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# ON TOPOLOGICAL SEQUENCE ENTROPY AND CHAOTIC MAPS ON INVERSE LIMIT SPACES

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Abstract. The aim of this paper is to prove the following results: a continuous map $f\colon [0,1] \to [0,1]$ is chaotic iff the shift map $\sigma_f\colon \varprojlim([0,1],f) \to \varprojlim([0,1],f)$ is chaotic. However, this result fails, in general, for arbitrary compact metric spaces. $\sigma_f\colon \varprojlim([0,1],f) \to \varprojlim([0,1],f)$ is chaotic iff there exists an increasing sequence of positive integers $A$ such that the topological sequence entropy $h_A(\sigma_f) > 0$. Finally, for any $A$ there exists a chaotic continuous map $f_A\colon [0,1] \to [0,1]$ such that $h_A(\sigma_{f_A}) = 0$.

## 1. Introduction

Let $(X,d)$ and $f\colon X \to X$ be a compact metric space and a continuous map respectively. Consider the space of sequences

$$\varprojlim(X,f) = \{\underline{x} = (x_0, x_1, \dots, x_n, \dots) : x_i \in X,\ f(x_i) = x_{i-1}, \text{ for } i = 1, 2, \dots\}.$$

This set is called the **inverse limit space** associated to $X$ and $f$. Define a new metric $\widetilde{d}$ on $\varprojlim(X,f)$ as

$$\widetilde{d}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i},$$

where $\underline{x} = (x_0, x_1, \dots, x_n, \dots)$ and $\underline{y} = (y_0, y_1, \dots, y_n, \dots)$. Then $(\varprojlim(X,f), \widetilde{d})$ is a compact metric space. Consider the natural projection $\pi : \varprojlim(X,f) \to X$ defined by $\pi(x_0, x_1, \dots, x_n, \dots) = x_0$. Note that $\widetilde{d}(\underline{x}, \underline{y}) \geq d(\pi(\underline{x}), \pi(\underline{y}))$ for all $\underline{x}, \underline{y} \in \varprojlim(X,f)$. The **shift map** is a homeomorphism $\sigma_f : \varprojlim(X,f) \to \varprojlim(X,f)$ defined by

$$\sigma_f(\underline{x}) = \sigma_f(x_0, x_1, \dots, x_n, \dots) = (f(x_0), x_0, x_1, \dots, x_n, \dots).$$

It is clear that $\pi \circ \sigma_f = f \circ \pi$.

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Inverse limit spaces have been studied in the setting of dynamical systems in a large number of papers. In [**6**], Shihai Li proved that some dynamical properties hold at the same time for $f$ and $\sigma_f$. In particular, he showed that $f$ is chaotic in Devaney's sense iff $\sigma_f$ is also like that. He also proved a similar result for a suitable definition of $w$-chaos. In this paper, a similar result is studied in case of the Li-Yorke's chaos. Recall briefly this definition of chaos.

A point $p \in X$ is **periodic** if there exists a positive integer $n$ such that $f^n(p) = p$. The smallest positive integer satisfying this condition is called the **period of** $p$. Denote by $\mathrm{Per}(f)$ the set of **periodic points of** $f$. A point $x \in X$ is said to be **asymptotically periodic** if there exists a $p \in \mathrm{Per}(f)$ such that $\limsup_{n\to\infty} d(f^n(x), f^n(p)) = 0$. A map $f\colon X \to X$ is said to be **chaotic in the sense of Li-Yorke**or simply **chaotic** if there exists an uncountable set $D \subset X \setminus \mathrm{Per}(f)$ such that

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0,$$
$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$

hold for all $x, y \in D$, $x \neq y$. $D$ is called a **scrambled set** of $f$.

The Li-Yorke's chaos on inverse limit spaces has been studied by Gu Rongbao in [**3**]. In that paper the author attempts to prove that a continuous map $f$ is chaotic iff the shift map $\sigma_f$ is chaotic. However, in the proof he uses implicitly that $f$ is surjective. As we will see later, this hypothesis on $f$ cannot be removed in the following theorem essentially proved in [**3**].

**Theorem 1.1.** *Suppose $f$ that is surjective. Then it is chaotic in the sense of Li-Yorke if and only if the map $\sigma_f$ is chaotic in the sense of Li-Yorke.*

When continuous maps $f\colon [0,1] \to [0,1]$ are concerned, the Li-Yorke's chaos is connected with the notion of topological sequence entropy. Let us recall the definition (see [**2**]). Let $A = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers. Given $\epsilon > 0$, we say that $E \subset X$ is an $(A, \epsilon, n, f)$-separated set if for any $x, y \in E$ with $x \neq y$ there exists $1 \leq k \leq n$ such that $d(f^{a_k}(x), f^{a_k}(y)) > \epsilon$. Denote by $s_n(A, \epsilon, f)$ the cardinality of any maximal $(A, \epsilon, n, f)$-separated set. The **topological sequence entropy** of $f$ is given by

$$h_A(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \epsilon, f).$$

In general,

$$h_A(f) \geq h_A(\sigma_f) \tag{1}$$

for every $A$. When $f$ is surjective we obtain the equality (see [**2**])

$$h_A(f) = h_A(\sigma_f). \tag{2}$$

The connection between the Li-Yorke's chaos and the topological sequence entropy is established in the following result (see [**1**] and [**5**]).

**Theorem 1.2.** *Let $c, d \in \mathbb{R}$ and let $f\colon [c,d] \to [c,d]$ be continuous. Then*

(a) *$f$ is chaotic iff there exists an increasing sequence of positive integers $A$ such that $h_A(f) > 0$.*
(b) *For any increasing sequence $A$ there exists a chaotic map $f_A\colon [c,d] \to [c,d]$ such that $h_A(f_A) = 0$.*

Theorem 1.2(a) does not hold in general for continuous maps on arbitrary compact metric spaces as it can be seen in [**8**]. In that paper, on $[0,1] \times [0,1]$ a chaotic map $f$ with $\sup_A h_A(f) = 0$ and a non-chaotic map $g$ with $\sup_A h_A(g) > 0$ are constructed.

The aim of this paper is to prove the following results: $f\colon [0,1] \to [0,1]$ is chaotic iff $\sigma_f$ is chaotic. Theorem 1.2 holds for maps $\sigma_f\colon \varprojlim([0,1], f) \to \varprojlim([0,1], f)$. Moreover, an example of a chaotic map $f$ for which $\sigma_f$ is not chaotic is given.

## 2. Positive Results for One-Dimensional Maps

Let $f\colon [0,1] \to [0,1]$ be continuous. Consider $[a,b] = \bigcap_{n\geq 0} f^n[0,1]$. Then $f|_{[a,b]}\colon [a,b] \to [a,b]$ is obviously surjective.

**Proposition 2.1.** *Under the above conditions $f$ is chaotic iff $f|_{[a,b]}$ is chaotic.*

*Proof.* It is clear that if $f|_{[a,b]}$ is chaotic then $f$ is chaotic. Suppose that $f$ is chaotic and let $D$ be a scrambled of $f$. It is easy to see that $f^n(D)$ is also a scrambled set of $f$. Let $D_n = f^n(D) \cap [a,b]$. If $\{f^n(x) : n \geq 0\} \cap [a,b] = \emptyset$, then $x$ is asymptotically periodic and then $x \notin D_n$ for all $n \in \mathbb{N}$. So, it must exist a positive integer $n_0$ such that $D_{n_0}$ is uncountable. Then, $f|_{[a,b]}$ is chaotic. $\square$

**Theorem 2.2.** *Let $f\colon [0,1] \to [0,1]$ be continuous. Then:*

(a) *$f$ is chaotic if and only if $\sigma_f$ is chaotic.*
(b) *$\sigma_f\colon \varprojlim([0,1], f) \to \varprojlim([0,1], f)$ is chaotic if and only if there exists an increasing sequence of positive integers $A$ such that $h_A(\sigma_f) > 0$.*
(c) *For any increasing sequence of positive integers $A$ there exists a chaotic map $f_A\colon [0,1] \to [0,1]$ such that $h_A(\sigma_{f_A}) = 0$.*

*Proof.* It is clear that

$$\varprojlim([0,1], f) = \{(x_0, x_1, \dots, x_n, \dots) : x_i \in [a,b],\ f(x_i) = x_{i-1}\} = \varprojlim([a,b], f).$$

First of all we prove (a). Assume that $f$ is chaotic. By Proposition 2.1, $f|_{[a,b]}$ is also chaotic. Applying Theorem 1.1 it follows that $\sigma_f$ is chaotic. Conversely, suppose that $\sigma_f$ is chaotic. Applying Theorem 1.1 it follows that $f|_{[a,b]}$ is chaotic. Proposition 2.1 proves that $f$ is chaotic.

Part (b). If $\sigma_f$ is chaotic, then it follows by (a) that $f$ is chaotic. Hence, by Proposition 2.1, $f|_{[a,b]}$ is chaotic. Applying Theorem 1.2 (a), there exists an

increasing sequence of positive integers such that $h_A(f|_{[a,b]}) > 0$. Since $f|_{[a,b]}$ is surjective, by (2), $h_A(\sigma_f) = h_A(f|_{[a,b]}) > 0$. Now suppose that $\sigma_f$ is non-chaotic. Assertion (a) states that $f$ is non-chaotic. Applying Theorem 1.2 and (1), we conclude that $h_A(\sigma_f) \leq h_A(f) = 0$ for any increasing sequence of positive integers $A$.

Part (c). Let $A$ be an arbitrary sequence of positive integers. By Theorem 1.2(b), there exists a chaotic map $f_A\colon [0,1] \to [0,1]$ such that $h_A(f_A) = 0$. Since $f_A$ is chaotic, by (a), $\sigma_{f_A}$ is also chaotic. By (2), $h_A(\sigma_{f_A}) \leq h_A(f_A) = 0$, and the proof ends. □

## 3. A Counterexample

As usual, $\mathbb{Z}$ will stand for the set of integers, while if $Z \subset \mathbb{Z}$ then $Z^n$ (resp. $Z^\infty$) will denote the set of finite sequences of length $n$ (resp. infinite sequences) of elements from $Z$. If $\theta \in \mathbb{Z}^n$ or $\alpha \in \mathbb{Z}^\infty$ then we will often describe them through their components as $(\theta_1, \theta_2, \dots, \theta_n)$ or $(\alpha_i)_{i=1}^\infty$, respectively. The **shift map** $\sigma\colon \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ is defined by $\sigma((\alpha_i)_{i=1}^\infty) = (\alpha_{i+1})_{i=1}^\infty$. If $\theta \in \mathbb{Z}^n$ and $\vartheta \in \mathbb{Z}^m$ (with $m \leq \infty$) then $\theta * \vartheta \in \mathbb{Z}^{n+m}$ (where $n+\infty$ means $\infty$) will denote the sequence $\lambda$ defined by $\lambda_i = \theta_i$ if $1 \leq i \leq n$ and $\lambda_i = \vartheta_{i-n}$ if $i > n$. In what follows we will denote $\mathbf{0} = (0, 0, \dots, 0, \dots)$ and $\mathbf{1} = (1, 1, \dots, 1, \dots)$, while if $\alpha \in \mathbb{Z}^\infty$, then $\alpha|_n \in \mathbb{Z}^n$ is defined by $\alpha|_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

This section is devoted to construct a chaotic map $f$ on a compact metric space for which $\sigma_f$ is non-chaotic. In order to do this we will need some information concerning so-called **weakly unimodal maps of type** $2^\infty$. Recall briefly the definition. We say that a continuous map $f\colon [0,1] \to [0,1]$ is **weakly unimodal** if $f(0) = f(1) = 0$, it is non-constant and there is $c \in (0,1)$ such that $f|_{(0,c)}$ and $f|_{(c,1)}$ are monotone. The map $f$ is said to be of **type** $2^\infty$ if it has periodic points of period $2^n$ for any $n \geq 0$ but no other periods.

Weakly unimodal maps of type $2^\infty$ (briefly, **w-maps**) were studied in [**4**]. In that paper it was proved that for any w-map $f$ it is possible to construct a family $\{K_\alpha(f)\}_{\alpha \in \mathbb{Z}^\infty}$ (or simply $\{K_\alpha\}_{\alpha \in \mathbb{Z}^\infty}$ if there is no ambiguity on $f$) of pairwise disjoint (possibly degenerate) compact subintervals of $[0,1]$ satisfying the following key properties (P1)–(P4):

(P1) The interval $K_{\mathbf{0}}$ contains all absolute maxima of $f$.

(P2) Define in $\mathbb{Z}^\infty$ the following total ordering: if $\alpha, \beta \in \mathbb{Z}^\infty$, $\alpha \neq \beta$ and $k$ is the first integer such that $\alpha_k \neq \beta_k$, then $\alpha < \beta$ if either Card $\{1 \leq i < k : \alpha_i \leq 0\}$ is even and $\alpha_k < \beta_k$ or Card $\{1 \leq i < k : \alpha_i \leq 0\}$ is odd and $\beta_k < \alpha_k$. Then $\alpha < \beta$ if and only if $K_\alpha < K_\beta$ (that is, $x < y$ for all $x \in K_\alpha$, $y \in K_\beta$).

(P3) Let $\alpha \in \mathbb{Z}^\infty$, $\alpha \neq \mathbf{0}$, and let $k$ be the first integer such that $\alpha_k \neq 0$. Define $\beta \in \mathbb{Z}^\infty$ by $\beta_i = 1$ for $1 \leq i \leq k-1$, $\beta_k = 1 - |\alpha_k|$ and $\beta_i = \alpha_i$ for $i > k$

Then $f(K_\alpha) = K_\beta$ and $f(K_{\mathbf{0}}) \subset K_{\mathbf{1}}$.

For any $n$ and $\alpha \in \mathbb{Z}^\infty$, let $K_{\alpha|_n}(f)$ (or just $K_{\alpha|_n}$) be the least interval including all intervals $K_\beta$, $\beta \in \mathbb{Z}^\infty$, such that $\alpha|_n = \beta|_n$. Then

(P4) For any $\alpha \in \mathbb{Z}^\infty$, $K_\alpha = \bigcap_{n=1}^\infty K_{\alpha|_n}$.

Additionally, for any fixed $n$ it can be easily checked that the intervals $K_\theta$, $\theta \in \mathbb{Z}^n$, are open and pairwise disjoint and (after replacing $\infty$ by $n$, $\mathbf{0}$ by $(0, 0, \dots, \overset{n}{0})$ and $\mathbf{1}$ by $(1, 1, \dots, \overset{n}{1})$), they also satisfy (P1)–(P3). Observe that if $\theta \in \{-1, 0, 1\}^n$ and we put $|\theta| := (|\theta_1|, |\theta_2|, \dots, |\theta_n|)$ then $f^{2^n}(K_\theta) \subset K_{|\theta|}$; in particular, $f^{2^n}(K_\theta) \subset K_\theta$ if $\theta \in \{0, 1\}^n$.

In the rest of this section $\widetilde{f}$ will denote a fixed w-map with the additional property that $\alpha \in \mathbb{Z}^\infty$ implies $K_\alpha(\widetilde{f})$ is non-degenerate if and only if there is an $n \geq 0$ such that $\sigma^n(\alpha) = \mathbf{0}$. An example of such a map is constructed in [**4**]; it is possible to show that the stunted tent map $\widetilde{f}(x) = \min\{1 - |2x - 1|, \mu\}$ ($\mu \approx 0.8249\dots$) from [**7**] is also a w-map with this property.

Bd $(Z)$, Cl $(Z)$ and Int $(Z)$ will respectively denote the boundary, the closure and the interior of $Z$.

Now, we are ready to construct our counterexample. Consider

$$X = \bigcup_{\alpha \in \{-1,0,1\}^\infty} \text{Bd } (K_\alpha).$$

Let us emphasize that Bd $(K_\alpha)$ consists of both endpoints of $K_\alpha$ if it is non-degenerate and of its only point if it is degenerate. Let $f = \tilde{f}|_X : X \to X$ be the restriction of the above-mentioned w-map $\tilde{f}$ to the set $X$. The following lemma shows that the above choices make sense.

**Lemma 3.1.** *$X$ is a compact set and $f\colon X \to X$ is a well-defined continuous map.*

*Proof.* Since

$$X = \left( \bigcap_{n=1}^\infty \bigcup_{\theta \in \{-1,0,1\}^n} \text{Cl } (K_\theta) \right) \setminus \bigcup_{\alpha \in \{-1,0,1\}^\infty} \text{Int } (K_\alpha),$$

by (P2) and (P4), $X$ is compact.

Recall that if $\mathbf{0} \neq \alpha \in \{-1, 0, 1\}^\infty$ then $\tilde{f}$ carries the interval $K_\alpha$ onto $K_\beta$ with $\beta$ defined as in (P3) (and hence belonging to $\{-1, 0, 1\}^\infty$). Moreover, $\tilde{f}$ is monotone on $K_\alpha$ because of (P1). So it maps the endpoints of $K_\alpha$ onto the endpoints of $K_\beta$. Similarly, since $K_{\mathbf{1}}$ is degenerate both endpoints of $K_{\mathbf{0}}$ are mapped onto its only point. The conclusion is that $f(X) \subset X$ and the map $f : X \to X$ is well–defined (and it is clearly continuous). $\square$

Let $X_1 = \bigcup_{\alpha \in \{0,1\}^\infty} \text{Bd } (K_\alpha)$. Note that, by (P3), $\bigcap_{n \geq 0} f^n(X) = X_1$. Let us see that $f$ is chaotic while $f|_{X_1}$ is non–chaotic.

**Theorem 3.2.** *$f|_{X_1}$ is non–chaotic and hence $\sigma_f$ is non–chaotic.*

*Proof.* Let $x, y \in X_1$ with $x \in K_\alpha$ and $y \in K_\beta$ for some $\alpha \neq \beta$, $\alpha, \beta \in \{0,1\}^\infty$. We will see that there exists a positive real number $M$ satisfying

$$\liminf_{i\to\infty} |f^i(x) - f^i(y)| \geq M,$$

and hence $x$ and $y$ cannot belong to the same scrambled set $D$. This proves that $\mathrm{Card}(D) \leq 2$ for each scrambled set $D$ of $f|_{X_1}$ and so $f|_{X_1}$ is non–chaotic.

Let $j$ be the first positive integer satisfying $\alpha_j \neq \beta_j$. Suppose, for example, that $\alpha_j = 0$ and $\beta_j = 1$. For any $\theta \in \{0,1\}^{j-1}$ consider the closed interval $A_{\theta*0}$ satisfying $K_{\theta*1} < A_{\theta*0} < K_{\theta*0}$ or $K_{\theta*0} < A_{\theta*0} < K_{\theta*1}$, and let $M = \min\{|A_{\theta*0}| : \theta \in \{0,1\}^{j-1}\} > 0$. By (P3) and (P2), $f^i(x) \in K_{\theta*1}$ and $f^i(y) \in K_{\theta*0}$ or viceversa for all $i \in \mathbb{N}$ and for all $\theta \in \{0,1\}^{j-1}$. This shows that

$$|f^i(x) - f^i(y)| \geq M$$

which concludes the proof. □

For any $\alpha \in \{-1,0,1\}^\infty$ let $\tau(\alpha|_n) = \sum_{i=1}^n |\alpha_i| 2^{i-1}$ for all $n \in \mathbb{N}$.

**Theorem 3.3.** *$f : X \to X$ is chaotic.*

*Proof.* Define on $\{-1,1\}^\infty$ the following relation: $\alpha \sim \beta$ if and only if there exists a positive integer $k$ such that $\sigma^k(\alpha) = \sigma^k(\beta)$. Obviously $\sim$ is an equivalence relation. Moreover, for $\alpha \in \{-1,1\}^\infty$ the class of $\alpha$ is given by

$$[\alpha] = \bigcup_{k=0}^{\infty} \{\sigma^{-k}(\sigma^k(\alpha))\} \cap \{-1,1\}^\infty.$$

Since $\{-1,1\}^\infty$ is uncountable and $[\alpha]$ is countable for all $\alpha \in \{-1,1\}^\infty$, the set containing all the equivalence classes $\{-1,1\}^\infty/\sim$ is uncountable.

Let $\mathcal{A}$ be a set containing one and only one representative $\alpha \in [\alpha]$ for all $[\alpha] \in \{-1,1\}^\infty/\sim$, and let $D$ be the set containing exactly one $x \in X \cap K_\alpha$ for all $\alpha \in \mathcal{A}$. We claim that $D$ is a scrambled set for $f$. In order to see this take $x, y \in D$, $x \in K_\alpha$ and $y \in K_\beta$ with $\alpha, \beta \in \mathcal{A}$, $\alpha \neq \beta$. Then there exists an increasing sequence of positive integers $(k_i)_{i=0}^\infty$ satisfying $\alpha_{k_i} \neq \beta_{k_i}$ and $\alpha_j = \beta_j$ if $j \neq k_i$ for all $i \in \mathbb{N}$. Note that $\tau(\alpha|_n) = \tau(\beta|_n)$ for all $n \in \mathbb{N}$. Suppose, for example, that $\alpha_{k_i} = 1$ and $\beta_{k_i} = -1$ for some $i$. Then, by (P3),

$$f^{\tau(\alpha|_{k_i})}(x) \in K_{(0,0,\ldots,\overset{k_i-1}{0},1)*\sigma^{k_i}(\alpha)} \text{ and } f^{\tau(\alpha|_{k_i})}(y) \in K_{(0,0,\ldots,\overset{k_i-1}{0},-1)*\sigma^{k_i}(\beta)}.$$

By (P2), $|f^{\tau(\alpha|_{k_i})}(x) - f^{\tau(\alpha|_{k_i})}(y)| \geq |K_{\mathbf{0}}|$ and then

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geq \limsup_{i\to\infty} |f^{\tau(\alpha|_{k_i})}(x) - f^{\tau(\alpha|_{k_i})}(y)| \geq |K_{\mathbf{0}}|.$$

Let now $j \neq k_i$ for all $i \in \mathbb{N}$, and suppose that $\alpha_j = \beta_j = 1$ (the case $\alpha_j = -1$ is analogous). Then $f^{\tau(\alpha|_j)}(y), f^{\tau(\alpha|_j)}(x) \in K_{(0,0,\ldots,0,\overset{j}{1})}$. For any $n \in \mathbb{N}$ let $K^+_{\mathbf{0}|_n}$ and $K^-_{\mathbf{0}|_n}$ be the right and left side components of $K_{\mathbf{0}} \setminus K_{\mathbf{0}|_n}$. Applying (P4), for any $\varepsilon > 0$ there exists a positive integer $n_\varepsilon$ such that $\max\{|K^+_{\mathbf{0}|_n}|, |K^-_{\mathbf{0}|_n}|\} < \varepsilon$ for all $n \geq n_\varepsilon$. By (P2) and (P3), $K_{(0,0,\ldots,0,\overset{j}{1})} \subset K^-_{\mathbf{0}|_j}$ or $K_{(0,0,\ldots,0,\overset{j}{1})} \subset K^+_{\mathbf{0}|_j}$, and so for $j \geq n_\varepsilon$ we conclude that $|f^{\tau(\alpha|_j)}(y) - f^{\tau(\alpha|_j)}(x)| < \varepsilon$. This proves that

$$\liminf_{n\to\infty} |f^n(y) - f^n(x)| = 0,$$

and the proof concludes. □

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