Acta Math. Univ. Comenianae
Vol. LXV, 1(1996), pp. 53–64

COEXISTENCE OF SINGULAR AND REGULAR SOLUTIONS FOR THE EQUATION OF CHIPOT AND WEISSLER

F. X. VOIROL

1. Introduction

In this paper, we are interested in the existence of positive solutions of

$$(P_{B_R}) \qquad \begin{cases} \Delta u - |\nabla u|^q + \lambda u^p = 0 & \text{on } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

where B_R is a ball in $\mathbb{R}^n$ of radius R and

$$p > 1,\ q = \frac{2p}{p+1},\ \lambda > 0.$$

This problem was introduced in 1989 by M. Chipot and F. Weissler (cf. [**CW**]) in connection with the study of the nonlinear parabolic equation

$$\begin{aligned} u_t &= \Delta u - |\nabla u|^q + |u|^p && \text{on } B_R \times (0,T), \\ u &= 0 && \text{on } \partial B_R \times (0,T), \\ u(x,0) &= u_0 && \text{on } B_R. \end{aligned}$$

One can show that the solutions of (P_{B_R}) are radially symmetric (using the technique of Gidas-Ni-Nirenberg [**GNN**]) and so we consider the solution u_a of

$$(P_a) \qquad \begin{cases} u'' + \dfrac{n-1}{r}u' - |u'|^q + \lambda |u|^p = 0 & \text{if } r > 0, \\ u(0) = a, \\ u'(0) = 0, \end{cases}$$

where $a > 0$.

We will denote by $z(a)$ the first zero of u_a if it exists; if $u_a > 0$ on $[0, +\infty)$, we will set $z(a) = +\infty$.

Received April 23, 1996.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 35J65.

We know from [**CW**] that z verifies the relation $z(a) = a^{-\frac{p-1}{2}} z(1)$; we have then only two possibilities

– either $z(a) = +\infty$ for all $a > 0$ and (P_{B_R}) has no solution for any R;

– or $z(a)$ is finite for all $a > 0$ and z is a decreasing function from $[0, +\infty)$ into $[0, +\infty)$ (cf. [**CW**, Lemma 4.7]); in this case, (P_{B_R}) has one and only solution for any R.

The range of λ is crucial for the behaviour of the map z.

In their paper, M. Chipot and F. Weissler show the following result:

Theorem. *If $q = \frac{2p}{p+1}$ and $p < \frac{n}{n-2}$ the equation*

$$(I) \qquad u''(r) + \frac{n-1}{r} u'(r) - |u'(r)|^q + \lambda |u(r)|^p = 0$$

has a solution in the form of $u(r) = kr^{-\frac{2}{p-1}}$ if and only if $\lambda \leq \lambda_{n,p}$ where

$$\lambda_{n,p} = \frac{(2p)^p}{(p+1)^{p+1}(2p - np + n)^p} = \frac{q^p}{(p+1)\,(2p - np + n)^p}.$$

When $n = 1$ the equation (I) becomes autonomous and if $u\colon r \mapsto k\, r^{-\frac{2}{p-1}}$ is a solution of (I), the function $u_1\colon r \mapsto (r + c)^{-\frac{2}{p-1}}$, is a solution too. If $\lambda \leq \lambda_{1,p}$, then it follows from the Cauchy theorem and a translation argument (see [**CW**]) that the problem (P_{B_R}) has no solution. This is also the case when $n \geq 1$ and $\lambda \leq \lambda_{1,p}$ (see [**CW**] or [**V**, Proposition I.7]).

In a more recent paper (see [**FQ**]), M. Fila and P. Quittner show that the condition $\lambda > \lambda_{n,p}$ implies that $z(a)$ is finite for all $a > 0$; but the case $\lambda = \lambda_{n,p}$ where we could have coexistence of the singular solution $u(r) = k\, r^{-\frac{2}{p-1}}$ and solutions of (P_a) with $z(a)$ finite was open. We solve this issue here. Indeed we show :

Theorem A. *Assume $q = \frac{2p}{p+1}$, $\lambda_{n,p} = \frac{(2p)^p}{(p+1)^{p+1}(2p-np+n)^p}$ and*

1) $n = 2$

2) $n \geq 3$ *and* $1 < p < \frac{n}{n-2}$.

Let u_a be the solution of (P_a). Then there exists $\lambda'_{n,p} < \lambda_{n,p}$ such that $z(a)$ is finite for $\lambda > \lambda'_{n,p}$.

Remark 1. When $\lambda = \lambda_{n,p}$, there exists only one solution of (I) of the form $u(r) = kr^{-\frac{2}{p-1}}$ (according to the proof of Proposition 5.5 in [**CW**]) and its graph cuts the one of the solution of (P_a) for any $a > 0$ (see [**V**, Proposition I.6]). In the case $\lambda'_{n,p} < \lambda < \lambda_{n,p}$ the equation (I) has two distinct solutions in the form $u(r) = kr^{-\frac{2}{p-1}}$ whose graphs cut those of the solutions of (P_a).

In the case $\frac{n}{n-2} \leq p < \frac{n+2}{n-2}$ there always exist singular solutions of (I). We show here the following theorem:

Theorem B. *Assume* $1 < p < \frac{n+2}{n-2}$, $n \geq 3$ *and* $q = \frac{2p}{p+1}$. *If* $\lambda \geq \Lambda_{n,p}$ *where*

$$\Lambda_{n,p} = \frac{1}{(p+1)^{p+1}} + \frac{n\,(p-1)2^p\,q^{p+1}}{(2p+2-np+n)^{p+1}},$$

then $z(a)$ *is finite for any* $a > 0$.

Remark 2. As in the case where $p < \frac{n}{n-2}$, the graphs of regular and singular solutions are crossing. In order to prove Theorems A and B, following [**FQ**], we introduce a two dimensional autonomous system. The main properties of this system are recalled in Section 2. The Sections 3, 4 and 5 are devoted to the cases $n = 2$, $n \geq 3$ and $p < \frac{n}{n-2}$, $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, respectively.

2. Transformation of the Problem to an Autonomous System

Let u be a solution of (P_a). We consider $(X, Y)\colon t \mapsto \big(X(t), Y(t)\big)$ defined by

$$\begin{cases} X(t) = -\dfrac{ru'}{u}, \\ Y(t) = r^2 u^{p-1}, \\ r(t) = e^t. \end{cases} \tag{2}$$

We will recall some results of [**FQ**] in Propositions 1 and 2.

First we find, since $r'(t) = r(t)$

$$\begin{aligned} X'(t) &= \frac{-(ru' + r^2u'')u + u'r^2u'}{u^2} \\ &= \Big(\frac{ru'}{u}\Big)^2 - \frac{ru'}{u} - \frac{r^2u''}{u} \\ &= X^2 + X - \frac{r^2}{u}\Big((-u')^q - \lambda u^p - \frac{(n-1)}{r}u'\Big) \end{aligned}$$

and we obtain

$$X'(t) = (2-n)X + X^2 + \lambda Y - X^{\frac{2p}{p+1}}\,Y^{\frac{1}{p+1}}.$$

On the other hand,

$$\begin{aligned} Y'(t) &= 2r^2u^{p-1} + r^2(p-1)u^{p-2}u'r \\ &= r^2u^{p-1}\Big(2 + r(p-1)\frac{u'}{u}\Big) \\ &= Y\big(2-(p-1)X\big). \end{aligned}$$

Since u verifies also $u(0) = a$ and $u'(0) = 0$, we have

$$\lim_{t\to-\infty} Y(t) = \lim_{t\to-\infty} X(t) = 0, \text{ according to (2), and } \lim_{t\to-\infty}\frac{Y(t)}{X(t)} = \frac{n}{\lambda}.$$

This last equality results from $\frac{Y(t)}{X(t)} = -\frac{ru^p}{u'}$. If $t \to -\infty$, then $r \to 0$ and $\frac{u'(r)}{r} \to -\lambda\frac{u^p(0)}{n}$ since $u'' + \frac{n-1}{r}u' = |u'|^q - \lambda u^p$ and $\lim_{r\to 0}\frac{u'(r)}{r} = u''(0)$. These results are summarized in the following proposition:

Proposition 1. *Let u be a solution of (P_a). If (X, Y) is defined by (2) then (X, Y) is a solution of the autonomous system*

$$\begin{cases} x'(t) = (2-n)x + x^2 + \lambda y - x^{\frac{2p}{p+1}} y^{\frac{1}{p+1}}, \\ y'(t) = y\big(2-(p-1)x\big) \end{cases} \tag{3}$$

and we have

$$\lim_{t\to-\infty} Y(t) = \lim_{t\to-\infty} X(t) = 0, \quad \lim_{t\to-\infty} \frac{Y(t)}{X(t)} = \frac{n}{\lambda}. \tag{4}$$

Let us recall also (according to a lemma in [**FQ**]) that an orbit of (3) starting when $t = t_0$ in the first quadrant $\{(x,y) \mid x \geq 0,\ y \geq 0\}$ stays in this quadrant when $t > t_0$. Moreover, there exists only one orbit coming from the origin O, its slope is $\frac{n}{\lambda}$.

The continuous dependence of solutions of (P_a) on λ implies that if $z(a)$ is finite for all a when $\lambda = \lambda_{n,p}$, then there exists $\lambda'_{n,p} < \lambda_{n,p}$ such that we have the same behaviour for all $\lambda \in (\lambda'_{n,p}, \lambda_{n,p}]$.

In the computation below we set for convenience $\lambda = \lambda_{n,p}$.

Moreover, define f and g by

$$f(x,y) = (2-n)x + x^2 + \lambda y - x^{\frac{2p}{p+1}} y^{\frac{1}{p+1}},$$
$$g(x,y) = y\big(2-(p-1)x\big).$$

We see that $g(x,y) = 0$ if $y = 0$ or if $x = x_1$ where $x_1 := \frac{2}{p-1}$.

We are going to study the set Γ defined by

$$\Gamma = \{(x,y) \mid x \geq 0,\ y \geq 0,\ f(x,y) = 0\}.$$

When $n = 2$ this set is one half of the parabola defined by $x \geq 0$ and $y = x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}$ (see Proposition 2). It cuts the straight line $x = x_1$ at one point only (see Figure 1).

We study also the position (with respect to Γ) of the orbit $\mathcal{O}$ of the system (3) corresponding to the map $t \mapsto \big(X(t), Y(t)\big)$. We show that

(a) $\mathcal{O}$ is located above Γ when $0 < X(t) < x_1$ because on the corresponding part of Γ the vector field is "vertical and oriented upwards",

(b) $\mathcal{O}$ cuts the straight line $x = x_1$ above Γ (by linearization of the vector field around the point of intersection of Γ with this straight line),

(c) $X(t)$ blows up in finite time (see Figure 1).

When $n \geq 3$, Γ is tangent to the straight line $x = x_1$ (for $\lambda = \lambda_{n,p}$) and located in the half-plane defined by $x \leq x_1$ (see Figure 2). We show next that for $0 < X(t) \leq x_1$, $\mathcal{O}$ is located above Γ, and that if $X(t) > x_1$ then $X'(t) > 0$ and $Y'(t) < 0$. Then we deduce that $X(t)$ blows up in finite time.

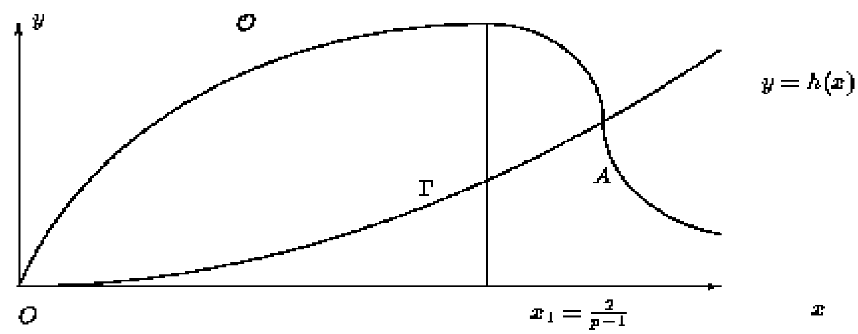

Figure 1.

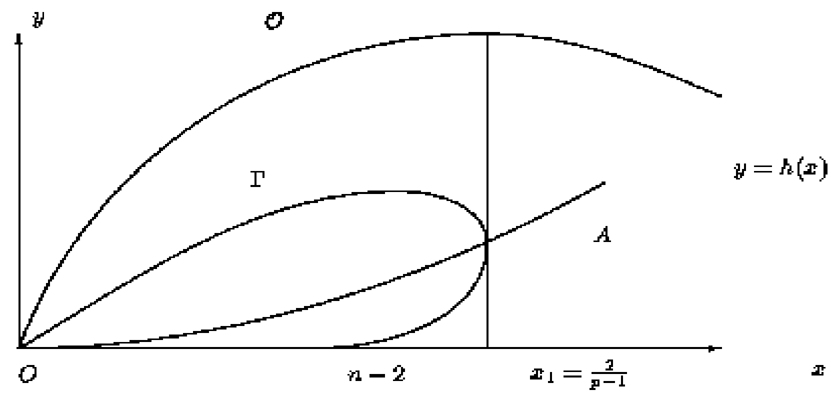

Figure 2.

Proposition 2. *Let $\lambda = \lambda_{n,\,p}$ and x be fixed, $x > 0$, then $f(x,y)$ has a unique minimum for $y = h(x) := x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}$. This minimum is*

$$m(x) := f\big(x, h(x)\big) = x(n-2)\Big(\frac{p-1}{2}x - 1\Big).$$

The vector field $\big(f(x,y), g(x,y)\big)$ has on the half straight line $x = x_1$ and $y \geq 0$ only one singular point $A = (x_1, y_1)$ with $y_1 = h(x_1)$.

Proof. Put $h(x) = x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}$. We have

$$\frac{\partial f}{\partial y}(x,y) = \lambda - \frac{1}{p+1}x^{\frac{2p}{p+1}}y^{-\frac{p}{p+1}}$$

so that

$$\frac{\partial f}{\partial y}(x,y) < 0 \;\text{ if }\; 0 < y < h(x) \;\text{ and }\; \frac{\partial f}{\partial y}(x,y) > 0 \;\text{ if }\; y > h(x).$$

Therefore, the map $y \mapsto f(x,y)$ has a unique minimum for $y = h(x)$. Its value is

$$f\big(x, h(x)\big) = (2-n)x + x^2\, \frac{\lambda^{\frac{1}{p}}(p+1)^{\frac{p+1}{p}} - p}{\lambda^{\frac{1}{p}}(p+1)^{1+\frac{1}{p}}}.$$

But

$$\lambda^{\frac{1}{p}}\big(p+1\big)^{\frac{p+1}{p}} = \frac{2p}{2p-np+n}$$

and

$$\frac{\lambda^{\frac{1}{p}}(p+1)^{\frac{p+1}{p}} - p}{\lambda^{\frac{1}{p}}(p+1)^{1+\frac{1}{p}}} = (p-1)\Big(\frac{n}{2}-1\Big).$$

We obtain finally

$$f\big(x,\ h(x)\big) = (2-n)x + x^2(p-1)\Big(\frac{n}{2}-1\Big) = x\ (n-2)\ \Big(\frac{p-1}{2}x - 1\Big).$$

The fact that the vector field $\big(f(x,y),\, g(x,y)\big)$ has on the half straight line defined by $x = x_1$, $y \geq 0$ only one singular point $A = \big(x_1, h(x_1)\big)$ can be deduced easily from the expression for $m(x)$. This completes the proof of Proposition 2.□

On the set $\mathcal{E}$ defined by $\mathcal{E} = \{(x,y) \,|\, 0 < x < x_1, y > 0\}$, we have $g(x,y) > 0$. If we consider an orbit defined by a map $t \mapsto \big(X_1(t), Y_1(t)\big)$ such that $\big(X_1(t_0), Y_1(t_0)\big)$ is in $\mathcal{E}$, we have a priori three possible behaviours since the vector field $\big(f(x,y), g(x,y)\big)$ has no singular point in $\mathcal{E}$:

1) either $Y_1(t) \to +\infty$ as $t \to \alpha$ (with $\alpha = +\infty$ or α real) and $X_1(t) < x_1$ for $t \geq t_0$;
2) either the orbit cuts the straight line $x = x_1$;
3) or this orbit has A as the limit-point as $t \to \infty$.

First note that the case 1) cannot occur since from the formulae (3) for $X_1'(t)$ and $Y_1'(t)$ we could deduce $\limsup_{t\to\alpha} \frac{Y_1'(t)}{X_1'(t)} \leq \frac{2}{\lambda}$. Since $\lim_{t\to\alpha} Y_1(t) = +\infty$, we would get $\lim_{t\to\alpha} X_1(t) = +\infty$ which yields a contradiction with $X_1(t) < x_1$.

3. The Case $n = 2$

In this section we prove Theorem A for $n = 2$.

According to Proposition 2, for any $x > 0$ the map $y \mapsto f(x,\, y)$ has a unique minimum and its value is $f\big(x, h(x)\big) = 0$ when $n = 2$. The set defined by $f(x,y) = 0$, $x \geq 0$, $y \geq 0$ is then one half of the parabola defined by $y = h(x) = x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}$. Moreover, if $x > 0$, $y > 0$ and $y \neq h(x)$ then $f(x,y) > 0$.

Since $\lim_{t\to-\infty} \frac{Y(t)}{X(t)} = \frac{n}{\lambda} = \frac{2}{\lambda}$, the orbit $\mathcal{O}$ defined by the map $t \mapsto \big(X(t), Y(t)\big)$ is in a neighbourhood of the origin above the parabola.

The vector field $\big(f(x,y),g(x,y)\big)$ has two singular points in the first quadrant of the plane: the origin $O=(0,0)$ and $A=(x_1,y_1)$ with $y_1=h(x_1)$.

If $0<x<x_1$ and $y=h(x)$, then $f(x,y)=0$ and $g(x,y)>0$, so that the orbit of $t\mapsto \big(X(t),Y(t)\big)$ can cut the line $x=x_1$ at (x_1,y) with $y\geq y_1$ or have A as the limit point. Let us show that in fact the first possibility occurs.

For this, linearize the vector field $(x,y)\mapsto \big(f(x,y),g(x,y)\big)$ at the point A. We have

$$\frac{\partial f}{\partial x}(x,y)=2x-\frac{2p}{p+1}x^{\frac{p-1}{p+1}}y^{\frac{1}{p+1}} \quad\text{and}\quad \frac{\partial f}{\partial y}(x,\ y)=\lambda-\frac{1}{p+1}x^{\frac{2p}{p+1}}y^{-\frac{p}{p+1}}.$$

Since $\big(\lambda(p+1)\big)^{\frac{p+1}{p}}=\big(\frac{p}{p+1}\big)^{p+1}$, we have $h(x)=x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}=x^2\big(\frac{p+1}{p}\big)^{p+1}$ and

$$\frac{\partial f}{\partial x}\big(x,h(x)\big)=2x\Big(1-\frac{p}{p+1}x^{-\frac{2}{p+1}}x^{\frac{2}{p+1}}\frac{p+1}{p}\Big)=0 \quad \text{for any } x.$$

Next, since $y\mapsto f(x_1,y)$ has its minimum for $y=y_1$ then $\frac{\partial f}{\partial y}(x_1,\ y_1)=0$. We have also $\frac{\partial g}{\partial x}(x_1,y_1)=-\frac{4}{p-1}\big(\frac{p+1}{p}\big)^{p+1}$ and $\frac{\partial g}{\partial y}(x_1,y_1)=0$. Thus we obtain

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}(x_1,y_1)=\begin{pmatrix} 0 & 0\\ -\frac{4}{p-1}\left(\frac{p+1}{p}\right)^{p+1} & 0\end{pmatrix}. \tag{5}$$

Let us now show that the orbit cuts the straight line $x=x_1$ at $A'=(x_1,y_1')$ with $y_1'>y_1$. Consider the set $\mathcal{E}'$ of the plane defined by

$$\mathcal{E}'=\{(x,y)\,|\,0<x<x_1,\ h(x)\leq y\leq y_1\}$$

and let us set

$$a=x-x_1,\quad b=y-y_1.$$

On the other hand, let $C_1=h'(x_1)$ be the slope of the tangent line to the parabola at the point $A=\big(x_1,h(x_1)\big)$. We can see (cf. Figure 1) that if (x,y) is in $\mathcal{E}'$, then $\frac{b}{a}<C_1$. Next, from (5) there exists $\epsilon>0$ such that $-\epsilon<a<0$ and $-\epsilon<b\leq 0$ imply, as $f\geq 0$ on $\mathcal{E}$,

$$0\leq f(x_1+a,\ y_1+b)<\frac{C_2}{4\ C_1(1+C_1)}(a+b)$$

where

$$C_2=-\frac{4}{p-1}\left(\frac{p+1}{p}\right)^{p+1}=\frac{\partial g}{\partial x}(x_1,y_1)$$

and

$$g(x_1+a,y_1+b)>\frac{a\,C_2}{2}. \tag{6}$$

If $(x, y) \in \mathcal{E}'$ is close to A then $-\epsilon < a < 0$ and $-\epsilon < b \leq 0$ so we can deduce from the fact that $aC_1 < b$ that

$$f(x_1 + a, y_1 + b) < \frac{C_2}{4C_1(1 + C_1)}(a + aC_1) = \frac{aC_2}{4C_1}$$

and, finally,

$$\frac{g(x_1 + a,\, y_1 + b)}{f(x_1 + a,\, y_1 + b)} > 2C_1$$

using (6). So we obtain that an orbit of the vector field $\big(f(x, y), g(x, y)\big)$ passing, for $t = t_0$, through a point (x, y) in $\mathcal{E}'$ such that

$$-\epsilon < a < 0 \quad \text{and} \quad -\epsilon < b \leq 0 \quad \text{where } \ a = x - x_1,\ b = y - y_1$$

(cf. Figure 2) cuts for $t_1 > t_0$ the straight line $x = x_1$ at (x_1, y_1'') with $y_1'' > y_1$. Since orbits cannot intersect, the orbit $\mathcal{O}$ has to cut the line $x = \frac{2}{p-1}$ above A.

Now $X(t) > x_1$ implies $g\big(X(t), Y(t)\big) < 0$ and $f\big(X(t), Y(t)\big) > 0$ except if $Y(t) = h\big(X(t)\big)$; then, if $t \geq t_1'$, $Y(t) \leq Y(t_1')$, on the other hand X is increasing. The first equation of (3) shows that there exist $\alpha > 0$ and $x_2 > 0$ such that $x > x_2$ implies $f\big(x(t), y(t)\big) > \alpha x^2$. We deduce from this that X blows up in a finite time T and $z(a) = e^T$.

4. The Case $n \geq 3$ and $1 < p < \frac{n}{n-2}$

Proposition 3. *Let $\lambda = \lambda_{n,p}$. The curve $\Gamma = \{(x, y) \,|\, x \geq 0,\ y \geq 0,\ f(x,\ y) = 0\}$ admits a tangent line at every point. The half straight line defined by $x = c$, $y \geq 0$ cuts Γ at one point if $c = \frac{2}{p-1}$, two points if $n - 2 \leq c < \frac{2}{p-1}$, one point if $0 \leq c < n - 2$.*

Moreover, $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are the graphs of some functions h_1 and h_2,

$$\Gamma_1 = \{(x, y) \in \Gamma \,|\, y \leq h(x)\}, \qquad \Gamma_2 = \{(x, y) \in \Gamma \,|\, y \geq h(x)\}$$

(see Figure 2).

Proof. One has $f(x, 0) = (2 - n)x + x^2$ so that $f(\cdot, 0) < 0$ on $(0, n - 2)$ and $f(\cdot, 0) > 0$ on $(n - 2, +\infty)$. On the other hand, according to Proposition 2, for x fixed, $x > 0$, the map $y \mapsto f(x,\ y)$ attains its unique minimum at $y = h(x)$; its value is

$$m(x) = x(n - 2)\Big(\frac{p - 1}{2}\, x - 1\Big).$$

This minimum is negative if $0 < x < x_1$, zero if $x = x_1$ and positive if $x > x_1$. A half line $x = C$, $y \geq 0$ has then in common with Γ

– one point if $0 < C < n-2$ or if $C = x_1$,

– two points if $n-2 \leq C < x_1$,

– no point if $x > x_1$.

Next, if $0 < x < x_1$ and $y \neq h(x)$, then $\frac{\partial f}{\partial y}(x,y) \neq 0$. For $A = \big(x_1, h(x_1)\big)$ we have $m'(x_1) = n-2$. Since $\frac{\partial f}{\partial y}(A) = 0$ and $m' = \frac{\partial f}{\partial x} + h' \frac{\partial f}{\partial y}$ it follows that $\frac{\partial f}{\partial x}(A) \neq 0$. This shows that for every point $M = (x,y)$ of Γ such that $y > 0$, either $\frac{\partial f}{\partial x}(M) \neq 0$, or $\frac{\partial f}{\partial y}(M) \neq 0$.

The above considerations show that there exist two functions $h_1\colon [n-2, x_1] \to \mathbb{R}$ and $h_2\colon [0, x_1] \to \mathbb{R}$ such that $f(x,y) = 0$ if and only if $y = h_1(x)$ or $y = h_2(x)$, h_1, h_2 verifying the following conditions:

1) if $n-2 < x < x_1$ then $h_1(x) < h(x)$;
2) if $0 < x < x_1$ then $h(x) < h_2(x)$;
3) $h(x_1) = h_1(x_1) = h_2(x_1)$.

Moreover, h_2 is differentiable at 0 since $f(x,y) = (2-n)x + \lambda y + o\big(\sqrt{x^2+y^2}\big)$, and $h_2'(0) = \frac{n-2}{\lambda}$. We can verify also that $h_1'(n-2) = 0$.

Since Γ is differentiable at (x_1, y_1), there exists $x_1' \in (0, x_1)$, such that h_2 is decreasing on $[x_1',\ x_1]$. □

Let us consider now the orbit $\mathcal{O}$ of $t \mapsto \big(X(t), Y(t)\big)$. It is located above the graph Γ_2 of h_2 in a neighbourhood of O since $h_2'(0) = \frac{n-2}{\lambda}$ and $\lim_{t\to-\infty} \frac{Y(t)}{X(t)} = \frac{n}{\lambda}$ according to Proposition 1. Since g is continuous, for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\varepsilon < x \leq x_1'$ implies $g\big(x,\, h_2(x)\big) > \eta$. Since, on the other hand, $0 < x < x_1'$ implies $f\big(x,\, h_2(x)\big) = 0$; the orbit $\mathcal{O}$ stays above Γ_2 when $0 < X(t) < x_1'$. Finally, $x_1' < X(t) < x_1$ implies $g\big(X(t),\, Y(t)\big) \geq 0$ and $Y(t) > h_2(x_1')$ (cf. Figure 2).

$\mathcal{O}$ cuts then the straight line $x = x_1$ since, on this straight line, the only singular point of the vector field is $A = \big(x_1, h_2(x_1)\big)$ and $h_2(x_1) < h_2(x_1')$ (let us recall that according to 3) above $h_2(x_1) = h(x_1)$). We can finish as in the case $n = 2$, noting that if $X(t) > x_1$ then $f\big(X(t), Y(t)\big) > 0$ and $g\big(X(t), Y(t)\big) < 0$ which implies that $Y(t)$ is bounded and that $X(t)$ blows up in a finite time T, hence $z(a) = e^T$.

5. The Case $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$

If $p \geq \frac{n}{n-2}$ then $n-2 \geq \frac{2}{p-1}$ and the previous method cannot be applied. Moreover, if $p \to \frac{n}{n-2}$ then $2p - np + n \to 0$ and $\lambda_{n,p} \to +\infty$. We introduce here another method which gives in both cases $\frac{n}{n-2} \leq p < \frac{n+2}{n-2}$ and $1 < p < \frac{n}{n-2}$ a new value

$$\Lambda_{n,p} = \frac{1}{(p+1)^{p+1}} + \frac{2^{2p+1}(p-1)\, n\, p^{p+1}}{\big((p+1)\,(2p+2-np+n)\big)^{p+1}}$$

such that $\lambda \geq \Lambda_{n,p}$ implies that $z(a)$ is finite for any $a > 0$. When $p < \frac{n}{n-2}$, $\Lambda_{n,p} < \lambda_{n,p}$ if p is near $\frac{n}{n-2}$.

The idea is to use the dissymmetry of the level lines of $f\colon (x,y)\mapsto f(x,y)$. In fact, if $0\le y<y_0$ (where y_0 is given in (8) below) and $\alpha\in(0,x_1)$ then we show

$$\begin{cases} f(x_1+\alpha,\,y) > f(x_1-\alpha,\,y),\\ g(x_1+\alpha,\,y) = -g(x_1-\alpha,\,y)\end{cases} \tag{7}$$

and we show also that $\lambda\ge\Lambda_{n,p}$ implies $\mathcal{O}$ is below the line $y=y_0$.

If $\mathcal{O}_1$ is the part of $\mathcal{O}$ in the set $\{(x,y)\,|\,0\le x\le x_1\}$, then $\mathcal{O}$ is above $S(\mathcal{O}_1)$ in the set $\{(x,y)\,|\,x\ge x_1\}$ where $S(\mathcal{O}_1)$ is the reflection of $\mathcal{O}_1$ with respect to the straight line $x=x_1$. This shows that $X(t)>x_1$ implies $X'(t)>0$ and we can conclude as in the proof of Theorem A (notice that $n-2<2x_1$).

First, we see that $\frac{\partial f}{\partial x}(x_1,y)=0$ if and only if

$$(2-n)+2x-\frac{2p}{p+1}\,x^{\frac{p-1}{p+1}}\,y^{\frac{1}{p+1}}=0$$

with $x=x_1$ i.e.

$$y=y_0:=\left(\frac{(2-n)(p-1)+4}{p-1}\right)^{p+1}\left(\frac{p+1}{2p}\right)^{p+1}\left(\frac{p-1}{2}\right)^{p-1}. \tag{8}$$

Now, we show the following lemma:

Lemma. *If $\beta\in(-y_0,0)$ and $\alpha\in(0,x_1)$ then*

$$f(x_1+\alpha,\,y_0+\beta)\;>\;f(x_1-\alpha,\,y_0+\beta).$$

Proof. We have

$$\frac{\partial f}{\partial x}(x_1,\,y_0)=(2-n)+\frac{4}{p-1}-\frac{2p}{p+1}\left(\frac{2}{p-1}\right)^{\frac{p-1}{p+1}}y_0^{\frac{1}{p+1}}=0 \tag{9}$$

and

$$\begin{aligned} &f(x_1+\alpha,\,y_0+\beta)\\ &\quad=(2-n)\Big(\frac{2}{p-1}+\alpha\Big)+\Big(\frac{2}{p-1}+\alpha\Big)^2+\lambda(y_0+\beta)\\ &\qquad-\Big(\frac{2}{p-1}+\alpha\Big)^{\frac{2p}{p+1}}(y_0+\beta)^{\frac{1}{p+1}}.\end{aligned}$$

Using the Taylor-Lagrange formula for the last term we obtain

$$\begin{aligned} &f(x_1+\alpha,\,y_0+\beta)\\ &=(2-n)\frac{2}{p-1}+(2-n)\alpha+\Big(\frac{2}{p-1}\Big)^2+\frac{4}{p-1}\alpha+\alpha^2+\lambda(y_0+\beta)\\ &\quad-\bigg[\Big(\frac{2}{p-1}\Big)^{\frac{2p}{p+1}}+\frac{2p}{p+1}\Big(\frac{2}{p-1}\Big)^{\frac{p-1}{p+1}}\alpha+\frac{1}{2}\,\frac{2p}{p+1}\,\frac{p-1}{p+1}\,\Big(\frac{2}{p-1}\Big)^{-\frac{2}{p+1}}\alpha^2\\ &\quad+\frac{1}{6}\frac{2p}{p+1}\frac{p-1}{p+1}\Big(\frac{-2}{p+1}\Big)\Big(\frac{2}{p-1}+\theta\alpha\Big)^{-\frac{p+3}{p+1}}\alpha^3\bigg]\\ &\quad\times\bigg[y_0^{\frac{1}{p+1}}+\frac{1}{p+1}(y_0+\theta'\beta)^{-\frac{p}{p+1}}\beta\bigg]\end{aligned}$$

where $0 \le \theta \le 1$ and $0 \le \theta' \le 1$.

Using (9) we see that the only terms which are not symmetric in α are

$$-\frac{2p}{p+1}\Big(\frac{2}{p-1}\Big)^{\frac{p-1}{p+1}}\alpha\,\frac{1}{p+1}(y_0+\theta'\beta)^{-\frac{p}{p+1}}\beta$$

and

$$-\frac{1}{6}\frac{2p}{p+1}\frac{p-1}{p+1}\Big(\frac{-2}{p-1}\Big)\Big(\frac{2}{p-1}+\theta\alpha\Big)^{-\frac{p+3}{p+1}}\alpha^3\left[y_0{}^{\frac{1}{p+1}}+\frac{1}{p+1}(y_0+\theta'\beta)^{-\frac{p}{p+1}}\beta\right].$$

The term

$$\left[y_0{}^{\frac{1}{p+1}}+\frac{1}{p+1}(y_0+\theta'\beta)^{-\frac{p}{p+1}}\beta\right]$$

equal to $(y_0+\beta)^{\frac{1}{p+1}}$ is positive and $(y_0+\theta'\beta)^{-\frac{p}{p+1}}$ too. Since $\beta<0$ we see that the signs of this expression and α are the same. This shows that $f(x_1+\alpha, y_0+\beta) > f(x_1-\alpha, y_0+\beta)$ provided $\beta \in (-y_0, 0)$ and $\alpha \in (0, x_1)$. □

It it easy to see that $g(x_1+\alpha, y_0+\beta) = -g(x_1-\alpha, y_0+\beta)$ for any β and α since

$$g\Big(\frac{2}{p-1}+\alpha,\, y_0+\beta\Big) = (y_0+\beta)\Big(2-(p-1)\big(\frac{2}{p-1}+\alpha\big)\Big) = -(y_0+\beta)(p-1)\alpha.$$

Consequently, (7) is verified.

Now, we use the fact (cf. Theorem 2 of [**FQ**]) that if $(X,Y)\colon t\mapsto \big(X(t),Y(t)\big)$ corresponds to u then $0\le X(t)\le x_1$ implies

$$Y(t) \le \frac{n}{\lambda-(p+1)^{-(p+1)}}X(t).$$

In particular, if $X(t)=x_1$ then $Y(t)\le \frac{n}{\lambda-(p+1)^{-(p+1)}}\frac{2}{p-1}$. Consequently, the orbit $\mathcal{O}$ corresponding to u cuts the straight line $x=x_1$ below $A_0 := (x_1, y_0)$ provided

$$\frac{n}{\lambda-(p+1)^{-(p+1)}}\frac{2}{p-1} \le y_0.$$

Since the last inequality is equivalent to the condition $\lambda \ge \Lambda_{n,p}$, we see that this is a sufficient condition to have $z(a)$ finite for any $a>0$.

References

[**AW**] Alfonsi L. and Weissler F. B., *Blow up in $\mathbb{R}^N$ for a parabolic equation with a damping nonlinear gradient term*, Nonlinear Diffusion Equations and their Equilibrium States (1992), 1–20 (L. A. Peletier et al., eds.), Birkhäuser, Boston.

[**C**] Chipot M., *On a class of nonlinear elliptic equations*, Proceedings of the Banach Center **27** (1992), 75–80.

[**CW**] Chipot M. and Weissler F. B., *Some blow up results for a nonlinear parabolic equation with a gradient term*, SIAM J. Math. Anal. **20** (1989), 886–907.

[**CW2**] ———, *On the elliptic problem* $\Delta u - |\nabla u|^q + \lambda u^p = 0$, Nonlinear Diffusion Equations and Their Equilibrium States Vol. I, W.-Ni (1988), 237–243 (L. A. Peletier, J. B. Serrin, eds.), Springer, New-York.

[**FLN**] De Figueiredo D. G., Lions P.-L. and Nussbaum R. D., *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. pures et appliquées **61** (1982), 41–63.

[**FQ**] Fila M. and Quittner P., *Radial positive solutions for a semilinear elliptic equation with a gradient term*, Adv. Math. Sci. Appl. **1** (1993), 39–45.

[**HW**] Haraux A. and Weissler F. B., *Non-uniqueness for a semilinear initial value problem*, Indiana Univ. Math. J. **31** (1982), 167–189.

[**GNN**] Gidas B., Ni W. M. and Nirenberg L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[**PSZ**] Peletier L. A., Serrin J. B. and H. Zou, *Ground states of a quasilinear equation*, Differential and Integral Equations **7** (1994), 1063–1082.

[**Q1**] Quittner P., *Blow-up for semilinear parabolic equations with a gradient term*, Math. Meth. Appl. Sci. **14** (1991), 413–417.

[**Q2**] ———, *On global existence and stationary solutions for two classes of semilinear parabolic problems*, Comment. Math. Univ. Carolinae **34** (1993), 105–124.

[**SZ**] Serrin J. B. and Zou H., *Existence and non existence results for ground states of quasilinear elliptic equations*, Archive Rat. Mech. Anal. **121** (1992), 101–130.

[**SYZ**] Serrin J. B., Yan Y. and Zou H., *A numerical study of the existence and non-existence of ground states and their bifurcation for the equations of Chipot and Weissler*, Preprint AHPCRC, University of Minnesota, 1993.

[**V**] Voirol F., Thesis, Université de Metz, 1994.

F. X. Voirol, Centre d'Analyse Non Linéaire, Université de Metz, URA-CNRS 399, Ile du Saulcy, 57045 Metz-Cedex 01, France