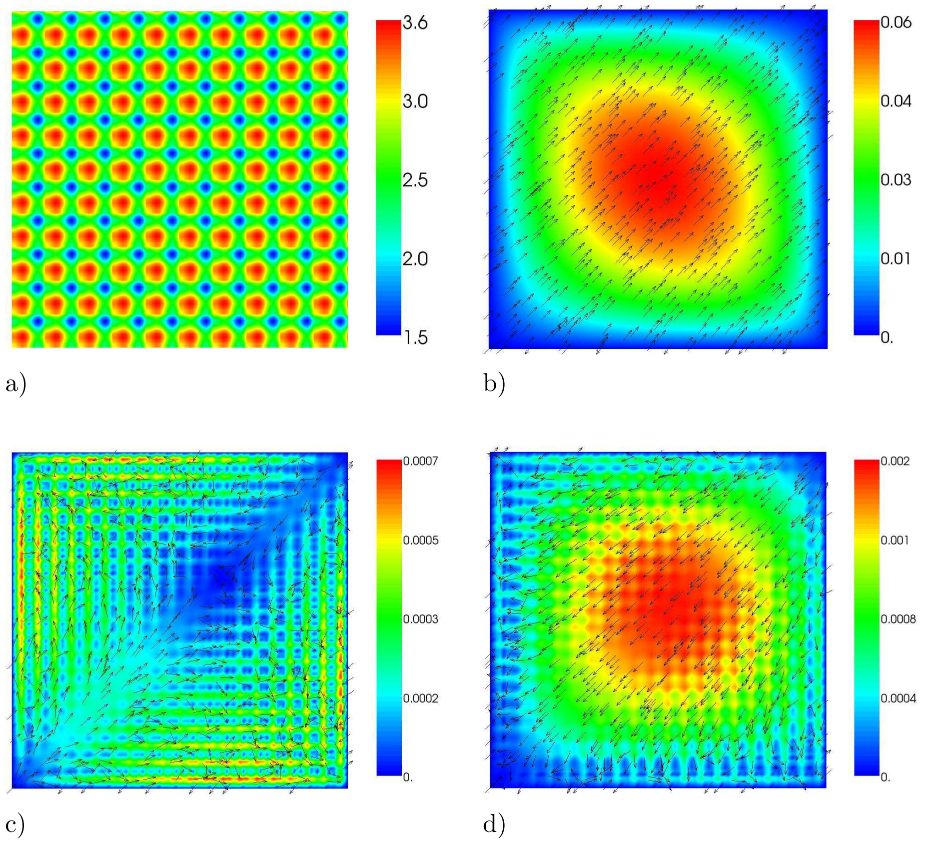


FIG. 4.1. *Example 1 : a) ν for $\varepsilon = 0.1$; b) $\boldsymbol{A}^\varepsilon$; c) difference between $\boldsymbol{A}$ and $\boldsymbol{A}^\varepsilon$ using ν^h; d) difference between $\boldsymbol{A}$ and $\boldsymbol{A}^\varepsilon$ using the Y-average of ν^ε*

4.2. Example 2. In the second example we consider that only the conductivity is periodic, such that

$$\epsilon^\varepsilon = 1, \quad \sigma^\varepsilon = \sin(2\pi x_1/\varepsilon) + \sin(2\pi x_2/\varepsilon) + 3, \quad \nu^\varepsilon = 0. \tag{4.3}$$

The source is $\boldsymbol{J}_a = (1, 0, 0)$. The meshes are set up as in Example 1. $\varepsilon = 0.05$. The relative error between $\boldsymbol{A}^\varepsilon$ and $\boldsymbol{A}$ is rather big - 0.232. From the Figure (4.2) we can see that the fine solution has significant periodic component. We can not expect the homogenized solution to resolve the microscopic nature of the fine solution. Thus, we consider the $Y-$averaged fine solution $\bar{\boldsymbol{A}}^\varepsilon$. The relative error between $\bar{\boldsymbol{A}}^\varepsilon$ and the homogenized solution is 0.064. Moreover, in all examples we use homogeneous Dirichlet boundary condition for the fine and the homogenized problem. Considering the natural Dirichlet boundary conditions $\boldsymbol{A} = (1, 0, 0)$ and $\boldsymbol{A}^\varepsilon = (1, 0, 0)$ on Γ (taking into account $\boldsymbol{J}_a$) gives the relative error 0.03.

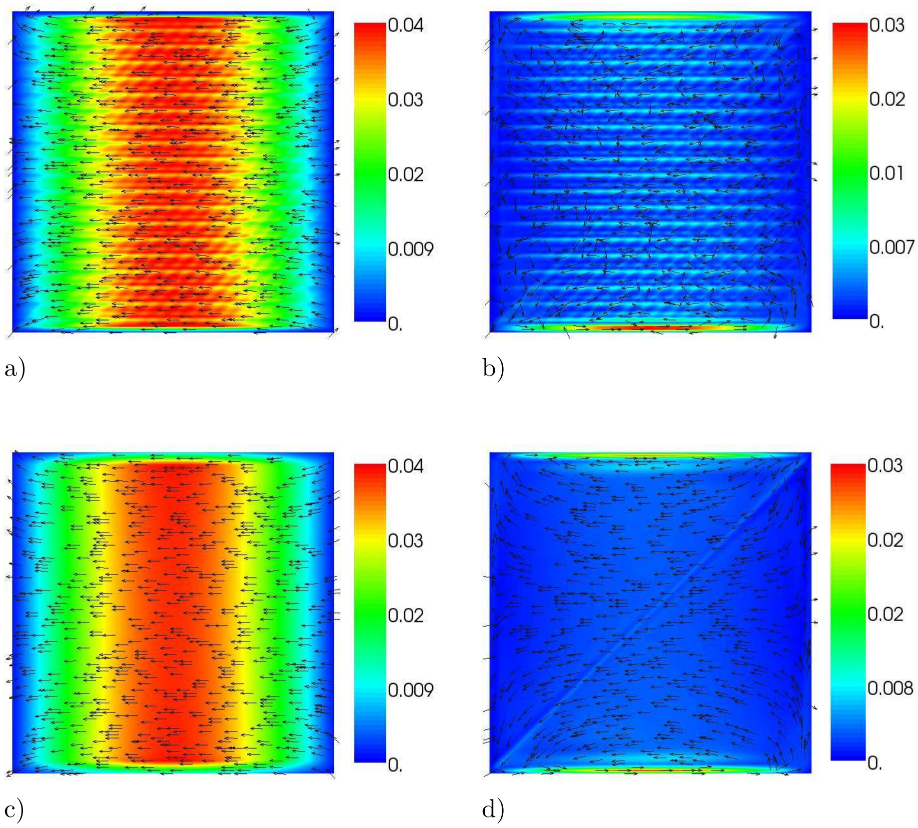


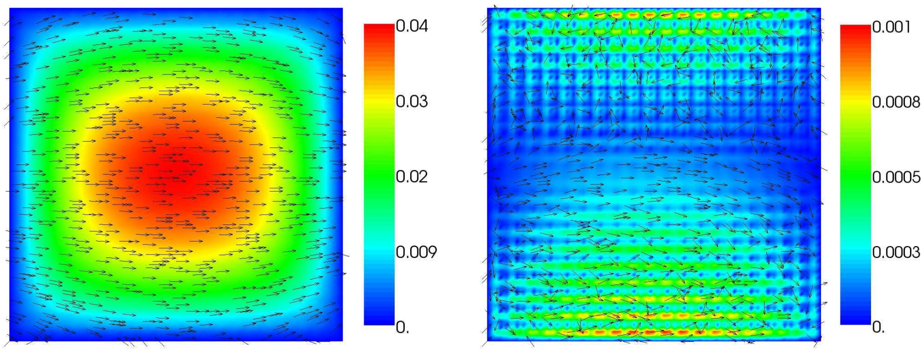
Fig. 4.2. *Example 2 : a) $\boldsymbol{A}^\varepsilon$; b) difference between $\boldsymbol{A}$ and $\boldsymbol{A}^\varepsilon$; c) $\bar{\boldsymbol{A}}^\varepsilon$ d) difference between Y-averaged $\boldsymbol{A}^\varepsilon$ and $\boldsymbol{A}$*

4.3. Example 3. Let both the permeability and the conductivity be periodic such that

$$\epsilon^\varepsilon = 1, \quad \sigma^\varepsilon = \sin(2\pi x_1/\varepsilon) + \sin(2\pi x_2/\varepsilon) + 3, \quad \nu^\varepsilon = \sin(2\pi x_1/\varepsilon) + \sin(2\pi x_2/\varepsilon) + 3. \tag{4.4}$$

Again, all the set up is the same as in the examples above. The results are depicted in Figure 4.3. The relative error between $\boldsymbol{A}^\varepsilon$ and $\boldsymbol{A}$ is 0.014.

Conclusion. We analyzed numerically homogenized eddy–currents model for linear periodic heterogeneous materials. The experiments confirmed that macroscopic electro–magnetic behavior of such materials is well described by the homogenized model. This was done for conductivity and permeability being periodic. Further investigation is necessary to confirm the theoretical result for general case when permittivity is periodic as well.



a) b)

FIG. 4.3. *Example 3 : a)* $\boldsymbol{A}^{\varepsilon}$*; b) difference between* $\boldsymbol{A}$ *and* $\boldsymbol{A}^{\varepsilon}$

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