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PERIODS OF PERIODIC POINTS FOR TRANSITIVE DEGREE ONE MAPS OF THE CIRCLE WITH A FIXED POINT

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Abstract. A map of a circle is a continuous function from the circle to itself. Such a map is transitive if there is a point with a dense orbit. For degree one transitive maps of the circle with a fixed point, we give all possible sets of periods and the best lower bounds for topological entropy in terms of the set of periods.

0. Introduction

A map of a space X is a continuous function $f\colon X \to X$. We say f is transitive if there is a point with a dense orbit. We denote by $P(f)$ the set of periods of the periodic points under f, and by $\operatorname{ent}(f)$ its topological entropy.

Consider the following ordering of the set $\mathbb{N}$ of natural numbers: 3, 5, 7, ... , $2\cdot 3$, $2\cdot 5$, ... , ... , $2^k\cdot 3$, $2^k\cdot 5$, ... , ... , 2^3, 2^2, 2, 1. Let $S(n)$ be the set consisting of n and all integers standing to the right of n in the above order, and $S(2^\infty)$ the set of all powers of 2. In [**9**], Sarkovskii showed that for maps of the real line, the sets of periods of periodic points are of the form $S(n)$ for some $n \in \mathbb{N} \cup \{2^\infty\}$.

Block [**2**] proved the following result for degree one maps of the circle.

Theorem 0.1. [**2**]. *Let f be a continuous degree one map of the circle with a fixed point. Then $P(f) = S(n) \cup \{j \in \mathbb{N} : j \geq k\}$ for some positive integer k and some $n \in \mathbb{N} \cup \{2^\infty\}$. (Note: One of the sets may be empty).*

In this paper, we consider transitive maps of the circle and show how the above result changes when we impose this dynamical restriction. Our main result is:

Theorem 3.1. *Let f be a transitive degree one map of the circle with a fixed point. Then $P(f) = \{1\} \cup \{j \in \mathbb{N} : j \geq k\}$ for some positive integer k and* $\operatorname{ent}(f) \geq \log$ (largest zero of $x^{k+1} - x^k - x - 1$).

Moreover, if $k = 2$ and there is a periodic point of periodic point of period two with rotation number zero, then $\operatorname{ent}(f) > \log 2$.

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Note that there is strictly inequality in the $\log 2$ entropy bound, and that 2 is greater than the largest zero of $x^3 - x^2 - x - 1$. The converse to Theorem 3.1 is also true, i.e., every possible $P(f)$ is realizable, and the entropy bounds are sharp. In Section 4, we discuss some examples.

1. Background

For a map f of a space X, and $n \geq 0$, f^n is defined by: $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. A point $x \in X$ is periodic of period n if $f^n(x) = x$ and n is the least integer for which this happens. $\mathrm{Orb}_f(x)$ (or $\mathrm{Orb}(x)$) denotes the orbit $\{f^n(x) : n \geq 0\}$ of the point x. A subset E is (f)-**invariant** if $f(E) \subseteq E$. $\mathrm{Int}\,(E)$ and $\mathrm{cl}(E)$ denote the interior and closure, respectively, of a set E. We denote by S^1 the circle $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ and $\mathbb{Z}$ denote the real and integer numbers, respectively.

The ambient space is S^1. An interval $[a,b]$, (a,b), $[a,b)$ or $(a,b]$ in S^1 is the closed, open or half-open arc, resp., from a counterclockwise to b. e is the natural projection from $\mathbb{R}$ onto S^1 ($e(x) = \exp(2\pi i x)$). A **lift** F of f is map of the real line for which $f(e(x)) = e(F(x))$ for all $x \in \mathbb{R}$. There are countably many lifts of f and any two differ by an integer. The **degree** of f, denoted deg f, is the integer n such that $F(x+1) = F(x) + n$ for all $x \in \mathbb{R}$ and for every lift F of f. Note that deg $f^k = (\text{deg } f)^k$.

In our proofs, we adopt the notion of f-covers from [**5**]. Let J and K be nondegenerate proper closed intervals. We say J f-**covers** K (n times) if there exist subintervals $\{L_i : i \leq i \leq n\}$ of J, with pairwise disjoint interiors, such that, for each i, $f(L_i) = K$. Note that if F is a lift of f, and J' and K' are interval lifts to R of J and K, resp., then J F-covers K if and only if $F(J')$ contains some integer translate $K' + m$ of K'.

Lemma 1.1. [**2**]. *Let $I = [a,b]$ be a proper closed interval of S^1. If $f(a) = c$ and $F(b) = d$ and $c \neq d$, then I f-covers either $[c,d]$ or $[d,c]$.*

f-covers can be used to infer the existence of certain periods and obtain estimates on topological entropy, a topological conjugacy invariant of continuous maps. More specifically, if P is a finite (but not necessarily invariant) subset of S^1, label the points in P $x_1, x_2, \ldots, x_n$ so that the intervals $I_1 = [x_1, x_2]$, … , $I_n = [x_n, x_1]$, we call P-**intervals**, have pairwise disjoint interiors.

The P-**graph** of f is the directed graph having as vertices the P-intervals, and with k arrows from I_i to I_j if and only if I_i f-covers I_j exactly k times. The following lemma states that closed walks in the P-**graph** force the existence of periodic orbits that move in the same order.

Lemma 1.2. [**5**]. *Let f be a map of the interval or the circle. If $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$ is a loop in the P-graph of f, then there exists a fixed point x of f^n such that $f^i(x) \in J_i$ for $i = 0, 1, \ldots, n-1$.*

The P-**matrix** of f is defined by $(A)_{ij} =$ (no. of arrows from I_i to I_j). The Perron-Frobenius Theorem [**8**] guarantees the existence of a non-negative eigenvalue, denoted $r(A)$, of maximum modulus. If A' is any proper submatrix of A, then $r(A) \geq r(A')$, with strict inequality if $A > 0$. In the remainder of this paper, we shall abuse notation and set $\log 0 = 0$.

Lemma 1.3. [**5, 6**]**.** *Let P be a finite subset of S^1, and A the P-matrix of f. Then* $\operatorname{ent}(f) \geq \log r(A)$, *with equality if P is invariant, and f is monotone between adjacent points of P.*

In the proof of Theorem 3.1 we show the existence of a subset called an n-horseshoe (for f), i.e., a collection $J_1, \dots, J_n$ of closed subintervals with pairwise disjoint interiors, such that for $1 \leq i \leq n$, J_i f-covers $J_1, \dots, J_n$. When such a collection exists, the set of endpoints of the J_i's yields a P-matrix with a proper sub-matrix whose entries are all ≥ 1. Lemma 1.3 and standard Perron-Frobenius arguments imply $\operatorname{ent}(f) \geq \log n$.

Lemma 1.4. [**3,** Lemma 2]**.** *Let $f\colon S^1 \to S^1$ have an n-horseshoe ($n \geq 2$). Then f has periodic points of all periods, and* $\operatorname{ent}(f) \geq \log n$.

2. Transitivity

In this section, we prove the following analogue of a result of Barge and Martin [**1,** Theorem 13] on transitive maps of the real line.

Theorem 2.1. *Let f be a transitive map of the circle with a fixed point. If f^2 is transitive, then f has a periodic point of odd period $k > 1$.*

Remark. This result is new only for $|\deg f| \leq 1$ [**5**].

The proof of the theorem is based on the following results of Coven and Mulvey [**7**]:

Theorem 2.2. [**7,** Corollary 3.4]**.** *For a transitive map of the circle with periodic points, the set of periodic points is dense.*

Lemma 2.3. [**7,** Lemma 5.2]**.** *If f has periodic points and f^n is transitive for every $n > 0$, then for every non-degenerate interval E in S^1, $\bigcup_{n \geq 0} f^n(E)$ misses at most one point, which must then be a fixed point.*

Theorem 2.4. [**7,** Theorem C]**.** *Let f be a continuous map of the circle. Then the following statements are equivalent:*
1) *There is an m such that f^{2m} is transitive, and f^m has a fixed point.*
2) *f^n is transitive for every $n > 0$, and f has periodic points.*
3) *f is topologically mixed (i.e., for every pair U, V of non-empty open sets, there is an N such that $f^n(U) \cap V \neq \emptyset$ for every $n \geq N$).*

We also use the following equivalent definitions of transitivity:
1) There is a point with a dense orbit.
2) The only closed invariant set K with $\text{int}\,(K) \neq \emptyset$ is the whole space
3) If $\text{int}\,(E) \neq \emptyset$, then $\text{cl}(\bigcup_{n\geq 0} f^n(E))$ is the whole space.

Proof of Theorem 2.1. **Notation:** If $U = [a, b]$, $V = [c, d]$ are non-overlapping closed intervals, we denote by $\langle U, V\rangle$ the open interval (b, c).

Since f is transitive, there is a point with a dense orbit. For this x, we have x, $f(x)$ and $f^2(x)$ distinct. Then by continuity of f, there is an interval J about x with J, $f(J)$ and $f^2(J)$ pairwise disjoint. By shrinking J, if necessary, we may assume that $f^2(J)$ contains no fixed point. Start at J and label the other two intervals as J' and J'' in the counterclockwise direction.

By Theorem 2.4, f^n is transitive for every $n > 0$. We then choose a periodic orbit $\text{Orb}(c)$ in the following way. By Lemma 2.3, either
(1) $\bigcup_{n\geq 0} f^{mn}(J) = S^1$ for every $m \geq 1$, or
(2) $\bigcup_{n\geq 0} f^{mn}(J) = S^1 - \{p\}$ for some $m \geq 1$ and some p fixed by f^m.

If (1) holds, use Theorem 2.2 to choose a periodic point c with period $t \geq 2$, such that $\text{Orb}(c)$ meets both $\langle J, J'\rangle$ and $\langle J'', J\rangle$. By shrinking J, we may assume that $\text{Orb}(c)$ does not meet J. Call c_1 and c_2 the points in $\text{Orb}(c)$ such that J lies in (c_1, c_2) and $f(J) \cup f^2(J) \cup \text{Orb}(c)$ lies in $[c_2, c_1]$.

If (2) holds, we may assume that p is not in $J \cup f(J) \cup f^2(J)$. Since $g = f^m$ is transitive, use Theorem 2.2 to choose c with g-period $t \geq 2$ so that for some points c_1, c_2 in $\text{Orb}_f(c)$, $p \in (c_2, c_1)$, and $J \cup f(J) \cup f^2(J) \cup \text{Orb}_f(c)$ lies in $[c_1, c_2]$. In either case, there exists $M > 0$ such that for all $n \geq M$, $\text{Orb}(c) \subseteq f^n(J)$. (For example, in (1) $\text{Orb}(c) \subseteq \bigcup_{n\geq 0} f^{tn}(J)$ andf^t fixed every point in $\text{Orb}(c)$. Let $M = t(k_1 + \cdots + k_t)$, where $c_i \in f^{tk_i}(J)$ for $c_i \in \text{Orb}(c)$. Make a similar determination in (2), since $\text{Orb}(c) \subset \bigcup_{n\geq 0} g^{tn}(J) = \bigcup_{n\geq 0} f^{mtn}(J)$ and f^{mt} fixes every point in $\text{Orb}(c)$.)

By Lemma 1.1, for each $n \geq M$, J f^n-covers either $[c_1, c_2]$ or $[c_2, c_1]$.

If (1) holds, choose an odd $n > M$. If J F^n-covers $[c_1, c_2]$, then J F^n-covers itself. By Lemma 1.2, J contains a periodic point of period a divisor of n. Since J does not contain a fixed point, this point is of some odd period > 1. If J f^n-covers $[c_2, c_1]$, then J f^n-covers $f^2(J)$. Therefore, $f^2(J)$ f^{n-2}-covers itself, and again, since $f^2(J)$ has no fixed point, it has a periodic point of add period > 1.

If (2) holds, choose $n \geq M$ such that n is also a multiple of m. Then since $p \in [c_2, c_1]$, J f^n-covers $[c_1, c_2]$, hence f^n-covers J, $f(J)$ and $f^2(J)$. If n is odd,then J has a point of odd period > 1. If n is even, then since $f(J)$ f^{n-1} covers itself, $f(J)$ has a point of odd period > 1. □

3. Degree One Maps

Let deg $f = 1$. If F is a lift of f and x is f-periodic of period n with $e(y) = x$, then $F^n(y) = y + k$ for some integer k. The number k/n, denoted $\rho_F(x)$, is the **rotation number** of X. This is independent of the choice of y, and if $F' = F+m$, then $\rho_{F'}(x) = \rho_F(x) + m$.

In this section , we prove our main result.

Theorem 3.1. *Let f be a transitive degree one map of the circle with a fixed point. Then $P(f) = \{1\} \cup \{j \in \mathbb{N} : j \geq k\}$ for some positive integer k and* $\text{ent}\,(f) \geq \log$ (largest zero of $x^{k+1} - x^k - x - 1$).

Moreover, if $k = 2$ and there is a periodic point of period two with rotation number zero, then $\text{ent}\,(f) > \log 2$.

Lemma 3.2 [**5**]. *Let f be a continuous degree one map of the circle with a fixed point. Then if f has a fixed point and a periodic point of period $n > 1$ having different rotation numbers, then f has periodic points of all periods larger than n, and* $\text{ent}\,(f) \geq \log$ (largest zero of $x^{n+1} - x^n - x - 1$).

Lemma 3.3. *If f is a transitive, degree one map of the circle with a fixed point, then f^n is transitive for every $n > 0$ (hence is topologically mixing).*

Proof. By Theorem 2.4, it is enough to show that f^2 is transitive. Suppose f^2 is not transitive.

Let p be a fixed point of f.

Claim: There is a nondegenerate proper closed interval K such that:
(i) $f^2(K) = K$
(ii) $K \cup f(K) = S^1$
(iii) $\text{int}\,[K \cap f(K)] = \emptyset$.

By [**7,** Lemma 2.1], (i)–(iii) hold for some closed proper subset K with nonempty interior. To see that K is an interval, let L be a nondegenerate component of K, and $L^* = \text{cl}\left[\bigcup_{n\geq 0} f^{2n}(L)\right]$. [**7**] shows that $L^* \subseteq K$ has finitely many components, each with non-empty interior, and they are permuted by f^2. Since $f^2|K$ is transitive and L^* is nondegenerate, closed and f^2-invariant, $L^* = K$. So K and $f(K)$ each has finitely many components, alternating on S^1.

Thus, p must be a common endpoint of components K_1 of K and K_2 of $f(K)$, and f must permute K_1 and K_2. Since $K_1 \cup K_2$ is closed, f-invariant and has non-empty interior, it must be the whole circle, f-invariant and has non-empty interior, it must be the whole circle, and $K = K_1$.

Now let p' be the lift of p to $[0, 1]$, F the lift of f that fixed p' (hence, also $p' + 1$). Then a left of K to $[p', p' + 1]$ has either p' or $p' + 1$ as an endpoint. A consideration of cases shows that no such lift can exist. □

Proposition 3.4. *Let f be a topologically mixing map of the circle, with a lift F such that for some $0 < x < y < 1$, $F(0) = F(y) = 0$ and $F(x) = y$. Then* $\operatorname{ent}(f) > \log 2$.

(*The same conclusion holds if instead $F(1) = F(x) = 1$ and $F(y) = x$.*)

Proof. We will prove the proposition for when $F(x) = y$ and $F(Y) = 0$. (The second case can be handled in a similar way.) To simplify notation, we will also call 0, x and y their respective projections in S^1.

By Theorem 2.4, f^n is transitive for every $n > 0$. Since $[0, y] \subseteq f([0, y])$, but $[0, y]$ cannot be f-invariant, there exists a non-degenerate interval J in $[y, 0]$ that is adjacent to $[0, y]$ (i.e., has y or 0 as an endpoint) such that $J \subseteq f[0, y]$.

Suppose for the moment that $J = [y, w]$. Let $P = \{0, x, y, w\}$. Since f is mixing, J meets $f^n(J)$ for all large enough n. Since x is not a fixed point, $x \in f^n(J)$ for infinitely many n. (If $x \notin f^n(J)$ for $n \geq N$, let $K = f^N(J)$. Then $x \notin \bigcup_{n \geq 0} f^n(K)$, hence by Lemma 2.3 must be fixed.)

Therefore, $\{x, 0\} \subseteq f^n(J)$ for some large enough n. Thus, by Lemma 1.1, J has to f^n-cover either $[0, x]$ or $[x, y]$.

With F as given, it is easy to see that in the P-graph of f^n there are at least 2^{n-1} arrows each from $[0, x]$ to itself and to $[x, y]$, and from $[x, y]$ to itself and to $[0, x]$; there is at least one arrow from either $[0, x]$ or $[x, y]$ to J (depending on which one covers J), and at least one arrow from J to either $[0, x]$ or $[x, y]$. In any case, the submatrix B corresponding to this subgraph is irreducible (i.e., $B^m > 0$ for some $m > 0$) and so by Perron-Frobenius arguments, $r(B^m) > 2^{mn}$. Since $k \cdot \operatorname{ent}(f) = \operatorname{ent}(f^k)$ for any $k \geq 0$, and the corresponding entries in the P-matrix of f^{mn} are greater than or equal to that in B^m, by Lemma 1.3, $mn \cdot \operatorname{ent}(f) = \operatorname{ent}(f^{mn}) \geq \log r(B^m) > \log 2^{mn}$, i.e., $\operatorname{ent}(f) > \log 2$.

The same argument is valid if $J = [w, 0]$. (Here $\min F \mid [0, y] < 0$, and we look at the lift to $[-1, 0]$ of J.) ☐

Proof of Theorem 3.1. If f has no point of period 2, then by Theorem 0.1,

$$P(f) = \{1\} \cup \{j \in \mathbb{N} : j \geq k\}$$

and $\{j \in \mathbb{N} : j \geq k\} \neq \emptyset$ by Lemma 3.3 and Theorem 2.1. Let k be the smallest period greater than 1 in $P(f)$. Suppose that x has f-period k. Then $\rho(x) \neq 0$; otherwise, for some lift F of f, the lifts $e^{-1}\{x\}$ of x are all F-periodic of period k. By [**9**], $2 \in P(F)$. But a point $z \in \mathbb{R}$ of F-period two either projects to a fixed point of f or to a point of f-period two. Since f has no point of period two, z project to an f-fixed point, and so $F(z) = z + j$, $j \in \mathbb{Z} - \{0\}$. Since $\deg f = 1$, $z = F^2(z) = F(z + j) = z + 2j$, a contradiction. Thus $\rho(x) \neq 0$, and by Lemma 3.2, $\operatorname{ent}(f) \geq \log$ (largest zero of $x^{k+1} - x^k - x - 1$).

If there is a point of period two with nonzero rotation number, then Lemma 3.2 again implies that f has points of all periods and $\operatorname{ent}(f) \geq \log$ (largest zero of $x^3 - x^2 - x - 1$).

We will show that if there is a point of period two with rotation number zero, then f has a 2-horseshoe. By Lemma 1.4 f has points of all periods, and $\operatorname{ent}(f) \geq \log 2$. Proposition 3.4 will be used to show strict inequality.

Let $a < b$ in $(0, 1)$ be a lift of a period two orbit having rotation number zero. Then $F(a) = b + n$, $F(b) = a - n$ for some integer n. If $n > 0$, or if $n < -1$, then the intervals $[0, a]$, $[a, b]$, $[b, 1]$ indicate a 3-horseshoe for f and we are done.

So assume that either:

(i) $F(a) = b - 1$, $F(b) = a + 1$; or

(ii) $F(a) = b$, $F(b) = a$.

In either case, there is a fixed point q of F in $[a, b]$. We need look only at case (ii) since if (i) holds then $b < a + 1$ and both lie in $[q, q + 1]$. Since $F(b) = a + 1$, $F(a + 1) = b$, using q in place of 0, b in place of a, and $a + 1$ in place of b puts us in (ii).

It is clear that f has a 2-horseshoe if for some s, t, u, $0 \leq s < t < u \leq 1$, either

$$(*) \qquad F(s), F(u) \leq s \ \text{ and } \ F(t) \geq u;$$

or

$$(**) \qquad F(s), F(u) \geq u \ \text{ and } \ F(t) \leq s.$$

Let $a_0 = a$, $b_0 = b$. Assume that F has no 2-horseshoe in $[a_0, b_0]$. Since $[a, b]$ cannot be F-invariant and $[a, b] \subseteq F[a, b] \subseteq \dots$, we have $F[a, b] = [a_1, b_1]$, where $a_1 \leq a_0$ and $b_0 \leq b_1$. Notice that if a_1 is attained in $[a_0, q]$ then $(**)$ holds for $\{a_0, z, q\}$ where $a_0 < z < q$ and $F(z) = a_1$. Similarly, $(*)$ holds if b_1 is attained in $[q, b_0]$. So $[a_0, q]$ must F-cover $[b_0, b_1]$ and $[q, b_0]$ must F-cover $[a_1, a_0]$, and at least one of these intervals is nondegenerate.

Now suppose $a_1 \leq 0$ (resp., $b_1 \geq 1$). Then $(*)$ (resp., $(**)$) holds for $\{0, a_0, x\}$ (resp., $\{y, b_0, 1\}$) where $q < x < b_0$ and $F(x) = 0$ (resp., $a_0 < y < q$ and $F(y) = 1$). So we may suppose $a_1 > 0$, $b_1 < 1$.

Suppose there exist $a_1, \dots, a_n$; $b_1, \dots, b_n$ such that:

(1) $F[a_{k-1}, b_{k-1}] = [a_k, b_k] = [a_k, a_{k-1}] \cup [a_{k-1}, b_{k-1}] \cup [b_{k-1}, b_k] \qquad (1 \leq k \leq n)$

where at least one outside interval is nondegenerate;

(2) Neither $(*)$ nor $(**)$ holds in $[a_k, b_k]$, $0 \leq k \leq n - 1$.

(3) $0 < a_n \leq \dots \leq a_1 \leq a_0$; $b_0 \leq b_1 \leq \dots \leq b_n < 1$.

Then b_k is only attained in $[a_{k-1}, a_{k-2}]$, and a_k is only attained in $[b_{k-2}, b_{k-1}]$ for all $k > 1$, (resp., in $[a_0, q]$ and $[q, b + 0]$ if $k = 1$). If this process can go on forever then by transitivity of f, $\operatorname{cl}(\bigcup_{n \geq 0} [a_n, b_n]) = [0, 1]$. Evidently, there are $\{c_n\}_{n \geq 0}$ and $\{d_n\}_{n \geq 0}$ in $[0, 1]$ with $\lim c_n = 0$, $\lim d_n = 1$, $\lim F(c_n) = 1$ and $\lim F(d_n) = 0$. Since 0 and 1 are F-fixed, this is impossible. Thus for some $n \geq 0$, $(*)$ or $(**)$ must hold in one of $[0, b_n]$, $[a_n, b_n]$, or $[a_n, 1]$.

To see that $\text{ent}(f) > \log 2$, note that if $(*)$ or $(**)$ holds then we may assume F satisfies the conditions of Proposition 3.4 by looking, if necessary, at another interval $[z, z+1]$ in place of $[0,1]$. (For example, if a_1 is attained in (a_0, q), use $[q-1, q]$.) □

4. Examples

Block's examples in [**2**] of degree one maps f_k $(k \geq 2)$ have $P(f_k) = \{1\} \cup \{j \in \mathbb{N} : j \geq k\}$, and $\text{ent}(f_k)$ equal to the bound of Theorem 3.1. By [**4,** Theorem 3.1] the irreducibility of each P_k-matrix implies transitivity of f_k. Notice that f_1 has no point of period two with rotation number zero.

We show that the $\log 2$ bound is also sharp by the following sequence of degree one transitive maps each one having a fixed point and a period-two point with rotation number zero.

Define the lift F_0 by $F_0(0) = 0$, $F_0(1/6) = -1/3$, $F_0(1/3) = 0$, $F_0(1/2) = 2/3$, $F_0(2/3) = 1/3$, $F_0(1) = 1$, and linear between these points, and let P_0 be (the projection of) $\{0, 1/3, 2/3\}$.

For $n \geq 1$, define F_n by $F_n(0) = 0$, $F_n(1/(2^{n+1} \cdot 3)) = -1/3$, $F_n(1/(2^n \cdot 3) = 0$, $F_n(1/(2^{n-1} \cdot 3)) = 1/(2^n \cdot 3)$, $\ldots$, $F_n(1/3) = 1/(2 \cdot 3)$, $F_n(1/2) = 2/3$, $F_n(2/3) = 1/3$, $F_n(1) = 1$, and linear between these points, and let P_n be (the projection of) $\{0, 1/(2^n \cdot 3), 1/(2^{n-1} \cdot 3), \ldots, 1/3, 2/3\}$.

Application of [**4,** Theorem 3.1] again implies that all the the f_n's are transitive. The induced P_n-graphs indicate a point of period two with rotation number zero for f_n. By Lemma 1.3, $\text{ent}(f_n) = \log r_n$, where r_n is the largest zero of the characteristic polynomial $p_n(x) = x^{n+1} \cdot (x-1) \cdot (x-2) - 2$, $(n \geq 0)$ of the P_n-matrix. It is an elementary argument to show that $r_1 > r_2 > \cdots > 2$, and $\lim_{n \to \infty} r_n = 2$.

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