UNIQUENESS AND INDEPENDENCE OF SUBMATRICES IN SOLUTIONS OF MATRIX EQUATIONS

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Suppose that there is an X satisfying $AX = B$, where A and B are $m \times n$ and $m \times k$ known matrices, respectively. Partition now the matrix equation in the block form

$$[\, A_1,\ A_2\,] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = A_1X_1 + A_2X_2 = B. \tag{1}$$

In this note we consider the following two basic problems related to this matrix equation:

(i) Under what conditions, the block X_1 or X_2 in solutions to (1) is unique?

(ii) Under what conditions, the block X_1 and X_2 in solutions to (1) are independent, that is, for any two solutions $\begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$ and $\begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix}$ of (1), the matrix $\begin{bmatrix} X_1' \\ X_2'' \end{bmatrix}$ is also a solution of (1)?

From the theory of generalized inverses of matrices (see, e.g., [2], [6]), the equation in (1) is consistent if and only if $AA^-B = B$. In this case, the general solution to (1) can be written as

$$X = A^-B + (\, I_n - A^-A\,)V, \tag{2}$$

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where A^- is a generalized inverse of A, that is, $AA^-A = A$, V is arbitrary matrix. Let X_1 and X_2 in X be $n_1 \times k$, and $n_2 \times k$ matrices, respectively. Then the general expressions of X_1 and X_2 can be written as

$$X_1 = P_1 A^- B + P_1(I_n - A^- A)V, \tag{3}$$

$$X_2 = P_2 A^- B + P_2(I_n - A^- A)V, \tag{4}$$

where $P_1 = [\, I_{n_1},\, 0\,]$ and $P_2 = [\, 0,\, I_{n_2}\,]$. If $\mathrm{rank} A < n$, then the solution to (1) is not unique. For simplicity, we use $\{X_1\}$ and $\{X_2\}$ to denote the collections of solutions X_1 and X_2 to (1), that is,

$$\{X_1\} = \{X_1 \,|\, X_1 = P_1 A^- B + P_1(I_n - A^- A)V_1 \}, \tag{5}$$

$$\{X_2\} = \{X_2 \,|\, X_2 = P_2 A^- B + P_2(I_n - A^- A)V_2 \}. \tag{6}$$

We are now ready to find the solution to the two problems in (i) and (ii).

Theorem 1. *Suppose that the quation in (1) is consistent. Then*

(a) *The block X_1 in the solution to (1) is unique if and only if*

$$\mathrm{rank}\, A = n_1 + \mathrm{rank}\, A_2, \tag{7}$$

or equivalently,

$$\mathrm{rank}\, A_1 = n_1 \quad \textit{and} \quad \mathscr{R}(A_1) \cap \mathscr{R}(A_2) = \{0\}, \tag{8}$$

where $\mathscr{R}(\cdot)$ denotes the range (column space) of a matrix.

(b) *The block X_2 in the solution to (1) is unique if and only if*

$$\mathrm{rank} A_2 = n_2 \quad \textit{and} \quad \mathscr{R}(A_1) \cap \mathscr{R}(A_2) = \{0\}. \tag{9}$$

Proof. It is obvious to see from the general expression of X_1 in (3) that X_1 is unique if and only if

$$P_1(I_n - A^- A) = 0. \tag{10}$$

From the following rank formula ([4])

$$\operatorname{rank}\begin{bmatrix} M \\ N \end{bmatrix} = \operatorname{rank} M + \operatorname{rank}(\, N - NM^{-}M\,), \tag{11}$$

it can be seen that (10) holds if and only if

$$\operatorname{rank}\begin{bmatrix} A \\ P_1 \end{bmatrix} = \operatorname{rank} A. \tag{12}$$

Substituting $P_1 = [\, I_{n_1},\ 0\,]$ into it and simplifying yields (7). Also observe that

$$\operatorname{rank} A \le \operatorname{rank}(A_1) + \operatorname{rank}(A_2) \le n_1 + \operatorname{rank}(A_2).$$

Thus (7) is equivalent to (8). Similarly one can show the result in Part (b). □

Theorem 2. *Suppose that the equation (1) is consistent. Then the two blocks X_1 and X_2 in the solution to (1) are independent, that is, for any $X_1 \in \{X_1\}$ and $X_2 \in \{X_2\}$, the corresponding matrix $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is also a solution of (1) if and only if*

$$\mathscr{R}(A_1) \cap \mathscr{R}(A_2) = \{0\}. \tag{13}$$

Proof. Substituting the general expressions of X_1 and X_2 in (5) and (6) into $AX - B$ gives

$$\begin{aligned} & AX - B \\ &= A_1X_1 + A_2X_2 - B \\ &= A_1P_1A^{-}B + A_1P_1(\, I_n - A^{-}A\,)V_1 + A_2P_2A^{-}B + A_2P_2(\, I_n - A^{-}A\,)V_2 - B \\ &= (\, A_1P_1 + A_2P_2\,)A^{-}B + A_1P_1(\, I_n - A^{-}A\,)V_1 + A_2P_2(\, I_n - A^{-}A\,)V_2 - B \\ &= [\, A_1P_1(\, I_n - A^{-}A\,),\ A_2P_2(\, I_n - A^{-}A\,)\,]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \end{aligned}$$

This equality implies that for any $X_1 \in \{X_1\}$ and $X_2 \in \{X_2\}$, the corresponding matrix $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is also a solution of (1) if and only if

$$A_1 P_1 (I_n - A^- A) = 0 \quad \text{and} \quad A_2 P_2 (I_n - A^- A) = 0.$$

From the rank formula (11), these two equalities are equivalent to

$$\operatorname{rank} \begin{bmatrix} A \\ A_1 P_1 \end{bmatrix} = \operatorname{rank} A \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} A \\ A_2 P_2 \end{bmatrix} = \operatorname{rank} A, \tag{14}$$

where

$$\operatorname{rank} \begin{bmatrix} A \\ A_1 P_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_1 & A_2 \\ A_1 & 0 \end{bmatrix} = \operatorname{rank} A_1 + \operatorname{rank} A_2$$

and

$$\operatorname{rank} \begin{bmatrix} A \\ A_2 P_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix} = \operatorname{rank} A_1 + \operatorname{rank} A_2.$$

Thus (14) is equivalent to (13). □

The result in Theorem 2 can be extended to the situation when X is partitioned into p blocks.

Theorem 3. *Suppose that $AX = B$ is consistent and partition it as*

$$AX = [\, A_1, \, A_2, \, \ldots, \, A_p \,] \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} = A_1 X_1 + A_2 X_2 + \cdots + A_p X_p = B. \tag{15}$$

Then the blocks $X_1, X_2, \cdots, X_p$ in the solution to (15) are independent if and only if

$$\operatorname{rank} [\, A_1, \, A_2, \, \ldots, \, A_p \,] = \operatorname{rank} A_1 + \operatorname{rank} A_2 + \cdots + \operatorname{rank} A_p. \tag{16}$$

It is well known that any linear matrix equation can equivalently be transformed into a linear matrix equation with the form $AX = B$ by the Kronecker product of matrices, see, e.g., [3]. Thus uniqueness and independence of submatrices in solutions of any linear matrix equation can be examined through the results in the above three theorems. For example, consider a matrix equation of the form

$$AXB + CYD = E, \tag{17}$$

where A, B, C and D are $m \times p_1$, $q_1 \times n$, $m \times p_2$, $q_2 \times n$, and $m \times n$ matrices, respectively. The consistency and solution of the matrix equation were previously examined, see, e.g., [1], [5], and [7].

From the Kronecker product of matrices, this equation can equivalently be written as

$$(B^T \otimes A)\,\mathrm{vec}X + (D^T \otimes C)\,\mathrm{vec}Y = \mathrm{vec}\,E. \tag{18}$$

Assume now that (17) is consistent. Then from Theorem 1(a), the solution X to (17) is unique if and only if

$$\mathrm{rank}\,(B^T \otimes A) = p_1 q_1 \quad \text{and} \quad \mathscr{R}(B^T \otimes A) \cap \mathscr{R}(D^T \otimes C) = \{0\}. \tag{19}$$

From Theorem 2, the solutions for X and Y to (17) are independent if and only if

$$\mathscr{R}(B^T \otimes A) \cap \mathscr{R}(D^T \otimes C) = \{0\}. \tag{20}$$

Notice a basic fact that $\mathrm{rank}\,(M \otimes N) = (\mathrm{rank}\,M)(\mathrm{rank}\,N)$, thus $\mathrm{rank}\,(B^T \otimes A) = p_1 q_1$ is equivalent to $\mathrm{rank}\,A = p_1$ and $\mathrm{rank}\,B = q_1$. It is shown in [9] that

$$\begin{aligned}
&\mathrm{rank}\,[\,A \otimes B,\ C \otimes D\,] \\
&\qquad \geq \mathrm{rank}\,(B)\,\mathrm{rank}\,[\,A,\ C\,] - \mathrm{rank}\,(B)\,\mathrm{rank}\,(C) + \mathrm{rank}\,(C)\,\mathrm{rank}\,(D), \\
&\mathrm{rank}\,[\,A \otimes B,\ C \otimes D\,] \\
&\qquad \geq \mathrm{rank}\,(A)\,\mathrm{rank}\,[\,B,\ D\,] - \mathrm{rank}\,(A)\,\mathrm{rank}\,(D) + \mathrm{rank}\,(C)\,\mathrm{rank}\,(D).
\end{aligned}$$

Hence if $\mathscr{R}(A) \cap \mathscr{R}(C) = \{0\}$ or $\mathscr{R}(B) \cap \mathscr{R}(D) = \{0\}$, then

$$\mathrm{rank}\,[\,A \otimes B,\ C \otimes D\,] = \mathrm{rank}\,(A \otimes B) + \mathrm{rank}\,(C \otimes D).$$

Thus if $\mathscr{R}(A) \cap \mathscr{R}(C) = \{0\}$ or $\mathscr{R}(B^T) \cap \mathscr{R}(D^T) = \{0\}$, then (20) holds.

Remarks. For any consistent linear matrix equation, one can investigate the uniqueness and independence of submatrices in solutions to the matrix equation. As continuation of this work, the uniqueness and independence of submatrices X_1, X_2, X_3, X_4 in solutions to the consistent matrix equation

$$[\,A_1,\, A_2\,]\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = C$$

are discussed in [10]. As applications, the uniqueness and independence of the submatrices G_1, G_2, G_3, G_4 in generalized inverse $M^- = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$ are also presented in [10]. Further, suppose $AX = C$ and $AXB = C$ are two consistent matrix equations over the field of complex numbers and write their solutions as $X = X_0 + iX_1$ (see [8]). Then it is natural to ask the uniqueness and independence of the two real matrices X_0 and X_1. Matrix equations have been basic objects for study in linear algebra. Besides $AX = B$ and $AXB = C$, some more general linear matrix equations have also been examined in the literature, for example, $[\,A_1XB_1,\, A_2XB_2\,] = [\,C_1,\, C_2\,]$, $A_1XB_1 + A_2XB_2 = C$, $A_1X_1B_1 + A_2X_2B_2 = C$, $A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 = C$. The uniqueness and independence of solutions to these equations are also worth investigating.

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