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THE MODIFICATION OF THE GALTON–WATSON PROCESS

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Abstract. In the paper we study the following modification of the classical Galton-Watson process. We suppose, starting with the first generation, that a particle of some generation "lives to see" the time of generating of the random number of new particles with the probability p. It is shown that the probability of the destruction of the population is equal to the smallest root of the equation $F(p \cdot s + q) = s$ $(q = 1 - p)$.

A branching process with the discrete time (also called Galton-Watson process) is the mathematical model of a population of particles (see [**1**]–[**3**]). It is supposed that each particle can "generate" at the close of its life (which is fixed) random number of new particles, independently with respect to others particles. Let U be the number of particles which can originate from some particle. Let

$$p_n = P(U = n) \qquad (n = 0, 1, 2, \dots) \tag{1}$$

and let $F(s) = \sum_{n=0}^{\infty} p_n \cdot s^n$ be the generating function of the random variable U. We suppose that there is only one particle at the beginning.

The first generation Y_1 of the population is created by particles originating from this particle. Analogously Y_n is the number of the particles of the n-generation i.e. the number of the particles which originate from the particles of the $(n-1)$-generation. Let $p_m^{(n)} = P(Y_n = m)$ $(m = 0, 1, 2, \dots)$ and $F_n(s) = \sum_{m=0}^{\infty} p_m^{(n)} \cdot s^m$. It is known (see [**1**]) that the probability of the destruction of the population is equal to the smallest positive root of the equation $F(s) = s$.

We shall modify this mathematical model in the following way. As in the Galton-Watson process let $G_1(s) = F(s)$ be the generating function of Z_1 — the number of the particles of the first generation. Starting with the first generation let every particle "live to see" the time of the generating of the new particles with the probability p $(0 < p < 1)$. Denote $q(k, m)$ the probability that among m particles of the $(n-1)$-generation there are k particles which "live to see" this time. Then the random variable Z_n (the number of the particles of the n-th generation) for

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$n \geq 2$ is the mixture of the random variables $X_{m,k}^{(n-1)} = \sum_{i=0}^{k} U_i$ where $U_0 = 0$ and U_i for $i \geq 1$ are independent random variables with the distribution given by (1) with the weights $\alpha_{m,k}^{(n-1} = p_m^{(n-1)} \cdot q(k,m)$; $m = 0, 1, \dots,$ $k = 0, 1, \dots, m$. Evidently

$$q(k,m) = \binom{m}{k} p^k \cdot q^{m-k} \,. \tag{2}$$

So for $n \geq 2$ the generating function of Z_n is $G_n(s) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \alpha_{m,k}^{(n-1)} \cdot F_{m,k}^{(n-1)}(s)$ where $F_{m,k}^{(n-1)}(s)$ is the generating function of the random variable $X_{m,k}^{(n-1)}$. From the assumptions it follows that $F_{m,k}^{(n-1)}(s) = F^k(s)$. With respect to (2) we have

$$\begin{aligned} G_n(s) &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} p_m^{(n-1)} \cdot \binom{m}{k} \cdot p^k \cdot q^{m-k} \cdot F^k(s) \\ &= \sum_{m=0}^{\infty} p_m^{(n-1)} \cdot (p \cdot F(s) + q)^m = G_{n-1}(p \cdot F(s) + q) \,. \end{aligned} \tag{3}$$

Then, for $n \geq 2$, $\mu = E(Z_n) = G_n'(1) = p \cdot \mu \cdot \mu_{n-1}$. It means that $\mu_n = p^{n-1} \cdot \mu_1^n$ with $\mu_1 = E(Z_1)$. Let $v_n = G_n(0)$. Then $v_n = G_{n-1}(p \cdot F(0) + q) \geq G(0) = v_{n-1}$, $n = 2, 3, \dots$. Denote $v = \lim_{n\to\infty} v_n$. We call the limit value v the probability of the destruction of the population. Now we prove four lemmas.

Lemma 1. *The generating function $G_n(s)$, $n \geq 2$, fulfills the following recurrent relation:*

$$G_n(s) = F(p \cdot G_{n-1}(s) + q) \,. \tag{4}$$

Proof. We prove relation (4) by the mathematical induction. For $n = 2$ equation (4) follows immediately from (3). Let (4) be fullfilled for some n. Then

$$G_{n+1}(s) = G_n(p \cdot F(s) + q) = F(p \cdot G_{n-1}(p \cdot F(s) + q) + q) = F(p \cdot G_n(s) + q). \ \square$$

Let $s = 0$ in (4). Then we have $v_n = F(p \cdot v_{n-1} + q)$. Passing to the limit $n \to \infty$ we obtain

$$v = F(p \cdot v + q) \,. \tag{5}$$

Thus the probability v of the destruction of the population is the solution the following equation

$$F(p \cdot s + q) = s \,. \tag{6}$$

Lemma 2. *The number* 1 *fulfills the equation* (6).

Proof. The proof is obvious. □

Lemma 3. *Let* $a > 0$ *fulfill* (6). *Then* $v \leq a$.

Proof. We again proceed by the mathematical induction in order to prove that $v_n \leq a$ for every n. Evidently $v_1 = F(0) \leq F(p \cdot a + q) = a$. Let $v_n \leq a$ for some n. Then $v_{n+1} = G_{n+1}(0) = F(p \cdot G_n(0) + q) = F(p \cdot v_n + q) \leq F(p \cdot a + q) = a$. Hence $v = \lim_{n\to\infty} v_n \leq a$. □

Lemma 4. *Let equation* (6) *have a solution in the interval* $(0,1)$. *Then* $p \cdot \mu_1 > 1$.

Proof. We shall show that if (6) has a root in the interval $(0,1)$ then $p_0 + p_1 < 1$, i.e. there exists $k \geq 2$ such that $p_k > 0$ $(\sum_{k=0}^{\infty} p_k = 1)$. Suppose to the contrary that $p_0 + p_1 = 1$. Then we have for every $s \in (0,1)$:

$$F(p \cdot s + q) = p_0 + p_1 \cdot (p \cdot s + q) > (p_0 + p_1 \cdot (p + q)) \cdot s = s\,.$$

But it contradicts to the assumption of the lemma. Hence we have shown that there exists $k \geq 2$ such that $p_k > 0$. It means that $F''(s) > 0$ for every $s \in (0,1)$, i.e. $F'(\cdot)$ is strongly increasing at the interval $(0,1)$. Let $s_1 \in (0,1)$ be a root of equation (6). As $F(1) = 1$ we have (according to Lagrange's theorem) that

$$1 = \frac{F(s_1 \cdot p + q) - 1}{s_1 - 1} = p \cdot \frac{F(s_1 \cdot p + q) - 1}{s_1 \cdot p + q - 1} = p \cdot F'(\hat{s}) \quad \text{where } \hat{s} \in (s_1 \cdot p + q, 1).$$

We have already shown that under the assumptions of the lemma the function $F'(\cdot)$ is strongly increasing and so

$$1 = p \cdot F'(\hat{s}) < p \cdot F'(1) = p \cdot \mu_1\,.$$ □

Lemma 5. *Let* $0 < p_0 < 1$. *If equation* (6) *has no root in the interval* $(0,1)$ *then* $p \cdot \mu_1 \leq 1$.

Proof. We shall show first that for every $s \in (0,1)$

$$s < F(s \cdot p + q)\,. \tag{7}$$

Let $0 < s < p_0$. Then $s < p_0 = F(0) \leq F(s \cdot p + q)$. It follows from the continuity of the function $G(s) = F(s \cdot p + q) - s$ and from the assumptions of the lemma, that the inequality (7) is also satisfied for $s \geq p_0$. Then we have for $s \in (0,1)$:

$$1 > \frac{1 - F(s \cdot p + q)}{1 - s} = p \cdot \frac{F(1) - F_s \cdot p + q)}{1 - (s \cdot p + q)}\,.$$

It means that

$$1 \geq \lim_{s \to 1} \frac{1 - F(s \cdot p + q)}{1 - (s \cdot p + q)} = p \cdot \mu_1\,.$$ □

We summarize the obtained results in the following theorem.

Theorem. *Let us consider the modified discrete branching process. Suppose that* $0 < p_0 < 1$*.*

(i) *It follows from Lemma* 3 *and equation* (5) *that the probability* v *of destruction is the smallest positive root of equation* (6)*.*

(ii) *It follows from Lemmas* 2, 4 *and from the part (i) of this theorem that if* $p \cdot \mu_1 \leq 1$ *then the probability of the destruction is equal* 1*.*

(iii) *It follows from Lemmas* 3, 5 *and from equation* (5) *that if* $p \cdot \mu_1 > 1$ *then the probability of the destruction is less then* 1 *and is equal to the smallest root of the equation* (6)*.*

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