Acta Math. Univ. Comenianae
Vol. LXXII, 1(2003), pp. 67–72

TOPOLOGICAL REPRESENTATIONS OF QUASIORDERED SETS

VĚRA TRNKOVÁ

Abstract. We prove that for every infinite cardinal number α there exists a space X with $|X| = \alpha$, metrizable whenever $\alpha \geq \mathfrak{c}$, strongly paracompact whenever $\omega \leq \alpha \leq \mathfrak{c}$, such that every quasiordered set $(Q, \leq)$ with $|Q| \leq \alpha$ can be represented by closed subspaces of X in the sense that there exists a system $\{X_q | q \in Q\}$ of non-homeomorphic closed subspaces of X such that

$$q_1 \leq q_2 \text{ if and only if } X_{q_1} \text{ is homeomorphic to a subset of } X_{q_2}.$$

In fact, stronger results are proved here.

1. Introduction and the Main Results

Every class $\mathcal{M}$ of continuous maps, closed with respect to the composition and containing all homeomorphisms, determines a relation $\preceq$ on the class **Top** of all topological spaces by the rule

$$X \preceq Y \quad \text{if and only if there exists} \quad f : X \to Y \quad \text{in} \quad \mathcal{M}.$$

Clearly, the relation $\preceq$ is reflexive and transitive but not antisymmetric, i.e. it is a quasiorder on **Top**. We say that a quasiordered set $(Q, \leq)$ has an *$\mathcal{M}$-representation within a class $\mathbb{C}$ of topological spaces* if there exists a system $\{X_q | q \in Q\}$ of non-homeomorphic spaces in $\mathbb{C}$ such that, for every q_1, $q_2 \in Q$,

$$q_1 \leq q_2 \quad \text{if and only if} \quad X_{q_1} \preceq X_{q_2}.$$

Investigations of $\mathcal{M}$-representations for the class $\mathcal{M}$ of all homeomorphic embeddings are of rather old origin. In 1926, C. Kuratovski and W. Sierpiński proved in [**4**] that the ordinal $\mathfrak{c}^+$ has such a representation within the class of subspaces of the real line and C. Kuratowski proved in [**3**] that the antichain on $2^{\mathfrak{c}}$ points also has such representation within this class. After more than sixty years, this field of problems was revisited in [**5**], [**6**], [**7**], [**8**]. In [**5**], such a representation was constructed for every poset(= partially ordered set) of cardinality at most $\mathfrak{c}$ and,

Received November 13, 2002.

2000 *Mathematics Subject Classification.* Primary 54B30, 54H10.

Key words and phrases. Homeomorphisms onto (closed,clopen) subspaces, quasiorders, ultrafilters on ω, metrizable spaces.

Financial support of the Grant Agency of the Czech Republic under the grants no. 201/00/1466 and no. 201/02/0148 is gratefully acknowledged. Also supported by MSM 113200007.

in [**8**], for the set $\exp \mathfrak{c}$ of all subsets of $\mathfrak{c}$ ordered by the inclusion. The result of [**8**] implies those of [**5**] and [**3**] because, for every infinite set A, every poset $(P, \leq)$ with $|P| \leq |A|$ and the antichain on $2^{|A|}$ points can be embedded into $(\exp A, \subseteq)$.

In [**6**], [**7**], representations of quosets(= quasiordered sets) are investigated (with respect to the homeomorphic embeddings). Given an infinite cardinal number α, the authors of [**7**] construct a T_0-space X with $|X| \leq \delta(\alpha)$, where $\delta(\alpha)$ denotes the smallest cardinal number δ for which there exist α distinct cardinals (not necessarily infinite) smaller than δ, such that every quoset $(Q, \leq)$ with $|Q| \leq \alpha$ has a representation by the subspaces of X (with respect to the homeomorphic embeddings). In the final comment, they say that it would be good to have such spaces with better separation axioms and a lower cardinality than $\delta(\alpha)$ (which is satisfactorily small for $\alpha = \omega$ but rather large for α uncountable). We present here such a space X with $|X| = \alpha$ and X strongly paracompact whenever $\omega \leq \alpha \leq \mathfrak{c}$ and metrizable whenever $\alpha \geq \mathfrak{c}$.

In fact, we present stronger results: we investigate also smaller systems of subspaces of X (e.g. all *closed* subspaces of X) and $\mathcal{M}$-representations also for other classes of maps, namely

- $\mathcal{M}_1$ = the class of all one-to-one continuous maps,
- $\mathcal{M}_2$ = the class of all homeomorphic embeddings,
- $\mathcal{M}_3$ = the class of all homeomorphisms onto closed subspaces,
- $\mathcal{M}_4$ = the class of all homeomorphisms onto clopen[1] subspaces.

For $i \leq j$, an *$\mathcal{M}_i\mathcal{M}_j$−representation of a quoset $(Q, \leq)$ within a class $\mathbb{C}$ of spaces* is any system $\{X_q | q \in Q\}$ of non-homeomorphic spaces in $\mathbb{C}$ such that

- if $q_1 \leq q_2$, then there exists $f : X_{q_1} \to X_{q_2}$ in $\mathcal{M}_j$ and
- if $q_1 \not\leq q_2$, then no $f : X_{q_1} \to X_{q_2}$ is in $\mathcal{M}_i$.

The following theorem is an easy application of the ideas of [**7**] and the well-known results (see below):

Theorem 1. *For any infinite cardinal α, $(\exp \alpha, \subseteq)$ has an $\mathcal{M}_1\mathcal{M}_4$-representation within the class of all clopen subspaces of a suitable space X with $|X| = \alpha$ which is*

- *strongly paracompact whenever $\omega \leq \alpha \leq \mathfrak{c}$,*
- *metrizable whenever $\alpha \geq \mathfrak{c}$.*

As mentioned above, all posets of cardinalities at most α and the antichain on 2^α points can be embedded into $(\exp \alpha, \subseteq)$, hence they have $\mathcal{M}_1\mathcal{M}_4$-representation within clopen subspaces of the above X.

To formulate the theorems about the representability of quosets, let T_α denote the quoset obtained from $(\exp \alpha, \subseteq)$ by splitting any element into 2^α distinct but mutually comparable elements. More precisely, T_α is the set $\exp \alpha \times \exp \alpha$ with the quasiorder $\leq$ given by the rule

$$(A_1, A_2) \leq (B_1, B_2) \quad \text{if and only if} \quad A_1 \subseteq B_1.$$

[1] closed-and-open

Theorem 2. *For every $\alpha \geq \mathfrak{c}$ there exists a metrizable space X such that $|X| = \alpha$ and T_α has an $\mathcal{M}_1\mathcal{M}_4$-representation within the clopen subspaces of X. For $\alpha = \mathfrak{c}$, X can be moreover strongly paracompact.*

Theorem 3. *For every α with $\omega \leq \alpha \leq \mathfrak{c}$ there exists a strongly paracompact space X such that $|X| = \alpha$ and T_α has an $\mathcal{M}_2\mathcal{M}_3$-representation within the set of closed subspaces of X.*

Proofs of these theorems are presented in the section below. Although, for $\alpha \geq \mathfrak{c}$, Theorem 2 implies the statement of Theorem 1, we give a separate proof of Theorem 1 because of its simplicity.

2. The Proofs

Proof of Theorem 1. a) Let $\omega \leq \alpha \leq \mathfrak{c}$: By [**2**], there exists a set of cardinality $\mathfrak{c}$ of non-principal ultrafilters on ω which are mutually incomparable in the Rudin-Keisler order of ultrafilters, i.e. there exists a system $\{\mathcal{F}_i | i \in \mathfrak{c}\}$ of non principal ultrafilters on ω such that, denoting by P_i the subspace $P_i = \omega \cup \{\mathcal{F}_i\}$ of the compactification $\beta\omega$, every continuous map $P_i \to P_j$ is constant on a set $F \in \mathcal{F}_i$ whenever $i \neq j$. Then the space $X = \coprod_{i\in\alpha} P_i$, where $\amalg$ denotes the coproduct(= the sum = the disjoint union as clopen subspaces), has the required properties: if $A \subseteq \alpha$, we put $X_A = \coprod_{i\in A} P_i$. Then, clearly, $\{X_A | A \subseteq \alpha\}$ forms an $\mathcal{M}_1\mathcal{M}_4$-representation of $(\exp\alpha, \subseteq)$.
b) Let $\alpha \geq \mathfrak{c}$: We put again $X = \coprod_{i\in\alpha} P_i$ and $X_A = \coprod_{i\in A} P_i$; but now, $\mathcal{P} = \{P_i | i \in \alpha\}$ is a system of metrizable spaces such that $|P_i| = \alpha$ and every continuous map $P_i \to P_j$ is constant whenever $i \neq j$ (and then $\{X_A | A \subseteq \alpha\}$ is an $\mathcal{M}_1\mathcal{M}_4$-representation of $(\exp\alpha, \subseteq)$ again). Such a system $\mathcal{P}$ does exist. More strongly,

$$(*)\quad \begin{cases} \text{for every cardinal nuber } \alpha \geq \mathfrak{c} \text{ there exists a set } \mathcal{P} \text{ of the cardinality } 2^\alpha \\ \text{consisting of metrizable spaces of the cardinality } \alpha \text{ such that if } X, Y \in \mathcal{P} \\ \text{and } f : X \to Y \text{ is a continuous map, then either } f \text{ is constant or } X = Y \\ \text{and } f \text{ is the identity.} \end{cases}$$

This is explicitly stated in [**12**, p. 510] where this construction is completly described (for all the corresponding proofs see [**9**, pp 139, 215–219 and 222–226], but $(*)$ is not explicitly stated there). The construction also implies that for $\alpha = \mathfrak{c}$, the spaces in $\mathcal{P}$ are separable; hence X, being a coproduct of $\mathfrak{c}$ metrizable separable spaces, is strongly paracompact. □

Proof of Theorem 2. We use the system $\mathcal{P}$ satisfying $(*)$ of the previous proof again and we use also a compact metric zero-dimensional space K of the cardinality at most $\mathfrak{c}$ homeomorphic to the coproduct of its three copies $K \amalg K \amalg K$ but not homeomorphic to $K \amalg K$. Such a space was constructed in [**1**]. Hence

1. if $P_1, P_2 \in \mathcal{P}, P_1 \neq P_2$, then there exists no continuous one-to-one map of any of the spaces $P_1, P_1 \times K, P_1 \times (K \amalg K)$ into any of the spaces $P_2, P_2 \times K, P_2 \times (K \amalg K)$ (because K is zero-dimensional while the spaces in $\mathcal{P}$ must be connected)

2. although $P_2 \times K$ is homeomorphic to a clopen subspace of $P_2 \times (K \amalg K)$ and vice versa, $P_2 \times K$ and $P_2 \times (K \amalg K)$ are not homeomorphic. In fact, since every continuous map $f : P_2 \to P_2$ has to be either the identity or a constant, by $(*)$, the existence of a homeomorphism of $P_2 \times K$ onto $P_2 \times (K \amalg K)$ would imply the existence of a homeomorphism of K onto $K \amalg K$.

Let $\widetilde{\mathcal{P}} = \{P_{i,j} | i \in \alpha, j = 1, 2\}$ be a subsystem of $\mathcal{P}$. Then our required space is

$$X = \coprod_{i \in \alpha} P_{i,1} \amalg \coprod_{i \in \alpha} (P_{i,2} \times K)$$

i.e. $|X| = \alpha$ and T_α has an $\mathcal{M}_1\mathcal{M}_4$-representation within clopen subspaces of X: for $(A_1, A_2) \in T_\alpha$ we put

$$X_{(A_1,A_2)} = \coprod_{i \in A_1} P_{i,1} \amalg \coprod_{i \in A_2} (P_{i,2} \times K) \amalg \coprod_{i \in \alpha \setminus A_2} (P_{i,2} \times h(K \amalg K)),$$

where h is a homeomorphism of $K \amalg K \amalg K$ onto K. Then, clearly, $\{X_{(A_1,A_2)} | (A_1, A_2) \in T_\alpha\}$ is an $\mathcal{M}_1\mathcal{M}_4$-representation of T_α. □

Proof of Theorem 3. As in part a) of the proof of Theorem 1, we use the incomparable ultrafilters again; but now, we denote the subspace $\omega \cup \{\mathcal{F}\}$ of $\beta\omega$ by $P_\mathcal{F}$. We use also the construction of [**11**] of a countable strongly paracompact space S homomorphic to $S \times S \times S$ but not to $S \times S$. We recall it briefly: first, for every triple $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ of non-principal ultrafilters on ω, the space $P_{\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2}$ is constructed in [**11**] as follows:

$$\widetilde{P}_0 = P_{\mathcal{F}_0}, \quad \widetilde{P}_1 = P_{\mathcal{F}_0} \amalg \coprod (\omega \times P_{\mathcal{F}_1}),$$

$$\widetilde{P}_n = P_{\mathcal{F}_0} \amalg (\omega \times P_{\mathcal{F}_1}) \amalg \coprod_{k=2}^{n} (\omega^k \times P_{\mathcal{F}_2}) \quad \text{for } n \geq 2.$$

Now, let $P_0 = P_{\mathcal{F}_0}$ and let P_n be the quotient space of $\widetilde{P}_n$ obtained by identifying each point $m \in \omega \subseteq P_{\mathcal{F}_0}$ with the point $(m, \mathcal{F}_1) \in \omega \times P_{\mathcal{F}_1}$ and, for $n > 1$, each point $(m_1, m_2) \in \omega \times \omega \subseteq \omega \times P_{\mathcal{F}_1}$ with the point $(m_1, m_2, \mathcal{F}_2) \in \omega^2 \times P_{\mathcal{F}_2}$ and, for $n > 2$, each point $(m_1, \ldots, m_k) \in \omega^{k-1} \times \omega \subseteq \omega^{k-1} \times P_{\mathcal{F}_2}$ with the point $(m_1, \ldots, m_k, \mathcal{F}_2) \in \omega^k \times P_{\mathcal{F}_2}$, $k = 3, 4, \ldots, n$. We may suppose that $P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$ and $P_{\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2}$ is $\bigcup_{n=0}^{\infty} P_n$ with the inductively generated topology (in the modern description, see [**14**], $P_{\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2}$ is precisely the space $Seq(u_t)$ with $u_t = \mathcal{F}_0$ whenever the length $|t|$ is 0, $u_t = \mathcal{F}_1$ whenever $|t| = 1$ and $u_t = \mathcal{F}_2$ in all the other cases).

Let $\{\mathcal{F}_{j,n} | j \in \{0, 1, 2\}; n \in \omega\}$ be a collection of pairwise incomparable non-principal ultrafilters on ω. As in [**11**], let us denote the space $P_{\mathcal{F}_{0,n},\mathcal{F}_{1,n},\mathcal{F}_{2,n}}$ by Q_n and its point $\mathcal{F}_{0,n}$ by O_n. For every map $a : \omega \to \omega$ put

$$\widetilde{Q}_a = \prod_{n \in \omega} Q_n^{a(n)}$$

(where $Q_n^{a(n)} = \{O_n\}$ whenever $a(n) = 0$) and denote by O_a its point with all the coordinates equal to the corresponding O_n's. Let Q_a be the subspace of $\widetilde{Q}_a$ consisting of all the points which differ from O_a only in finitely many coordinates. The space S (homeomorphic to $S \times S \times S$ but not to $S \times S$) is a coproduct of ω copies of Q_a for every a in a countable set $A \subseteq \omega^\omega$ satisfying $A = A + A + A$ and $A \cap (A + A) = \emptyset$ (where $A + A = \{a + b | a, b \in A\}$, $\big(a + b\big)(n) = a(n) + b(n)$); such a set A does exist, see [**10**] .

Now, let a cardinal number α with $\omega \leq \alpha \leq \mathfrak{c}$ be given. Let $\{\mathcal{F}_i, \mathcal{F}_{i,j,n} | i \in \alpha; j \in \{0, 1, 2\}; n \in \omega\}$ be a system of mutually incomparable non-principal ultrafilters on ω. We put

$$X = \coprod_{i \in \alpha} P_{\mathcal{F}_i} \amalg \coprod_{i \in \alpha} S_i$$

where S_i is the space obtained by the above described construction from the system $\{\mathcal{F}_{i,j,n} | j \in \{0, 1, 2\}; n \in \omega\}$. Then X has the required properties, i.e. T_α has an $\mathcal{M}_2\mathcal{M}_3$-representation within closed subspaces of X. In fact, for $(A_1, A_2) \in T_\alpha$, we put

$$X_{(A_1,A_2)} = \coprod_{i \in A_1} P_{\mathcal{F}_i} \amalg \coprod_{i \in A_2} S_i \amalg \coprod_{i \in \alpha \setminus A_2} h_i(S_i \times S_i \times \{s_i\})$$

where h_i is a homeomorphism of $S_i \times S_i \times S_i$ onto S_i and s_i is an arbitrarily chosen point in S_i. Then $\{X_{(A_1,A_2)} | (A_1, A_2) \in T_\alpha\}$ is an $\mathcal{M}_2\mathcal{M}_3$-representation of T_α by closed subspaces of X. This follows easily from the incomparability of the above ultrafilters $\mathcal{F}_i$, $\mathcal{F}_{i,j,n}$ using the following Lemma 5 of [**11**]:

> Let $\{R_n | n \in \omega\}$ be arbitrary spaces and $\pi_k : \prod_{n \in \omega} R_n \to R_k$ be the projections. For any non-principal ultrafilter $\mathcal{F}$ on ω and any homeomorphism h of $P_\mathcal{F}$ into the space $\prod_{n \in \omega} R_n$ there exists $n \in \omega$ such that $\pi_n \circ h$ is nonconstant on any $F \in \mathcal{F}$.

Clearly, for every clopen subset $\mathcal{U}$ of S_i, every point $x \in \mathcal{U}$ lies in a copy of $P_{\mathcal{F}_{i,j,n}}$ contained in $\mathcal{U}$, for some $j \in \{0, 1, 2\}$ and $n \in \omega$, such that the copy is closed in S_i and x plays the rôle of the point $\mathcal{F}_{i,j,n}$ in it. Hence, for every $i \in \alpha$, there exists no homeomorphism of S_i (or of P_i or $h_i(S_i \times S_i \times \{s_i\})$) into the coproduct of all the other summands in the definition of X (or of $X_{(A_1,A_2)}$). Thus if $A_1 \not\subseteq B_1$, then $X_{(A_1,A_2)}$ does not admit any homeomorphism into $X_{(B_1,B_2)}$; and $X_{(A,B_1)}$ is not homeomorphic to $X_{(A,B_2)}$ whenever $B_1 \neq B_2$ because, for $i \in (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$, S_i is not homeomorphic to $h_i(S_i \times S_i \times \{s_i\})$. □

Concluding remarks. Questions and results concerning $\mathcal{M}$-representations (or $\mathcal{M}\mathcal{M}'$-representations with $\mathcal{M} \supseteq \mathcal{M}'$) within various classes of spaces form a very extensive field. Some results of this kind can be found in [**13**], along with an initial attack on "simultaneous representations" (i.e. representations of more than one quasiordered set by a single system of spaces).

References

1. Ketonen J., *The structure of countable Boolean algebras*, Annals of Math. **108** (1978), 41–81.

2. Kunen K., *Ultrafilters and independent sets*, Trans. Amer. Math. Soc. **172** (1972), 299–306.

3. Kuratowski C., *Sur la puissance de l'ensemble des "nombres de dimension" au sens de M. Fréchet*, Fund. Math. **8** (1926), 201–208.
4. Kuratowski C. and Sierpiński W., *Sur un problème de M. Fréchet concernant les dimensions des ensembles linéaires*, Fund. Math. **8** (1926), 193–200.
5. Mathews P. T. and McMaster T. B. M., *Families of spaces having prescribed embeddability order-type*, Rend. 1st Mat. Univ. Trieste **25** (1993), 345–352.
6. McCluskey A. E. and McMaster T. B. M., *Realising quasiordered sets by subspaces of 'continuum-like' spaces*, Order **15** (1999), 143–149.
7. McCluskey A. E. and McMaster T. B. M., *Representing quasiorders by embeddability ordering of families of topological spaces*, Proc. Amer. Math. Soc. **127** (1999), 1275–1279.
8. McCluskey A. E., McMaster T. B. M. and Watson W. S., *Representing set-inclusion by embeddability (among the subspaces of the real line)*, Topology its Appl. **96** (1999), 89–92.
9. Pultr A. and Trnková V., *Combinatorial, Algebraic and Topological Representation of Groups, Semigroups and Categories*, North Holland 1980.
10. Trnková V., *Representation of semigroups by products in a category*, J. Algebra **34** (1975), 191–204.
11. V. Trnková, *Homeomorphisms of products of countable spaces*, DAN SSSR **263** (1982), 47–51 (in Russian); English translation AMS: Soviet. Math. Dokl. *25* (1982), 304–307.
12. Trnková V., *Amazingly extensive use of Cook continuum*, Math. Japonica **51** (2000), 499–549.
13. Trnková V., *Quasiorders on topological categories*, Proceedings of the Ninth Prague Topological Symposium, Contributed papers from the symposium held in Prague, Czech Republic, August 19–25, 2001, pp. 327–336.
14. Vaughan J., *Two spaces homeomorphic to Seq(p)*, preprint 2001.

Věra Trnková, Math. Institute of Charles University, Sokolovská 83, 18675 Praha 8, Czech Republic, *e-mail*: trnkova@karlin.mff.cuni.cz