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POINTWISE WEIGHTED VECTOR ERGODIC THEOREM IN $L^1(X)$

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Abstract. In this paper we prove the almost everywhere convergence of weighted multiparameter averages of linear surjective isometries in $L^1(X)$ and power bounded in $L^p(X), 1 < p \leq \infty$.

Throughout this paper X will be denoted a Banach space with norm $\|.\|$ and (Ω, β, μ) a σ-finite measure space. For $1 \leq p < \infty$, $L^p(X) = L^p(\Omega, X) = L^p((\Omega, \beta, \mu), X)$ denoted the usual Banach space of $X-$valued strongly measurable functions f on Ω with the norm given by

$$\begin{aligned} \|f\|_p &= \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty \text{ if } 1 \leq p < \infty, \\ \|f\|_\infty &= ess \sup \{|f(\omega)| \,; \omega \in \Omega\} < \infty \text{ if } p = \infty. \end{aligned}$$

Let $d \geq 1$ be an integer, and let $T_1, \dots, T_d$ be linear surjective isometries on $L^1(\Omega, X)$ such that each T_i is power bounded in $L^\infty(\Omega, X)$. Thus T_i, $1 \leq i \leq d$, can be considered to be power bounded in $L^p(\Omega, X)$ for each $1 < p < \infty$, by the Riesz convexity theorem.We will be concerned with classes of weights $\left\{a(\mathbf{k}); \mathbf{k} \in \mathbf{Z}_d^+\right\}$ such that the limit of averages

$$\lim_{|N| \to \infty} \frac{1}{|\mathbf{N}|} \sum_{k=0}^{N-1} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f$$

exists a.e. for all $f \in L^1(X)$, where $\mathbf{T}^{\mathbf{k}} = T_1^{k_1} \dots T_d^{k_d}$ with $\mathbf{N} = (N_1, \dots, N_d)$ and $\mathbf{k} = (k_1, \dots, k_d)$, $|\mathbf{N}| = N_1 \dots N_d$, etc. . . .

The class of weights we will consider are the Besicovich sequences in $\mathbf{Z}_d^+$. In the case $d = 1$, Besicovitch sequences are defined to be the class of sequences $a(k)$ such that given $\varepsilon > 0$, there is a trigonometric polynomial ψ_ε such that

$$\limsup_{n} \frac{1}{n} \sum_{k=0}^{n-1} |a(k) - \psi_\varepsilon(k)| < \varepsilon$$

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and bounded Besicovitch weights are bounded weights in this class. Let us define the d-dimensional analogs of Besicovitch sequences: We say that the sequence $\{a(\mathbf{k})\}$ to be r-Besicovich if for every $\varepsilon > 0$ there is a sequence of trigonometric polynomials in d variables such that

$$\limsup_{N\to\infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=0}^{\mathbf{N}-1} |a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})|^r < \varepsilon.$$

We denote this class by $B(r)$. We say that $\{a(\mathbf{k})\}$ to be r-bounded Besicovitch sequence if $\{a(\mathbf{k})\} \in B(r) \cap l^\infty$. Let $\alpha = \sup_{\mathbf{k}} a_k$. We call that the Banach space X is without 1-projections iff there is no projection P on X such that $\|x\| = = \|Px\| + \|x - Px\|$ for all $x \in X$.

In the vector case, R. V. Chacon proved the a.e. convergence of the averages $\frac{1}{n}\sum_{k=0}^{n-1} T^k f$ when X being reflexive Banach space, T is linear operator on $L^1(X)$ contraction in both $L^1(X)$ and in $L^\infty(X)$ and $f \in L^1(X)$. Yoshimoto [**13**] remarked that Chacon's theorem remains true if the operator T is contraction in $L^1(\Omega, X)$ but power bounded in $L^\infty(\Omega, X)$.

In [**8**] it was shown that Chacon's theorem remains true for the weighted averages $\frac{1}{n}\sum_{k=0}^{n-1} a(k)T^k f$ where $a(k)$ is a 1-Besicovitch bounded sequence.

In the real case ($X = \mathbf{R}$) and using linear modulus of non positive operator, R. Jones and J. Olsen proved [**11**] the a.e convergence of the averages $\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=0}^{\mathbf{N}-1} a(\mathbf{k})\mathbf{T}^{\mathbf{k}} f$ with $\mathbf{N} = (n, \dots, n)$ and $f \in L^1$ for Dunford-Schwartz operators. Akcoglu and Chacon proved [**1**] the almost everywhere convergence of the Cesaro average $A_n(T)f = \frac{1}{n}\sum_{i=0}^{n-1} T^i f$ for all $f \in L^1(\Omega, \mathbf{R})$, when T is a linear operator on $L^1(\Omega, \mathbf{R})$ contrction in $L^1(\Omega, \mathbf{R})$ and in $L^q(\Omega, \mathbf{R})$ for some $q \in]1, +\infty[$.

Using the linear modulus A. Brunel proved [**4**] the almost everywhere convergence of the multiparameter averages $A_n(T_1, \dots, T_d)f = A_n(T_1)\dots A_n(T_d)f$ when $T_1, \dots, T_d$ are linear commuting operators on $L^1(\Omega, \mathbf{R})$ contraction in both $L^1(\Omega, \mathbf{R})$ and in $L^\infty(\Omega, \mathbf{R})$.

Our aim is to prove that if the operators $T_1, \dots, T_d$ (possibly not commuting) are linear surjective isometries on $L^1(X)$ and contractions (or power bounded) in $L^\infty(X)$ then, we have a.e. convergence of the averages $\frac{1}{|\mathbf{N}|}\sum_{k=0}^{N-1} a(\mathbf{k})\mathbf{T}^{\mathbf{k}} f$ with $\mathbf{N} = (n, \dots, n)$ for all $f \in L^1(X)$.

This result maybe considered to be a multidimensional version of Chacon's theorem [**5**] for surjective isometries in $L^1(X)$ and for the weighted averages, of course, the extension of Chacon's theorem to multidimensional case remains an open problem.

Using a representation of surjective isometries due to S. Guerre and Y. Raynaud [**10**] we give a vector multiparameter version of Jones-Olsen's result. In fact, the difficulty in vector case is that: for a vector operator T on $L^1(X)$ we cannot always find a DPO (dominated positive linear operator) τ, which is a contraction on $L^1 = L^1(\mathbf{R})$ (analog to the linear modulus) and verify that for all $f \in L^1(X)$

$$\|Tf\| \leq \tau(\|f\|). \tag{1}$$

(We give a counter example proving that the operators τ does not exist in general). In [**10**] Guerre and Raynaud proved in proposition 6.1 that if an isometry T admits a DPO, τ then we have $\|Tf\| = \tau(\|f\|)$ for all $f \in L^p(X)$.

Our main result is the following:

Theorem 1. *Let $T_1, \dots, T_d$ be commuting linear surjective isometries on $L^1(X)$ and power bounded in $L^\infty(X)$. $a(\mathbf{k})$ is a d-dimensional r-Besicovitch bounded sequences. Then, for $f \in L^1(X)$ we have the almost everywhere convergence of the averages $\frac{1}{|\mathbf{N}|}\sum_{k=0}^{N-1} a(\mathbf{k})\mathbf{T}^{\mathbf{k}} f$ with $\mathbf{N} = (n, \dots, n)$ as $n \to \infty$.*

We need the following lemma:

Lemma 1. *Let $T_1, \dots, T_d$ be commuting linear surjective isometries on $L^1(X)$ and power bounded in $L^\infty(X)$. Let $F^* = \sup_N \left\| \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=0}^{N-1} a(\mathbf{k})\mathbf{T}^{\mathbf{k}} f \right\|$ then, for all $f \in L^1(X)$ and any $a > 0$, there exists a positive real χ_d such that*

$$a\mu\{F^* > a\} \le \chi_d \int_\Omega \|f\|\, d\mu.$$

Proof. We shall use the norm resolution [**10**, p. 368]. Let Z is Boole algebra (they are commuting). We define a measure μ_Z on $\mathcal{P}_1(Z)$ and a surjective mapping $N_Z : Z \to \mathcal{P}_1(Z)$ which is the norme resolution on $\mathcal{P}_1(Z)$ verifying

(i) $\forall z \in Z$, $\|z\| = |N_Z(z)|_{L^1_+(\mathcal{P}_1(Z))}$.

(ii) $\forall (u,v) \in Z^2, \forall(\alpha,\beta) \in R^2$

$$N_Z(\alpha u + \beta v) \le |\alpha|\, N_Z(u) + |\beta|\, N_Z(v).$$

Guerre and Raynaud showed [**10**] that for each linear surjective isometry $T : Z \to Z$ that there is a positive surjective isometry

$$\tau : L^1_+(\mathcal{P}_1(Z)) \to L^1_+(\mathcal{P}_1(Z))$$

such that

$$N_Z(Tf) = \tau(N_Z f). \tag{2}$$

Let $Z = L^1(X) = L(\Omega, \beta, \mu, X)$. In [**9**] it was proved that if X is separable space (without loss of generality we can suppose that X is separable) then

$$\mathcal{P}_1(Z) = \beta \otimes \mathcal{P}_1(X).$$

Let $\mu_Z = \mu \otimes \mu_X$ with these notations we can write for $f \in L^1(X)$

$$N_Z(f) \in L^1(\beta \otimes \mathcal{P}_1(X)) = L^1_\beta\left(L^1_{\mathcal{P}_1(X)}\right)$$
$$N_Z(f)(\omega) = N_X(f(\omega))$$

and

$$\|f(\omega)\|_X = |N_X(f(\omega))|_{L^1_+(\mathcal{P}_1(Z))} = |N_z(f)(\omega)|_{L^1_+(\mathcal{P}_1(Z))}.$$

Denote by $N = N_Z = N_{L^1(X)}$, (2) shows that for all $j \in \mathbf{N}$ we have

$$\tau^j(Nf) = N(T^j f)$$

and since X is reflexive Banach space then it has a finite number of 1-projections (because otherwise $X \supset l^1$). Let

$$\mathcal{P}_1(X) = \Pi = \{1, \ldots, K\}.$$

We can decompose the space X as

$$X = X_1 \oplus^1 X_2 \oplus^1 \ldots \oplus^1 X_K$$

where X_i is a Banach space without 1-projections for $i = 1, \ldots, K$. If $x \in X$ then $x = (x_i)_{1\leq i\leq K}$ and theorem of x will be

$$\|x\|_X = \sum_{i=1}^{K} \|x_i\|_{X_i}$$

where$\|x_i\|_{X_i}$ is the norm in the space X_i. We can write that the measure μ_X on the set $\{1, \ldots, K\}$ is a countable measure The space $L^1(\Omega \times \{1, \ldots, K\})$ identifies to the space $L^1(\Omega, l^1_K)$ and the norm resolutions $N = N_Z : L^1(X) \to L^1(\Omega \times \times\{1, \ldots, K\})$ and $N_X : X \to L^1_+(\{1, \ldots, K\}) = l^1_K$ are related by

$$N_X\left(f(\omega)\right)(i) = N(f)(\omega, i) = \|f_i(\omega)\|_{X_i}$$

and as $N_X\left(x)\right)(i) = \|x_i\|_{X_i}$ we obtain

$$\begin{aligned} \|f_i(\omega)\|_{X_i} &= \sum_{i=1}^{K} \|f_i(\omega)\|_{X_i} = \sum_{i=1}^{K} N_X\left(f(\omega)\right)(i) \\ &= \sum_{i=1}^{K} Nf(\omega, i) = \|Nf(\omega, .)\|_{L^1(\Pi)} \end{aligned} \tag{3}$$

(3) gives the norm in X in terms of the norm resolution on $L^1(\Omega \times \{1, \ldots, K\})$. In its definition the operator τ acts in $L^1(\Omega \times \{1, \ldots, K\}) = L^1(\Omega, l^1_K)$ and verifies $\forall j \in \mathbf{N}$

$$\tau^j(Nf) = \tau^j\left[\left(\|f_i\|_{X_i}\right)_{1\leq i\leq K}\right] = \left(\left\|T^j f_i\right\|_{X_i}\right)_{1\leq i\leq K} = N(T^j f).$$

By (2) we have for $\varphi = Nf \in L^1(\Omega' = \Omega \times \{1, \ldots, K\})$ with $f \in L^1(\Omega, X)$

$$\begin{aligned} \|\tau\varphi\|_{L^1(\Omega')} &= \|\tau\left(Nf\right)\|_{L^1(\Omega')} = \|N(Tf)\|_{L^1(\Omega,X)} \\ &= \|Nf\|_{L^1(\Omega')} = \|f\|_{L^1(\Omega,X)} = \|\varphi\|_{L^1(\Omega')} \end{aligned}$$

which proves that τ is isometry on $L^1(\Omega, l^1_K)$. In what follows we will prove that $\sup_N \left\|\frac{1}{|\mathbf{N}|}\sum_{k=0}^{N-1} a(\mathbf{k})\mathbf{T^k} f\right\|$ is finite a.e. for all $f \in L^1(\Omega, X)$. □

To complete the proof we need the following proposition:

Proposition 1. *Let T be a surjective isometry in $L^1(X)$ and contracting in $L^q(X)$ $(1 \leq q \leq \infty)$, then its DPO τ is a contraction on $L^q(l^1_k)$ and if T is power bounded in $L^q(X)$ then is power bounded in $L^q(\Omega, \{1, \ldots, k\}) = L^q(\Omega, l^1_k)$.*

Proof. Let $q \in]1,+\infty[$. If T is a contraction in $L^q(X)$,we can write

$$\begin{aligned}
\|\tau(Nf)\|_{L^q(\Omega')} &= \left[\int\left(\sum_{i=1}^{K}\tau(Nf)(\omega,i)\right)^q d\omega\right]^{1/q} \\
&= \left[\int\left(\sum_{i=1}^{K}N(Tf)(\omega,i)\right)^q d\omega\right]^{1/q} \\
&= \left[\int\left(\sum_{i=1}^{K}\|(Tf(\omega))_i\|_{X_i}\right)^q d\omega\right]^{1/q} = \left[\int \|Tf(\omega)\|_X^q\, d\omega\right]^{1/q} \\
&= \|Tf\|_{L^q(X)} \leq \|f\|_{L^q(X)} = \|\|f\|_X\|_{L^q} \\
&= \left\|\sum_{i=1}^{K}\|f\|_i\right\|_{L^q} = \|Nf\|_{L^q(l_k^1)}.
\end{aligned}$$

If T is power bounded in $L^q(X)$ we can write

$$\begin{aligned}
\left\|\tau^j(Nf)\right\|_{L^q(\Omega')} &= \left\|N(T^jf)\right\|_{L^q(l_K^1)} = \left\|T^jf\right\|_{L^q(X)} \\
&\leq \|f\|_{L^q(X)} = \|Nf\|_{L^q(l_K^1)}.
\end{aligned}$$

Clearly

$$\|(.)\|_{L^q(l_K^q)} \leq \|(.)\|_{L^q(l_K^1)} \leq K^{\frac{q-1}{q}}\|(.)\|_{L^q(l_K^q)}$$

and then we get

$$\left\|\tau^j(Nf)\right\|_{L^q(l_K^q)} \leq \left\|\tau^j(Nf)\right\|_{L^q(l_K^1)} \leq cK^{\frac{q-1}{q}}\|Nf\|_{L^q(l_K^q)}$$

which proves that τ is power bounded in $L^q(\Omega')$, $1 < q \leq \infty$.

We have two cases to consider:

Case 1. The space X is without 1-projections.

By [**10**] if T_j, $j = 1,\dots,d$, are surjective isometries on $L^1(X)$ and contraction (resp. power bounded) in $L^q(X)$, $1 < q \leq \infty$, then, by Proposition 1.6 in [**10**] (resp. Proposition 2) they are majorizable by (DPO) τ_j, $j = 1,\dots,d$, which are isometries in L^1 and contraction in L^q (resp. power bounded in $L^q(\Omega')$, $1 < q \leq \infty$.

To the operators $\tau_1,\dots,\tau_d$ we associate the Brunel operator U which is a contraction in L^1 and a contraction (resp. a power bounded) in $L^q, 1 < q \leq \infty$. Moreover, the operator U verifies [**12**, p. 213]

$$\begin{aligned}
&\sup_n\left\|\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1}a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f\right\| = \sup_n\left\|\frac{1}{\mathbf{n}^d}\sum_{\mathbf{k}=0}^{\mathbf{N}-1}a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f\right\| \\
&\leq \sup_n\left\|\frac{1}{\mathbf{n}^d}\sum_{k=0}^{\mathbf{N}-1}a(\mathbf{k})\tau^{\mathbf{k}}f\right\| \quad \leq \alpha\sup_n\left\|\frac{1}{\mathbf{n}^d}\sum_{k=0}^{\mathbf{N}-1}\tau^{\mathbf{k}}\|f\|\right\| \\
&\leq \alpha\chi_d\sup_n A_n(U)\|f\|.
\end{aligned}$$

In this case we obtain by applying the maximal weak inequality to the operator U (which is Dunford-Schwarz operators)

$$a\mu\left\{F^* > a\right\} \leq a\mu\left\{\sup_n A_n(U)\left\|f\right\| > a/\alpha\chi_d\right\} \leq \alpha\chi_d \int_\Omega \left\|f\right\| d\mu.$$

Case 2. The space X is reflexive Banach space:

The operators $\tau_1, \ldots, \tau_d$ are isometries in $L^1(\Omega' = \Omega \otimes \Pi)$ and power bounded in $L^\infty(\Omega')$ apply the Proposition 2 with $q = \infty$ to obtain that Brunel's operator U associated to the family $\tau_1, \ldots, \tau_d$ is also a contraction in L^1 and power bounded in L^∞. Using (3) and the fact that μ_X is a countable measure we get

$$\begin{aligned}
\sup_n \left\| \frac{1}{\mathbf{n}^d} \sum_{k=0}^{N-1} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f(\omega) \right\| &\leq \sup_n \sum_{i=1}^{K} N\left(\frac{1}{\mathbf{n}^d} \sum_{k=0}^{N-1} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f \right)(\omega, i) \\
&\leq \sum_{i=1}^{K} \sup_n \left(\frac{1}{\mathbf{n}^d} \sum_{k=0}^{N-1} a(\mathbf{k}) \tau^{\mathbf{k}} \right)(Nf)(\omega, i) \\
&\leq \chi_d \sum_{i=1}^{K} \sup_n A_n(U) N f(\omega, i).
\end{aligned} \tag{4}$$

By Yoshimoto [**13**], we can write

$$\begin{aligned}
a\mu\left\{F^* > a\right\} &\leq a\mu\left\{\sum_{i=1}^{K} \sup_n A_n(U) N f(\omega, i) > a/\chi_d\right\} \\
&\leq a\mu\left\{\bigcup_{i=1}^{K}\left\{\sup_n A_n(U) N f(\omega, i) > a/k\chi_d\right\}\right\} \\
&\leq \sum_{i=1}^{K} a\mu\left\{\sup_n A_n(U) N f(\omega, i) > a/k\chi_d\right\} \\
&\leq \chi_d \sum_{i=1}^{K} \int_{\Omega'} N f(\omega, i) d\left(\mu \otimes \mu_X\right) \\
&= \chi_d \left\|Nf\right\|_{L^1(\Omega')} \\
&= \alpha\chi_d \left\|f\right\|_{L^1(\Omega, X)} = \alpha\chi_d \int_\Omega \left\|f\right\| d\mu.
\end{aligned}$$

□

Before giving the proof of our main result, we state a result of Jones-Olsen in [**11**, pp. 351]

Theorem 2. *For all $r \geq 1$ we have $B(r) \cap l^\infty = B(1) \cap l^\infty$.*

By this theorem it suffices to prove theorem 1 for the 1-Besicovitch bounded sequences.

Proof of Theorem 1. We have to prove that the averages $\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f$ converge a.e. on a dense set on $L^1(X)$. Let $L^\infty(X)$ such a set. For every $\varepsilon > 0$ we have

$$\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f = \frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} (a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k}))\,\mathbf{T}^{\mathbf{k}}f + \frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} \psi_\varepsilon(\mathbf{k})\mathbf{T}^{\mathbf{k}}f$$

and then for all $f \in L^\infty(X)$ we have

(5)

$$\begin{aligned}\left\|\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f\right\| &\leq \left\|\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} (a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k}))\,\mathbf{T}^{\mathbf{k}}f\right\| + \left\|\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} \psi_\varepsilon(\mathbf{k})\mathbf{T}^{\mathbf{k}}f\right\| \\ &\leq \frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} |a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})|\,\left\|\mathbf{T}^{\mathbf{k}}f\right\| + \left\|\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} \psi_\varepsilon(\mathbf{k})\mathbf{T}^{\mathbf{k}}f\right\|.\end{aligned}$$

In the case 1 we have

$$\begin{aligned}\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} |a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})|\,\left\|\mathbf{T}^{\mathbf{k}}f\right\| &\leq \frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} |a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})|\,\tau^k\,\|f\| \\ &\leq \|f\|_\infty \frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} |a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})| < \varepsilon\,\|f\|_\infty.\end{aligned}$$

In the case 2 we have

$$\begin{aligned}\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} |a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})|\,\left\|\mathbf{T}^{\mathbf{k}}f\right\| &\leq \frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} |a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})|\left[\sum_{i=1}^{K} N\left(\mathbf{T}^{\mathbf{k}}f\right)(.,i)\right] \\ &= \frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} |a(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})|\left[\sum_{i=1}^{K} \tau^{\mathbf{k}} Nf(.,i)\right] \\ &\leq \varepsilon\left[\sum_{i=1}^{K} \tau\,\|Nf(.,i)\|_\infty\right].\end{aligned}$$

We have seen in [**7**, pp. 28] that a. e. convergence holds for the averages $\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} \mathbf{T}^{\mathbf{k}}f$. Let $(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$, with $|\lambda_i| = 1$, $i = 1, 2, \ldots, d$ then the theorem holds when $a(k) = \lambda_1^{k_1} \ldots \lambda_d^{k_d}$ since $\widetilde{\mathbf{T}}^{\mathbf{k}} = \lambda_1^{k_1} \ldots \lambda_d^{k_d}\mathbf{T}^{\mathbf{k}}$ is also a d-parameter sequences of surjective isometries operators on $L^1(X)$ and power bounded in $L^\infty(X)$ when $a(\mathbf{k}) = 1$. Clearly the a.e. convergence holds for finite linear combinations of such sequences, and hence holds for trigonometric ploynomial in d variables, which proves the convergence a.e. of the second term of (5) and then we have a.e. convergence of

$$\frac{1}{|\mathbf{N}|}\sum_{k=0}^{\mathbf{N}-1} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}f$$

for all $f \in L^{\infty}(X)$. The Banach principle combining with Lemma 1 end the proof of Theorem 1 in the contraction (resp. power bounded) case by applying Akcoglu-Chacon's Theorem (resp. Yoshimoto's) Theorem to the operator U.

Remark 1. In the case when X is without 1-projections we obtain that Akcoglu-Chacon's theorem [**2**] can be extended to vector case and for linear surjective isometries.

Remark 2. If $p = 2$, Burkholder [**12**] constructed a surjective isometry in L^2 for which the pointwise ergodic theorem is false. For this reason we can prove that if X is reflexive Banach lattice, and if $T_1, \ldots, T_d$ are (non-commuting) surjective isometries on $L^p(X)$, $1 < p \neq 2 < \infty$, then $\forall f \in L^p(X)$:

$$\begin{aligned} \left\| \sup_{\vec{N}} \left\| \frac{1}{|\mathbf{N}|} \sum_{k=0}^{N-1} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f \right\| \right\|_{L^p(\Omega,X)} &\leq \left\| \left\| \sup_{\vec{N}} N_X \left(\frac{1}{|\mathbf{N}|} \sum_{k=0}^{N-1} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f \right) \right\|_{L^p(\Pi)} \right\|_{L^p(\Omega,X)} \\ &= \left\| \sup_{\vec{N}} N_X \left(\frac{1}{|\mathbf{N}|} \sum_{k=0}^{N-1} a(\mathbf{k}) \mathbf{T}^{\mathbf{k}} f \right) \right\|_{L^p(\Pi \otimes \Omega)} \\ &\leq \left\| \sup_{\vec{N}} \frac{1}{|\mathbf{N}|} \sum_{k=0}^{N-1} a(\mathbf{k}) \tau^{\mathbf{k}} (Nf) \right\|_{L^p(\Pi \otimes \Omega)} \\ &\leq \alpha \left(\frac{p}{p-1} \right)^d \|Nf\|_{L^p(\Pi \otimes \Omega)} \\ &= \alpha \left(\frac{p}{p-1} \right)^d \|f\|_{L^p(\Omega,X)} . \end{aligned}$$

The least inequality is true by applying Akcoglu's theorem d times successively on the operators $\tau_1, \ldots, \tau_d$.

For the commuting case. Using Brunel operator we obtain the following strong estimates

$$\left\| \sup_{\vec{N}} \left\| A_{\vec{N}}(T_1, \ldots, T_d) f \right\| \right\|_{L^p(\Omega,X)} \leq \chi_d \frac{p}{p-1} \|f\|_{L^p(\Omega,X)} .$$

Remark 3. 1. If T is power bounded in $L^q(X)$ then its *DPO*; τ is power bounded in $L^q(\Omega')$. But we don't know if τ is a contraction in $L^q(\Omega') = L^q(l_K^q)$ when T is in $L^q(X)$. For this reason in the case "X has 1-projection", we cannot obtain an extension of Akcoglu-Chacon's theorem.

2. If $\mathcal{P}_1(X)$ is singleton and if T is a contraction in $L^q(X)$, then τ is contraction in $L^q(\Omega')$, $1 < q < \infty$.

Now, we can find the results of [**7**] and [**8**] as corollary:

Corollary 1. *Let $T_1, \ldots, T_d$ be d commuting linear surjective isometries on $L^1(X)$ and power bounded in $L^{\infty}(X)$. Then, for $f \in L^p(X)$ we have the almost* *everywhere convergence of the averages*

$$A_n(T_1,\dots,T_d)f = \frac{1}{n^d}\sum_{i_1=0}^{n-1}\dots\sum_{i_d=0}^{n-1} T^{i_1}\dots T^{i_d} f$$

as $n \to \infty$.

Corollary 2. *Let* X *be a reflexive Banach space and let* $\theta_1,\dots,\theta_d$ *be* d *preserving measure transformations on a* $\sigma-$*finite space. If* $T_j f = f o \theta_j$, $j = 1,\dots,d$, *and if the transformations are commuting then the averages* $A_n(T_1,\dots,T_d)f$ *converge a.e. for all* $f \in L^1(X)$.

In the following we give an example of vector operator which is not majorizable by a dominated positive operator contraction in L^1.

Example 1. Let $\Omega = \{1,2\}$ be probability space, $\mu(1) = \mu(2) = 1/2$, and let $X = \mathbf{R}^2$ (reflexive Banach space) with norm $\|(x,y)\| = |x| + |y|$. The Banach space $L^1\left(\{1,2\},\mathbf{R}^2\right)$ will be of dimension 4. The operator T will be represented by a square matrix of order 4. Let $T = (a_{ij})_{1\le i,j\le 4}$ if such an operator (DPO) τ exists then it will verify for all $\varphi \in L^1$

$$\tau_0\varphi = \sup\{\|Tf\|;\ \|f\| \le \varphi\} \le \tau\varphi.$$

We shall prove that for an operator T, τ_0 is not contracting in L^1. From this we deduce that τ is also not contracting. The condition $\|Tf\|_1 \le \|f\|_1$ implies that $\sum_{i=1}^4 |a_{ij}| \le 1$ for $j = 1,\dots,4$. The operator τ_0 is contracting in L^1 if

$$(*) \qquad |a_{11}| + |a_{21}| + |a_{32}| + |a_{42}| \le 1$$

$$(**) \qquad |a_{12}| + |a_{22}| + |a_{31}| + |a_{41}| \le 1.$$

But if we take $a_{11} = 1/2$; $a_{21} = a_{31} = a_{41} = 10^{-1}$ and $a_{12} = a_{22} = a_{32} = a_{42} = 1/4$ and 0 elsewhere then while the operator T is $L^1(X)-$contraction we have the conditions $(*)$ and $(**)$ are not satisfied. This implies that τ_0 is not L^1-contraction. Consequently, τ is not L^1-contraction. A similar calculation shows that τ_0 is not L^p-contraction although the operator T is. Since $X = l_2^1$ has two projections, by its definition, DPO and as τ_0 is not contraction in $L^1(\Omega')$, we then get that T does not have a *DPO*. By Chacon's theorem the Cesaro averages $A_n(T)f$ converge a.e. for all $f \in L^1(X)$ while the operator T is not majorilized by a DPO.

Remark 4.

1. By a similar argument of that of Brunel [**4**] Theorem 1 remains true when X is a reflexive Banach lattice space and the operators $T_1,\dots,T_d$ are positive linear operators contraction in both $L^1(X)$ and in $L^\infty(X)$.
2. The commutation of $\tau_1,\dots,\tau_d$ is not needed to the construction of the Brunel operator U.
3. If X is without1-projections the contraction of the operator T in $L^\infty(X)$ assumed in [**5**] can be replaced by the contraction in some $L^q(X)$ with $q \in]1,+\infty[$.

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