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# AFFINE COMPLETE ALGEBRAS ABSTRACTING KLEENE AND STONE ALGEBRAS

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Abstract. Boolean algebras are affine complete by a well-known result of G. Grätzer. Various generalizations of this result have been obtained. Among them, a characterization of affine complete Stone algebras having a smallest dense element was given by R. Beazer. In this paper, generalizations of Beazer's result are presented for algebras abstracting simultaneously Kleene and Stone algebras.

## 1. Introduction

G. Grätzer in [**6**] proved that all finitary functions on a Boolean algebra $B$ preserving the congruences of $B$ (he called such functions "Boolean", we shall use the usual term "compatible") are polynomials. Later on, in [**7**] he characterized bounded distributive lattices in which all compatible functions are polynomials. These were the first results leading to the study of affine complete algebras. H. Werner [**16**] calls an algebra $A$ **affine complete** if all finitary compatible functions on $A$ are polynomials. Further, an (infinite) algebra $A$ is said to be **locally affine complete**, if for every $n \geq 1$, every $n$-ary compatible function on $A$ can be interpolated on any finite subset $F \subseteq A^n$ by a polynomial of $A$. For various generalizations of Grätzer's results see [**11**]–[**15**] and [**1**], [**2**], [**9**].

R. Beazer in [**1**] characterized affine complete Stone algebras having a smallest dense element. This result is partially generalized to the class of all distributive $p$-algebras in [**9**]. Since Stone algebras form a subvariety of the MS-algebras introduced by T. S. Blyth and J. C. Varlet (see [**3**], [**4**]), it is natural to ask for a generalization of Beazer's result to MS-algebras. In this paper, investigations in this direction are presented.

We deal with the subvariety $K_2$ of MS-algebras whose members ($\mathrm{K}_2$-algebras) include Kleene algebras and Stone algebras. We first establish a characterization of locally affine complete $\mathrm{K}_2$-algebras (Theorem 1). We show that this characterization can be essentially simplified if $L$ is an infinite Stone algebra (Theorem 2). Theorem 2 can be considered as an affirmative answer to the "local version"

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of a question of R. Beazer (see Remark 3). For the class of so-called principal $\mathrm{K}_2$-algebras (which contains the class of Stone algebras having a smallest dense element investigated in [**1**]), an analogous characterization of affine complete members can be established (Theorem 3). Beazer's result in [**1**] immediately follows from this characterization (Corollary 6). Furthermore, several other consequences are presented, one of which asserts that finite Boolean algebras are the only affine complete finite $\mathrm{K}_2$-algebras.

## 2. Preliminaries

An **MS-algebra** is an algebra $\langle L; \vee, \wedge, ^\circ, 0, 1\rangle$ where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $^\circ$ is a unary operation such that for all $x, y \in L$,
(1) $x \leq x^{\circ\circ}$,
(2) $(x \wedge y)^\circ = x^\circ \vee y^\circ$,
(3) $1^\circ = 0$.

One can show that the following rules of computation hold in $L$:

$$\begin{aligned} (x \vee y)^\circ &= x^\circ \wedge y^\circ, \\ x^{\circ\circ\circ} &= x^\circ, \\ 0^\circ &= 1\,. \end{aligned}$$

The class of all MS-algebras is equational. The subvariety $\mathrm{K}_2$ of MS-algebras, which we deal with, is defined by the two additional identities,
(4) $x \wedge x^\circ = x^{\circ\circ} \wedge x^\circ$ and
(5) $(x \wedge x^\circ) \vee y \vee y^\circ = y \vee y^\circ$,
and the lattice of its subvarieties is drawn on Figure 1.

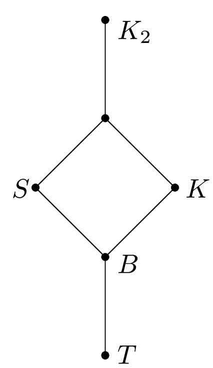


**Figure 1.**

The subvarieties of $K_2$ denoted by $T$, $B$, $S$, $K$ are the classes of all trivial, Boolean, Stone and Kleene algebras, respectively and are characterized in $K_2$ by the identities $T : x = y$, $B : x \vee x^\circ = 1$, $S : x \wedge x^\circ = 0$ and $K : x = x^{\circ\circ}$, respectively.

Let $L$ be an algebra from the subvariety $K_2$. Then

(i) $L^{\circ\circ} = \{x \in L;\ x = x^{\circ\circ}\}$ is a Kleene subalgebra of $L$;
(ii) $L^{\wedge} = \{x \wedge x^\circ;\ x \in L\}$ is an ideal of $L$;
(iii) $L^{\vee} = \{x \vee x^\circ;\ x \in L\}$ is a filter of $L$.

If $L$ is a Stone algebra, then the operation $^\circ$ is that of pseudocomplementation, $L^{\wedge} = \{0\}$, $L^{\circ\circ}$ is the Boolean algebra $B(L)$ of all closed elements of $L$ and $L^{\vee}$ is the filter $D(L)$ of all dense elements of $L$ (see [**8**] or [**1**]).

By a function on an algebra $L$ we always mean a finitary function. Functions on $L$ preserving the congruences of $L$ are called **compatible**. Furthermore, a partial function on an algebra $L$ is said to be compatible, if it preserves the congruences of $L$ where defined. The set of all total compatible (order-preserving) functions on a lattice $L$ will be denoted by $\mathcal{C}(L)$ ($\mathcal{OF}(L)$).

The members of the variety $K_2$ are called $\mathrm{K}_2$**-algebras**. We shall say that a $\mathrm{K}_2$-algebra $L$ is **principal**, if the filter $L^{\vee}$ is principal, i.e. $L^{\vee} = [d)$ for some element $d \in L$. A simple construction of principal $\mathrm{K}_2$-algebras is presented in [**10**].

For other basic results on MS-algebras we refer the reader to [**3**] and [**4**].

## 3. Affine Completeness

We start with some preliminary results.

**Proposition 1** ([**7**; Corollaries 1, 3]). *Let $L$ be a bounded distributive lattice. Then the following conditions are equivalent:*

(i) *$L$ is affine complete;*
(ii) *$\mathcal{C}(L) \subseteq \mathcal{OF}(L)$;*
(iii) *$L$ contains no proper Boolean interval.*

**Proposition 2** ([**5**; Theorem 4, Corollary 1]). *For any distributive lattice $L$ the following conditions are equivalent:*

(i) *$L$ is locally affine complete;*
(ii) *$\mathcal{C}(L) \subseteq \mathcal{OF}(L)$;*
(iii) *$L$ contains no proper Boolean interval.*

**Lemma 1.** *Let $(D; \vee, \wedge, 0, 1)$ be a bounded distributive lattice. Let $f', g' : D^n \to D$ be partial compatible functions on $D$ with domains $F$ and $G$ ($F, G \subseteq D^n$), respectively and let $S = F \cap G$. Let $S \cap \{0,1\}^n \neq \varnothing$ and $h(S \cap \{0,1\}^n) = h(S)$[1] for every $0,1$-homomorphism $h$ from $D$ onto a 2-element lattice. Then $f' \equiv g'$ identically on $S$ iff $f' \equiv g'$ identically on $S \cap \{0,1\}^n$.*

[1]Here $h(S)$ denotes the set $\{(h(x_1), \dots, h(x_n));\ (x_1, \dots, x_n) \in S\}$

*Proof.* Let $f' \equiv g'$ identically on $S \cap \{0,1\}^n$. Suppose on the contrary that there exists an $n$-tuple $(d_1, \dots, d_n) \in S$ such that $f'(d_1, \dots, d_n) = a \neq b = g'(d_1, \dots, d_n)$. Since $a, b \in D$ and $D$ is a subdirect product of copies of 2-element lattices, there exists a "projection" $h\colon D \to \{\underline{0}, \underline{1}\}$, which is a $0,1$-homomorphism between $D$ and some lattice $\mathbf{2} = \{\underline{0}, \underline{1}\}$, such that $h(a) \neq h(b)$. Define functions $f_2',\ g_2'\colon h(S) \to \{\underline{0}, \underline{1}\}$ by the following rules:

$$f_2'(h(x_1), \dots, h(x_n)) = h(f'(x_1, \dots, x_n)),$$
$$g_2'(h(x_1), \dots, h(x_n)) = h(g'(x_1, \dots, x_n)).$$

Obviously, $f_2',\ g_2'$ are well-defined. Furthermore, $f_2' \equiv g_2'$ identically on $h(S)$, because $h(S) = h(S \cap \{0,1\}^n)$ and $f' \equiv g'$ identically on $S \cap \{0,1\}^n$. Therefore $h(a) = h(f'(d_1, \dots, d_n)) = f_2'(h(d_1), \dots, h(d_n)) = g_2'(h(d_1), \dots, h(d_n)) = h(g'(d_1, \dots, d_n)) = h(b)$, a contradiction. Hence $f' \equiv g'$ identically on $S$ and the proof is complete. □

The following lemma states a canonical form of any polynomial function on an MS-algebra and generalizes a similar result for Stone algebras (see [**1**; Lemma 1]).

**Lemma 2.** *Any polynomial function $p(x_1, \dots, x_n)$ on an* MS-*algebra $L$ can be represented in the form*

$$p(x_1, \dots, x_n) = \bigvee_{\tilde{\imath}, \tilde{\jmath} \in \{0,1,2,3\}^n,\ \tilde{\imath} < \tilde{\jmath}} [\alpha(i_1, j_1, \dots, i_n, j_n) \wedge x_1^{i_1} \wedge x_1^{j_1} \wedge \cdots \wedge x_n^{i_n} \wedge x_n^{j_n}]$$

*and dually, in the form*

$$p(x_1, \dots, x_n) = \bigwedge_{\tilde{\imath}, \tilde{\jmath} \in \{0,1,2,3\}^n,\ \tilde{\imath} < \tilde{\jmath}} [\beta(i_1, j_1, \dots, i_n, j_n) \vee x_1^{i_1} \vee x_1^{j_1} \vee \cdots \vee x_n^{i_n} \vee x_n^{j_n}]$$

*where the join $\bigvee$ and the meet $\bigwedge$ are taken over all $n$-tuples $\tilde{\imath} = (i_1, \dots, i_n)$, $\tilde{\jmath} = (j_1, \dots, j_n) \in \{0,1,2,3\}^n$, the coefficients $\alpha(i_1, \dots, j_n)$, $\beta(i_1, \dots, j_n) \in L$ and $x^0, x^1, x^2$ and $x^3$ denote $1$ ($0$ in the dual form), $x, x^\circ$ and $x^{\circ\circ}$, respectively (i.e. $x^0$ means that the variable $x$ can be omitted in a given conjunction (disjunction)).*

*Proof.* It follows from the facts that

$$(x \vee y)^\circ = x^\circ \wedge y^\circ,\ (x \wedge y)^\circ = x^\circ \vee y^\circ,\ x^{\circ\circ\circ} = x^\circ,\ x \leq x^{\circ\circ} \quad \text{for any } x, y \in L$$

and that the lattice $L$ is distributive. □

**Proposition 3.** *Let $L$ be an* MS-*algebra. If $L$ is (locally) affine complete, then so is $L^{\circ\circ}$.*

*Proof.* Let $L$ be a (locally) affine complete MS-algebra. Let $f'\colon (L^{\circ\circ})^n \to L^{\circ\circ}$ be a compatible function (and $F \subseteq (L^{\circ\circ})^n$ be a finite set). Define a function

$f\colon L^n \to L$ by $f(x_1,\dots,x_n) = f'(x_1^{\circ\circ},\dots,x_n^{\circ\circ})$. Obviously $f$ is compatible, since $f'$ is compatible, so $f$ can be represented (on the set $F$) by a polynomial $p(x_1,\dots,x_n)$ of $L$. Hence, for all $\tilde{\mathrm{x}} = (x_1,\dots,x_n) \in (L^{\circ\circ})^n$ ($\tilde{\mathrm{x}} \in F$), we have $f'(\tilde{\mathrm{x}}) = f(\tilde{\mathrm{x}}) = p(\tilde{\mathrm{x}}) = p(\tilde{\mathrm{x}})^{\circ\circ}$, since $f'(\tilde{\mathrm{x}}) \in L^{\circ\circ}$. Using Lemma 2, all constants in $p(x_1,\dots,x_n)^{\circ\circ}$ are elements of $L^{\circ\circ}$. Thus $f'$ can be represented (on $F$) by a polynomial of $L^{\circ\circ}$. □

**Lemma 3.** *Let $L$ be a $\mathrm{K}_2$-algebra, $a \in L$ and $b \in L^\vee$. If $b \le a \vee a^\circ$ then $a = a^{\circ\circ} \wedge (a \vee b)$.*

*Proof.* By the distributivity of $L$, $a^{\circ\circ} \wedge (a \vee b) = a \vee (a^{\circ\circ} \wedge b)$. It suffices to show that $a^{\circ\circ} \wedge b = a \wedge b$. Put $x = a^{\circ\circ} \wedge b$, $y = a^\circ \wedge b$, $z = a \wedge b$. Using the identity (4) and the hypothesis we get

$$\begin{aligned} x \wedge y &= a^{\circ\circ} \wedge a^\circ \wedge b = a \wedge a^\circ \wedge b = z \wedge y \qquad \text{and} \\ x \vee y &= (a^{\circ\circ} \vee a^\circ) \wedge b = b = (a \vee a^\circ) \wedge b = z \vee y\,. \end{aligned}$$

Now $x = z$ follows immediately from the distributivity of $L$. □

**Lemma 4.** *Let $L$ be a $\mathrm{K}_2$-algebra and $x, y \in L$. If $x, y \in L^\vee$ then $x^\circ \le y$. If $x, y \in L^\wedge$ then $x^\circ \ge y$.*

*Proof.* If $x, y \in L^\vee$ then $x = a \vee a^\circ$, $y = b \vee b^\circ$ for some $a, b \in L$. Thus $x^\circ = a^\circ \wedge a^{\circ\circ} \le b \vee b^\circ = y$ by (5). The second statement can be shown analogously. □

**Lemma 5.** *Let $L$ be a $\mathrm{K}_2$-algebra, $f\colon L^n \to L$ be a compatible function on $L$, $F \subseteq L^n$ and $b \in L^\vee$. Let $f'_F\colon [b,1]^{3n} \to [b,1]$ be a partial function such that*

$$f'_F(x_1 \vee b,\dots,x_n \vee b, x_1^\circ \vee b,\dots,x_n^\circ \vee b, x_1^{\circ\circ} \vee b,\dots,x_n^{\circ\circ} \vee b) = f(x_1,\dots,x_n) \vee b \quad (\tilde{x} \in F)$$

*and $f'_F$ is undefined elsewhere. Then $f'_F$ is a well-defined partial compatible function of the lattice $[b,1]$.*

*Proof.* For any lattice congruence $\theta_b$ of $[b,1]$ we define an equivalence relation $\theta$ on $L$ by $x \equiv y(\theta)$ iff

(a) $x \vee b \equiv y \vee b\ (\theta_b)$ and $x^\circ \vee b \equiv y^\circ \vee b\ (\theta_b)$ and $x^{\circ\circ} \vee b \equiv y^{\circ\circ} \vee b\ (\theta_b)$.

It is easy to verify that $\theta$ is a congruence of the algebra $L$. Therefore, if some pairs $(x_i, y_i)$, $i = 1,\dots,n$ satisfy (a), then $x_i \equiv y_i(\theta)$, and since $f$ is compatible, $f(x_1,\dots,x_n) \equiv f(y_1,\dots,y_n)(\theta)$. Hence, $f(\tilde{\mathrm{x}}) \vee b \equiv f(\tilde{\mathrm{y}}) \vee b\ (\theta_b)$ again by (a). Thus $f'_F$ preserves the congruences of $[b,1]$ where defined. To show that $f'_F$ is well-defined, it suffices to take $\theta_b = \omega$, the smallest congruence of $[b,1]$. □

**Definition 1.** A $K_2$-algebra $L$ satisfies the condition **(FD)** if for any compatible function $f\colon L^n \to L$, any element $b \in L^\vee$ and any finite set $F \subseteq L^n$, the partial compatible function $f'_F$ defined above can be extended to a total compatible function of the lattice $[b, 1]$.

**Convention.** In what follows, the $3n$-tuples $(x_1, \dots, x_n, x_1^\circ, \dots, x_n^\circ, x_1^{\circ\circ}, \dots, x_n^{\circ\circ})$ will be shortly written as $(\tilde{x}, \tilde{x}^\circ, \tilde{x}^{\circ\circ})$, and the $3n$-tuples $(x_1 \vee b, \dots, x_n \vee b, x_1^\circ \vee b, \dots, x_n^\circ \vee b, x_1^{\circ\circ} \vee b, \dots, x_n^{\circ\circ} \vee b)$ will be abbreviated as $(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b)$.

**Theorem 1.** *Let $L$ be a $K_2$-algebra. The following two conditions are equivalent:*

(1) *$L$ is locally affine complete;*

(2) (i) *$L^\vee$ is locally affine complete distributive lattice and*

(ii) *$L^{\circ\circ}$ is locally affine complete Kleene algebra and*

(iii) (FD).

*Proof.* (1) $\Longrightarrow$ (2)(i). To show that the lattice $L^\vee$ is locally affine complete, it suffices to show (by Proposition 2) that $\mathcal{C}(L^\vee) \subseteq \mathcal{OF}(L^\vee)$. Suppose to the contrary that there exists a compatible function $f'\colon (L^\vee)^n \to L^\vee$ which is not order-preserving, i.e. $f(u) > f(v)$ for some $u, v \in (L^\vee)^n$, $u < v$. Define a function $f\colon L^n \to L$ as follows: $f(x_1, \dots, x_n) = f'(x_1 \vee x_1^\circ, \dots, x_n \vee x_n^\circ)$ for any $(x_1, \dots, x_n) \in L^n$. Obviously, $f \restriction (L^\vee)^n = f'$ (Lemma 4) and $f$ is compatible on $L$. By hypothesis, for any finite set $F \subseteq L^n$, the function $f$ can be interpolated on $F$ by a polynomial function of $L$. Thus, using Lemma 2, for all $\tilde{x} \in F \subseteq (L^\vee)^n$ we can write

$$\text{(b)}\quad f'(\tilde{x}) = f(\tilde{x}) = \bigwedge_{\tilde{\imath}, \tilde{\jmath} \in \{0,1,2,3\}^n,\ \tilde{\imath} < \tilde{\jmath}} [\beta(i_1, j_1, \dots, i_n, j_n) \vee x_1^{i_1} \vee x_1^{j_1} \vee \dots \vee x_n^{i_n} \vee x_n^{j_n}]$$

Since $\{f'(\tilde{x});\ \tilde{x} \in F\}$ is a finite subset of $L^\vee$, there exists an element $d \in L^\vee$ such that $f'(\tilde{x}) = f'(\tilde{x}) \vee d$ for all $\tilde{x} \in F$. Furthermore, by Lemma 4, the terms $x_i^\circ$ can be omitted in (b). Hence for all $\tilde{x} \in F$ we get

$$f'(\tilde{x}) = \bigwedge_{\tilde{\imath}, \tilde{\jmath} \in \{0,1,3\}^n,\ \tilde{\imath} < \tilde{\jmath}} [(\beta(i_1, j_1, \dots, i_n, j_n) \vee d) \vee x_1^{i_1} \vee x_1^{j_1} \vee \dots \vee x_n^{i_n} \vee x_n^{j_n}].$$

Now it is evident that $f'$ is an order-preserving function on $F$. For $F = \{u, v\}$ this contradicts $f(u) > f(v)$.

(1) $\Longrightarrow$ (2)(ii) This follows from Proposition 3.

(1) $\Longrightarrow$ (2)(iii) Let $f\colon L^n \to L$ be a compatible function on $L$, $F \subseteq L^n$ be a finite set, $b \in L^\vee$ and $f'_F$ be the partial compatible function defined in Lemma 5. Obviously, the function $f_1\colon L^n \to [b, 1]$, $f_1(\tilde{x}) = f(\tilde{x}) \vee b$ is compatible on $L$. Thus $f_1$ can be interpolated on $F$ by a polynomial $p(x_1, \dots, x_n)$ of the algebra $L$. Using the formulas $(x \wedge y)^\circ = x^\circ \vee y^\circ$, $(x \vee y)^\circ = x^\circ \wedge y^\circ$ and $x^{\circ\circ\circ} = x^\circ$,

the polynomial $p(\tilde{x})$ can be rewritten as $l(\tilde{x}, \tilde{x}^\circ, \tilde{x}^{\circ\circ})$ where $l(x_1, \dots, x_{3n})$ is a lattice polynomial of $L$. Furthermore, if $a_1, \dots, a_m$ are all constant symbols appearing in $l$, then $l(x_1, \dots, x_{3n})$ is a term $t(x_1, \dots, x_{3n}, a_1, \dots, a_m)$ of the algebra $L_1 = (L; \vee, \wedge, a_1, \dots, a_m)$. Hence for any $\tilde{x} \in F$, $f'_F(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b) = f_1(\tilde{x}) = l(\tilde{x}, \tilde{x}^\circ, \tilde{x}^{\circ\circ}) = t(\tilde{x}, \tilde{x}^\circ, \tilde{x}^{\circ\circ}, a_1, \dots, a_m)$. Since $f_1(\tilde{x}) = f_1(\tilde{x}) \vee b$ and the mapping $\varphi\colon L \to L^\vee$ defined by $\varphi(x) = x \vee b$ is a lattice homomorphism, we have $t(\tilde{x}, \tilde{x}^\circ, \tilde{x}^{\circ\circ}, a_1, \dots, a_m) = \varphi(t(\tilde{x}, \tilde{x}^\circ, \tilde{x}^{\circ\circ}, a_1, \dots, a_m)) = t(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b, a_1 \vee b, \dots, a_m \vee b) = l'(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b)$, where $l'(x_1, \dots, x_{3n})$ is now a lattice polynomial of the lattice $L^\vee$. Hence the partial function $f'_F$ can be extended to a total polynomial function $l'(x_1, \dots, x_{3n})$ of the lattice $L^\vee$. Thus (FD) holds in $L$.

(2) $\Longrightarrow$ (1) Let $f\colon L^n \to L$ be a compatible function of $L$ and $F$ be a finite subset of $L^n$. The finiteness of $F$ guarantees that there exists an element $b \in L^\vee$ such that $f(\tilde{x}) \vee f(\tilde{x})^\circ \in [b, 1]$ for all $\tilde{x} \in F$. Thus by Lemma 3,

$$f(\tilde{x}) = f(\tilde{x})^{\circ\circ} \wedge (f(\tilde{x}) \vee b) \qquad \text{for all } \tilde{x} \in F. \tag{c}$$

Obviously, the function $f_1\colon (L^{\circ\circ})^n \to L^{\circ\circ}$ defined by $f_1(x_1^{\circ\circ}, \dots, x_n^{\circ\circ}) = f(x_1, \dots, x_n)^{\circ\circ}$ is well-defined because $x \to x^{\circ\circ}$ is an endomorphism of $L$. We show that $f_1$ is compatible on $L^{\circ\circ}$. Let $\theta_1$ be a congruence of $L^{\circ\circ}$, $x_i$, $y_i \in L^{\circ\circ}$, $i = 1, \dots, n$ and $x_i \equiv y_i(\theta_1)$. Evidently, the relation $\theta$ defined on L by $x \equiv y(\theta)$ iff $x^{\circ\circ} \equiv y^{\circ\circ}(\theta_1)$ is a congruence of $L$ extending $\theta_1$. So we have $x_i \equiv y_i(\theta)$, $i = 1, \dots, n$. Since $f$ is compatible, we get $f(\tilde{x}) \equiv f(\tilde{y})(\theta)$. Thus $f(\tilde{x})^{\circ\circ} \equiv f(\tilde{y})^{\circ\circ}(\theta_1)$, and so $f_1$ is compatible on $L^{\circ\circ}$. By hypothesis, $f_1$ can be interpolated on the finite set $\{(x_1^{\circ\circ}, \dots, x_n^{\circ\circ});\ \tilde{x} \in F\}$ by a polynomial $k(x_1, \dots, x_n)$ of $L^{\circ\circ}$. Thus for all $\tilde{x} \in F$ we get $f(x_1, \dots, x_n)^{\circ\circ} = f_1(x_1^{\circ\circ}, \dots, x_n^{\circ\circ}) = k(x_1^{\circ\circ}, \dots, x_n^{\circ\circ})$, and so in (c), $f(\tilde{x})^{\circ\circ}$ can be replaced by a polynomial of the algebra $L$.

Now consider the partial function $f'_F\colon [b, 1]^{3n} \to [b, 1]$ defined in Lemma 5. Using (FD), $f'_F$ can be extended to a total compatible function $f_2$ of the lattice $[b, 1]$. By hypothesis and Proposition 2, $f_2$ can be represented on the finite set $\{(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b);\ \tilde{x} \in F\}$ by a lattice polynomial $l(x_1, \dots, x_{3n})$. Hence for any $\tilde{x} \in F$ we have

$$f(\tilde{x}) \vee b = f'_F(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b) = f_2(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b) = l(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b),$$

and consequently,

$$f(\tilde{x}) = k(\tilde{x}^{\circ\circ}) \wedge l(\tilde{x} \vee b, \tilde{x}^\circ \vee b, \tilde{x}^{\circ\circ} \vee b).$$

This proves that the algebra $L$ is locally affine complete. □

**Remark 1.** One can easily show that the local affine completeness of the algebra $L$ also yields the local affine completeness of the lattice $L^\wedge$.

**Lemma 6.** *Let $L$ be a Stone algebra, $b \in L^\vee$ and $x, y \in L$. Then*

$$x^\circ \vee b = y^\circ \vee b \qquad \text{iff} \qquad x^{\circ\circ} \vee b = y^{\circ\circ} \vee b\,.$$

*Proof.* Let $x^\circ \vee b = y^\circ \vee b$. Since $y^\circ \vee y^{\circ\circ} = 1$ holds in $L$, we have

$$\begin{aligned} x^{\circ\circ} \vee b &= (y^\circ \vee b \vee y^{\circ\circ}) \wedge (x^{\circ\circ} \vee b) = [(x^\circ \vee b) \wedge (x^{\circ\circ} \vee b)] \vee [y^{\circ\circ} \wedge (x^{\circ\circ} \vee b)] \\ &= [(x^\circ \wedge x^{\circ\circ}) \vee b] \vee [y^{\circ\circ} \wedge (x^{\circ\circ} \vee b)] = (y^{\circ\circ} \vee b) \wedge (x^{\circ\circ} \vee b) \end{aligned}$$

using (5).

Hence $x^{\circ\circ} \vee b \le y^{\circ\circ} \vee b$. Similarly, $y^{\circ\circ} \vee b \le x^{\circ\circ} \vee b$. The converse statement can be proved analogously. □

**Lemma 7.** *Let $L$ be a $\mathrm{K}_2$-algebra, $b \in L^\vee$ and $x \in L$ such that $(x \vee b, x^\circ \vee b, x^{\circ\circ} \vee b) \in \{b, 1\}^3$. Then $x^\circ \vee b = 1$ implies $x \vee b = x^{\circ\circ} \vee b = b$. Furthermore, $x \vee b = 1$ yields $x^\circ \vee b = b$ and $x^{\circ\circ} \vee b = 1$.*

*Proof.* If $b = 1$ then the statement is obvious. So let $b \ne 1$. Let $x^\circ \vee b = 1$. If also $x^{\circ\circ} \vee b = 1$, then $1 = (x^\circ \vee b) \wedge (x^{\circ\circ} \vee b) = (x^\circ \wedge x^{\circ\circ}) \vee b = b$ by (5), a contradiction. Hence $x \vee b = x^{\circ\circ} \vee b = b$. Now let $x \vee b = 1$. Then again, $x^\circ \vee b = 1$ would mean that $1 = (x \wedge x^\circ) \vee b = b$; a contradiction. Therefore $x^\circ \vee b = b$. □

**Remark 2.** If $L$ is a finite $\mathrm{K}_2$-algebra then in Theorem 1 as well as in the next results, the term "locally affine complete" can be replaced by the term "affine complete". However, in the following results the finite case would not be interesting to investigate. Therefore we confine our considerations to infinite algebras.

**Theorem 2.** *Let $L$ be an infinite Stone algebra. The following conditions are equivalent:*

(i) *$L$ is locally affine complete;*
(ii) *$L^\vee$ is locally affine complete distributive lattice;*
(iii) *No proper interval of $L^\vee$ is Boolean.*

*Proof.* If $L$ is a Stone algebra then $L^\wedge = \{0\}$ and $L^{\circ\circ}$ is a Boolean algebra. Therefore by Theorem 1 it suffices to show that the local affine completeness of $L^\vee$ $(= D(L))$ yields (FD).

So let $L^\vee$ contain no proper Boolean interval. For an $n$-ary compatible function $f$ on $L$, a finite set $F \subseteq L^n$ and an element $b \in L^\vee$ take the function $f'_F$ from Lemma 5. Take its partial extension $f' = f'_{L^n}$ with the domain $S = \{(\tilde{\mathrm{x}} \vee b, \tilde{\mathrm{x}}^\circ \vee b, \tilde{\mathrm{x}}^{\circ\circ} \vee b);\ \tilde{\mathrm{x}} \in L^n\}$ (see again Lemma 5). Define a polynomial $p(x_1, \dots, x_{3n})$ of the lattice $[b, 1]$ by

$$p(x_1, \dots, x_{3n}) = \bigvee_{\tilde{\mathrm{a}} \in S \cap \{b,1\}^{3n}} (f'(a_1, \dots, a_{3n}) \wedge y_1 \wedge \dots \wedge y_{3n}),$$

where $y_i = \begin{cases} x_i & \text{if } a_i = 1, \\ 1 & \text{if } a_i = b. \end{cases}$

We show that $f' \equiv p$ on $S \cap \{b,1\}^{3n}$.

Take any $\tilde{x} = (x_1, \dots, x_{3n}) \in S \cap \{b,1\}^{3n}$. Assume first that $\tilde{a} \in S \cap \{b,1\}^{3n}$, $\tilde{a} \neq \tilde{x}$ and $a_j \neq x_j$ for some $j$, $n < j \leq 3n$. Hence $a_j$ and $x_j$ are elements of the set $\{x^\circ \vee b;\ x \in L\}$ or the set $\{x^{\circ\circ} \vee b;\ x \in L\}$. By Lemma 6 we can suppose that $a_j \neq x_j$ for some $n < j \leq 2n$. If $a_j = 1$ then evidently $f'(a_1, \dots, a_{3n}) \wedge y_1 \wedge \dots \wedge y_{3n} = b$, if $a_j = b$ then $x_j = x^\circ \vee b = 1$ for some $x \in L$, thus by Lemmas 6, 7 $a_{j+n} = 1$ and again $f'(a_1, \dots, a_{3n}) \wedge y_1 \wedge \dots \wedge y_{3n} = b$. Now let $\tilde{a} \in S \cap \{b,1\}^{3n}$, $\tilde{a} \neq \tilde{x}$ and $a_j > x_j$ for some $j$, $1 \leq j \leq n$. Again it is clear that $f'(a_1, \dots, a_{3n}) \wedge y_1 \wedge \dots \wedge y_{3n} = b$. Hence we have shown that

$$p(x_1, \dots, x_{3n}) = \bigvee_{\tilde{a} \in S \cap \{b,1\}^{3n},\ \tilde{a} \leq \tilde{x}} (f'(a_1, \dots, a_n, x_{n+1}, \dots, x_{3n})\,.$$

Take any $\tilde{a} \in S \cap \{b,1\}^{3n}$ such that $a_i \leq x_i$ for $i = 1, \dots, n$, $a_j \neq x_j$ for some $1 \leq j \leq n$ and $a_i = x_i$ for $i = n+1, \dots, 3n$. We show that $f'(a_1, \dots, a_n, x_{n+1}, \dots, x_{3n}) \leq f'(x_1, \dots, x_{3n})$. Denote $z_k = a_k$ if $a_k = x_k$, otherwise $z_k = z$, $1 \leq k \leq n$. We define a total function of one variable $g\colon [b,1] \to [b,1]$ by $g(z) = f'(z_1, \dots, z_n, x_{n+1}, \dots, x_{3n})$. Obviously, $g$ is compatible on $[b,1]$ and $f'(a_1, \dots, a_n, x_{n+1}, \dots, x_{3n}) = g(b)$, $f'(x_1, \dots, x_{3n}) = g(1)$. Hence we need to show that $g(b) \leq g(1)$. For any $z \in [b,1]$ we have $g(b) \equiv g(z)$ $(\theta_{\mathrm{lat}}(b,z))$ and $g(z) \equiv g(1)$ $(\theta_{\mathrm{lat}}(z,1))$. Therefore

$$g(z) \vee z = g(b) \vee z \qquad \text{and}$$
$$g(z) \wedge z = g(1) \wedge z\,.$$

Thus for any $z \in [g(1), g(b) \vee g(1)]$, $g(z)$ is the relative complement of $z$ in this interval. Consequently, $[g(1), g(b) \vee g(1)]$ is a Boolean interval of $[b,1]$. By hypothesis this yields $g(b) \leq g(1)$, as required.

Hence $p \equiv f'$ on $S \cap \{b,1\}^{3n}$. To apply Lemma 1 to the functions $f'$, $p$, it remains to show that $h(S \cap \{b,1\}^{3n}) = h(S)$ for any $0,1$-lattice homomorphism $h$ from $[b,1]$ onto a 2-element lattice $\mathbf{2} = \{\underline{0}, \underline{1}\}$. Note that for any $x \in L$ we have

$$h(x^\circ \vee b) \vee h(x^{\circ\circ} \vee b) = h(x^\circ \vee x^{\circ\circ} \vee b) = h(1) = \underline{1}\,,$$
$$h(x^\circ \vee b) \wedge h(x^{\circ\circ} \vee b) = h((x^\circ \wedge x^{\circ\circ}) \vee b) = h(b) = \underline{0}\,,$$

and analogously,

$$h(x \vee b) \wedge h(x^\circ \vee b) = h(b) = \underline{0}\,.$$

So the triples $(h(x \vee b), h(x^\circ \vee b), h(x^{\circ\circ} \vee b))$ as components of every $3n$-tuple in $h(S)$ are only of the form $(\underline{0}, \underline{1}, \underline{0})$ or $(\underline{1}, \underline{0}, \underline{1})$ or $(\underline{0}, \underline{0}, \underline{1})$. Thus when finding the associated triples (their preimages in $h$) $(x \vee b, x^\circ \vee b, x^{\circ\circ} \vee b) \in S \cap \{b,1\}^3$, it suffices to take $x$ equal to $0, 1$ and $b$, respectively. Therefore $h(S) = h(S \cap \{b,1\}^{3n})$ and, by Lemma 1, $p(x_1, \dots, x_{3n})$ is a total compatible extension of the partial function $f'$, hence the required extension of the partial function $f'_F$. □

**Lemma 8.** *Let $L$ be an infinite* $\mathrm{K}_2$*-algebra. If $L^\vee$ has a smallest element then $L^\wedge$ has a greatest element. If $L^\vee$ is finite then $L^\wedge$ is finite. If $L$ is Kleene algebra then $L^\vee \cong (L^\wedge)^d$.*

*Proof.* The first statement follows from the fact that the mapping $x \to x^\circ$ is a dual endomorphism of $L^\vee$ onto $L^\wedge$. Let $L^\vee$ be finite. The mapping $x \to x^\circ$ is a dual embedding of $L^\wedge$ into $L^\vee$. Thus $L^\wedge$ is finite. If $L$ is a Kleene algebra then $x \to x^\circ$ define a dual isomorphism between $L^\vee$ and $L^\wedge$. $\square$

**Corollary 1.** *Let $L$ be an infinite* $\mathrm{K}_2$*-algebra such that $L^\wedge$ is finite. The following conditions are equivalent:*

(i) *$L$ is locally affine complete;*
(ii) *$L$ is locally affine complete Stone algebra.*

*Proof.* If $L$ is locally affine complete, then also $L^\wedge$ is locally affine complete (see Remark 1). By Proposition 2 this yields $|L^\wedge| = 1$ since $L^\wedge$ is finite. Thus $x \wedge x^\circ = 0$ for all $x \in L$ and $L$ is a Stone algebra. $\square$

**Corollary 2.** *Let $L$ be an infinite Kleene algebra such that $L^\wedge(L^\vee)$ is finite. Then $L$ is (locally) affine complete if and only if $L$ is a Boolean algebra.*

**Example 1.** Let $D$ be a dense-in-itself chain with 1, e.g. $D$ is the interval $(0, 1]$ in the real numbers. If we adjoin a new zero $\underline{0}$ and put $a^\circ = \underline{0}$ for all $a \in D$, then we obviously obtain a Stone algebra $L$ (see Figure 2). By Theorem 2, $L$ is locally affine complete because $L^\vee = D$ has no proper Boolean interval. Now, let $D = [0, 1]$ and $0^\circ = 0$, $a^\circ = \underline{0}$ for every $a > 0$ (see Figure 3). We obtain a $\mathrm{K}_2$-algebra $L$ in which $L^\vee = D$ has no proper Boolean interval again. But $L$ is not (locally) affine complete because $L$ is not a Stone algebra using Corollary 1. $\square$

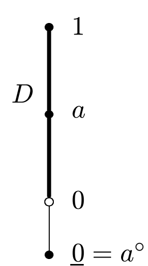


**Figure 2.**

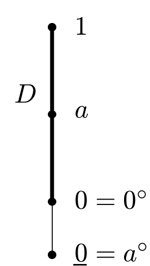


**Figure 3.**

To achieve similar results concerning affine completeness, we ought to confine our considerations to principal $\mathrm{K}_2$-algebras.

So let $(L; \vee, \wedge, ^\circ, 0, 1)$ be a principal $\mathrm{K}_2$-algebra such that $L^\vee = [d)$.

**Definition 2.** We shall say that $L$ satisfies the condition **(D)** if for any compatible function $f\colon L^n \to L$, the partial function $f' = f'_{L^n}$ (see Lemma 5) with the domain $S = \{(\tilde{x} \vee d, \tilde{x}^\circ \vee d, \tilde{x}^{\circ\circ} \vee d);\ \tilde{x} \in L^n\}$ can be extended to a total compatible function of the lattice $L^\vee$.

Repeating the proof of Theorem 1 with $d$ playing the role of the element $b$ everywhere and (D) used instead of (FD), we get the following generalization of R. Beazer's result.

**Theorem 3.** *Let $L$ be a principal* $\mathrm{K}_2$*-algebra such that $L^\vee = [d)$. Then the following two conditions are equivalent:*
(1) *$L$ is affine complete;*
(2) (i) *$L^\vee$ is an affine complete distributive lattice and*
(ii) *$L^{\circ\circ}$ is an affine complete Kleene algebra and*
(iii) (D).

**Corollary 3.** *Let $L$ be a* $\mathrm{K}_2$*-algebra such that $L^\wedge$ is finite. Then $L$ is affine complete if and only if $L$ is an affine complete Stone algebra.*

*Proof.* This can be done in the same way as that of Corollary 1. □

**Corollary 4.** *Let $L$ be a* $\mathrm{K}_2$*-algebra such that $L^\vee$ is finite. Then $L$ is affine complete if and only if $L$ is a Boolean algebra.*

**Corollary 5.** *Finite Boolean algebras are the only finite affine complete* $\mathrm{K}_2$*-algebras.*

Analogously as in Theorem 2, one can show that affine completeness of a principal $\mathrm{K}_2$-algebra $L$ yields (D). Hence from Theorem we immediately get R. Beazer's characterization of affine complete Stone algebras having a smallest dense element, i.e. (in our terminology) principal Stone algebras:

**Corollary 6 ([1; Theorem 4]).** *Let $L$ be a principal Stone algebra. Then the following conditions are equivalent:*

(i) *$L$ is affine complete;*
(ii) *$L^\vee$ is affine complete;*
(iii) *No proper interval of $L^\vee$ is a Boolean algebra.*

**Remark 3.** R. Beazer in [**1**] asked whether the equivalence (i) and (ii) in this result holds also for $L$ not having a smallest dense element (i.e. if $L$ is not principal). Theorem 2 can be considered as a positive answer to this question in its "local version".[2]

[2] The author together with M. Ploščica have shown (in an unpublished paper) that the mentioned equivalence does not hold in general.

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