ON THE CONVERGENCE OF THE ISHIKAWA ITERATION IN THE CLASS OF QUASI CONTRACTIVE OPERATORS

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ABSTRACT. A convergence theorem of Rhoades [18] regarding the approximation of fixed points of some quasi contractive operators in uniformly convex Banach spaces using the Ishikawa iterative procedure, is extended to arbitrary Banach spaces. The conditions on the parameters $\{\alpha_n\}$ that define the Ishikawa iteration are also weakened.

1. Introduction

In the last four decades, numerous papers were published on the iterative approximation of fixed points of self and nonself contractive type operators in metric spaces, Hilbert spaces or several classes of Banach spaces, see for example the recent monograph [1] and the references therein. While for strict contractive type operators, the Picard iteration can be used to approximate the (unique) fixed point, see e.g. [1], [14], [22], [23], for operators satisfying slightly weaker contractive type conditions, instead of Picard iteration, which does not generally converge, it was necessary to consider other fixed point iteration procedures. The Krasnoselskij iteration [15], [5], [12], [13], the Mann iteration [16], [8], [17] and the Ishikawa iteration [10] are certainly the most studied of these fixed point iteration procedures, see [1].

Let E be a normed linear space and $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ a sequence or real numbers.

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The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n\,, \quad n = 0, 1, 2, \dots \tag{1.1}$$

is called the *Mann iteration* or *Mann iterative procedure*, in light of [16].

The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n\,, \quad n = 0, 1, 2, \dots \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n\,, \quad n = 0, 1, 2, \dots\,, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$, and $x_0 \in E$ arbitrary, is called the *Ishikawa iteration* or *Ishikawa iterative procedure*, due to [10].

Remark 1. For $\alpha_n = \lambda$ (constant), the iteration (1.1) reduces to the so called *Krasnoselskij iteration*, while for $\alpha_n \equiv 1$ we obtain the *Picard iteration* or method of successive approximations, as it is commonly known, see [1]. Obviously, for $\beta_n \equiv 0$ the Ishikawa iteration (1.2) reduces to (1.1).

The classical Banach's contraction principle is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

Theorem B. *Let (X, d) be a complete metric space and $T : X \to X$ a strict contraction, i.e. a map satisfying*

$$d(Tx, Ty) \le a\, d(x, y)\,, \quad \text{for all } x, y \in X\,, \tag{1.3}$$

where $0 < a < 1$ is constant. Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = T x_n\,, \quad n = 0, 1, 2, \dots \tag{1.4}$$

converges to p, for any $x_0 \in X$.

Theorem B has many applications in solving nonlinear equations, but suffers from one drawback – the contractive condition (1.3) forces T be continuous on X. In 1968 R. Kannan [11], obtained a fixed point theorem which extends Theorem B to mappings that need not be continuous, by considering instead of (1.3) the next condition: there exists $b \in \left(0, \dfrac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X\,. \tag{1.5}$$

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see for example, Rus [22], and references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [6], is based on a condition similar to (1.5): there exists $c \in \left(0, \dfrac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X \tag{1.6}$$

It is known, see Rhoades [19] that (1.3) and (1.5), (1.3) and (1.6), respectively, are independent contractive conditions.

In 1972, Zamfirescu [24] obtained a very interesting fixed point theorem, by combining (1.3), (1.5) and (1.6).

Theorem Z. *Let (X, d) be a complete metric space and $T : X \to X$ a map for which there exist the real numbers a, b and c satisfying $0 < a < 1$, $0 < b$, $c < 1/2$ such that for each pair x, y in X, at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq a\, d(x, y)$;

(z_2) $d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big]$;

(z_3) $d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \ldots$$

converges to p, for any $x_0 \in X$.

One of the most general contraction condition for which the unique fixed point can be approximated by means of Picard iteration, has been obtained by Ciric [7] in 1974: there exists $0 < h < 1$ such that

$$d(Tx, Ty) \le h \cdot \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}, \quad \text{for all } x, y \in X. \tag{1.7}$$

Remarks. A mapping satisfying (1.7) is commonly called quasi contraction. It is obvious that each of the conditions (1.3) (1.5), (1.6) and (z_1)-(z_3) implies (1.7). An operator T which satisfies the contractive conditions in Theorem Z will be called a *Zamfirescu operator* (alternatively, we shall say that T satisfies condition Z, see Rhoades [17]).

One of the most studied class of quasi-contractive type operators is that of Zamfirescu operators, for which all important fixed point iteration procedures, i.e., the Picard [24], Mann [17] and Ishikawa [18] iterations, are known to converge to the unique fixed point of T. Zamfirescu showed in [24] that an operator satisfying condition Z has a unique fixed point that can be approximated using the Picard iteration. Later, Rhoades [17], [18] proved that the Mann and Ishikawa iterations can also be used to approximate fixed points of Zamfirescu operators.

The class of operators satisfying condition Z is independent, see Rhoades [17], of the class of strictly (strongly) pseudocontractive operators, extensively studied by several authors in the last years. For the case of pseudocontractive type operators, the pioneering convergence theorems, due to Browder [4] and Browder and Petryshyn [5], established in Hilbert spaces, were successively extended to more general Banach spaces and to weaker conditions on the parameters that define the fixed point iteration procedures, as well as to several classes of weaker contractive type operators. For a recent survey and a comprehensive bibliography, we refer to the author's monograph [1].

As shown by Rhoades ([18], Theorem 8), in a uniformly Banach space E, the Ishikawa iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) and $x_0 \in K$ converges (strongly) to the fixed point of T, where $T : K \to K$ is a mapping satisfying condition Z, K is a closed convex subset of E, and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of numbers in $[0,1]$ such that

(i) $$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty\,.$$

In [3] the author proved the following convergence theorem in arbitrary Banach spaces, for the Mann iteration associated to operators satisfying condition Z, extending in this way another result of Rhoades ([17], Theorem 4).

Theorem 1. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ an operator satisfying condition Z. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration defined by (1.1) and $x_0 \in K$, with $\{\alpha_n\} \subset [0,1]$ satisfying*

(ii) $$\sum_{n=0}^{\infty} \alpha_n = \infty\,.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Concluding paper [3], we wondered there if, in the light of Theorem 1, Theorem 8 in [18] could be also extended from uniformly convex Banach spaces to arbitrary Banach spaces.

The next section answers this question in the affirmative.

2. The main result

Theorem 2. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ an operator satisfying condition Z. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by (1.2) and $x_0 \in K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying (ii).*

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof. By Theorem Z, we know that T has a unique fixed point in K, say p. Consider $x, y \in K$. Since T is a Zamfirescu operator, at least one of the conditions (z_1), (z_2) and (z_3) is satisfied. If (z_2) holds, then

$$\begin{aligned} \|Tx - Ty\| &\leq b\big[\|x - Tx\| + \|y - Ty\|\big] \\ &\leq b\Big\{\|x - Tx\| + \big[\|y - x\| + \|x - Tx\| + \|Tx - Ty\|\big]\Big\}. \end{aligned}$$

So

$$(1-b)\|Tx - Ty\| \leq b \cdot \|x - y\| + 2b\|x - Tx\|,$$

which yields (using the fact that $0 \leq b < 1$)

$$\|Tx - Ty\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Tx\|. \tag{2.1}$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \leq \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Tx\|. \tag{2.2}$$

Denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{2.3}$$

Then we have $0 \leq \delta < 1$ and, in view of (z_1), (2.1) and (2.2) it results that the inequality

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\| \tag{2.4}$$

holds for all $x, y \in K$.

Now let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by (1.2) and $x_0 \in K$ arbitrary. Then

$$\begin{aligned} \|x_{n+1}-p\| &= \left\|(1-\alpha_n)x_n+\alpha_n Ty_n-(1-\alpha_n+\alpha_n)p\right\| = \\ &= \left\|(1-\alpha_n)(x_n-p)+\alpha_n(Ty_n-p)\right\| \le \\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|Ty_n-p\| . \end{aligned} \tag{2.5}$$

With $x := p$ and $y := y_n$, from (2.4) we obtain

$$\|Ty_n-p\| \le \delta\cdot\|y_n-p\| , \tag{2.6}$$

where δ is given by (2.3).
Further we have

$$\begin{aligned} \|y_n-p\| &= \left\|(1-\beta_n)x_n+\beta_n Tx_n-(1-\beta_n+\beta_n)p\right\| \\ &= \left\|(1-\beta_n)(x_n-p)+\beta_n(Tx_n-p)\right\| \\ &\le (1-\beta_n)\|x_n-p\|+\beta_n\|Tx_n-p\| . \end{aligned} \tag{2.7}$$

Again by (2.4), this time with $x := p$; $y := x_n$, we find that

$$\|Tx_n-p\| \le \delta\|x_n-p\| \tag{2.8}$$

and hence, by (2.5) – (2.8) we obtain

$$\|x_{n+1}-p\| \le \left[1-(1-\delta)\alpha_n(1-\delta\beta_n)\right]\cdot\|x_n-p\| ,$$

which, by the inequality

$$1-(1-\delta)\alpha_n(1-\delta\beta_n) \le 1-(1-\delta)^2\alpha_n,$$

implies

$$\|x_{n+1}-p\| \le \left[1-(1-\delta)^2\alpha_n\right]\cdot\|x_n-p\| , \quad n=0,1,2,\ldots . \tag{2.9}$$

By (2.9) we inductively obtain

$$\|x_{n+1} - p\| \leq \prod_{k=0}^{n} \left[1 - (1-\delta)^2 \alpha_k\right] \cdot \|x_0 - p\|\,, \quad n = 0, 1, 2, \ldots \,. \tag{2.10}$$

Using the fact that $0 \leq \delta < 1$, $\alpha_k, \beta_n \in [0,1]$, and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, by (ii) it results that

$$\lim_{n\to\infty} \prod_{k=0}^{n} \left[1 - (1-\delta)^2 \alpha_k\right] = 0\,,$$

which by (2.10) implies

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0\,,$$

i.e., $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. □

Remarks. 1) Condition (i) in Theorem 1 is slightly more restrictive than condition (ii) in our Theorem 2, known as a *necessary* condition for the convergence of Mann and Ishikawa iterations. Indeed, in virtue of (i) we cannot have $\alpha_n \equiv 0$ or $\alpha_n \equiv 1$ and hence

$$0 < \alpha_n(1 - \alpha_n) < \alpha_n\,, \quad n = 0, 1, 2, \ldots\,,$$

which shows that (i) always implies (ii).

But there exist values of $\{\alpha_n\}$ satisfying (ii), e.g., $\alpha_n \equiv 1$, such that (i) is not true.

2) Since the Kannan's and Chattejea's contractive conditions are both included in the class of Zamfirescu operators, by Theorem 2 we obtain corresponding convergence theorems for the Ishikawa iteration in these classes of operators.

Corollary 1. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ a Kannan operator, i.e., an operator satisfying* (1.5)*. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by* (1.2) *and $x_0 \in K$, with $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ satisfying* (ii)*.*

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Corollary 2. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ a Chatterjea operator, i.e., an operator satisfying* (1.6)*. Then the Ishikawa iteration $\{x_n\}_{n=0}^{\infty}$ defined by* (1.2) *and $x_0 \in K$, with $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ satisfying* (ii) *converges strongly to the fixed point of T.*

Remarks. 1) It is quite obvious that Theorem 1 is properly contained in Theorem 2, and it is obtained for $\beta_n \equiv 0$.

On the other hand, due to the fact that, except for (ii), no other conditions are required for $\{\alpha_n\}$, $\{\beta_n\}$, by Theorem 2 we obtain, in particular, the convergence theorem regarding the convergence of Picard iteration in the class of Zamfirescu operators [24] (for $\alpha_n \equiv 1$, $\beta_n \equiv 0$), as well as a convergence theorem for the Krasnoselskij iteration (for $\beta_n \equiv 0$ and $\alpha_n = \lambda \in [0,1]$).

2) Since the contractive condition of Kannan (1.5) is a special case of that of Zamfirescu, Theorems 2 and 3 of Kannan [12] are special cases of Theorem 2, with $\alpha_n = 1/2$ and $\beta_n = 0$. Theorem 3 of Kannan [13] is the special case of Theorem 2 with $\alpha_n = \lambda$, $0 < \lambda < 1$ and $\beta_n = 0$. However, note that all the results of Kannan [12], [13] are obtained in uniformly Banach spaces, like Theorem 8 of Rhoades [18].

3) In paper [2], the author compared the rate of convergence of Picard and Mann iterations in the class of Zamfirescu operators.

Using the inequality (2.10) and the corresponding one obtained in [3] for the Mann iteration, i.e.,

$$\|y_{n+1} - p\| \leq \prod_{k=0}^{n} \left[1 - (1-\delta)\alpha_k\right] \|y_0 - p\|,$$

where $\{y_n\}_{n=0}^{\infty}$ is the Mann iteration defined by (1.1) and $y_0 \in K$ (arbitrary), we can compare these two iteration procedures in what concern their convergence rate. In view of our paper [2] and based on the proofs of Theorems

1 and 2, it results that, in the class of Zamfirescu operators, the Mann iteration is always faster than the Ishikawa iteration.

Thus we can compare all Picard, Mann and Ishikawa iterations in the class of Zamfirescu operators: the conclusion is that the Picard iteration converges faster than both Mann and Ishikawa iterations.

Conclusions. Our Theorem 2 improves Theorem 8 in Rhoades [**18**] by extending it from uniformly convex Banach spaces to arbitrary Banach spaces and simultaneously by weakening the assumptions on the sequence $\{\alpha_n\}$ that defines the Ishikawa iteration.

Moreover, many other results in literature are also extended in this way, e.g.:

1) The convergence theorems of two mean value fixed point iteration procedures for Kannan operators [**12**], [**13**] are extended to the larger class of Zamfirescu operators and simultaneously from uniformly convex Banach spaces to arbitrary Banach spaces and to the Ishikawa iteration;

2) The fixed point theorem of Chatterjea is extended from the Picard iteration to the Ishikawa iteration. This also contains, as a particular case, the corresponding convergence theorem for Mann and Krasnoselskij iterations;

3) While the convergence of Picard iteration in the class of Zamfirescu operators cannot be deduced by Theorem 8 of Rhoades [**18**], our main result also include, as a particular case, the convergence of both Picard and Krasnoselskij iterations.

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