WEIGHTED ENDPOINT ESTIMATES FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS

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Abstract. In this paper, we prove weighted endpoint estimates for multilinear Littlewood-Paley operators.

1. Introduction and Results

Let \( \psi \) be a fixed function on \( \mathbb{R}^n \) which satisfies the following properties:
\[
(1) \int \psi(x)dx = 0, \\
(2) |\psi(x)| \leq C(1 + |x|)^{-(n+1)}, \\
(3) |\psi(x + y) - \psi(x)| \leq C|y|(1 + |x|)^{-(n+2)} \quad \text{when} \quad 2|y| < |x|;
\]

Let \( m \) be a positive integer and \( A \) be a function on \( \mathbb{R}^n \). The multilinear Littlewood-Paley operator is defined by
\[
g^A_\mu(f)(x) = \left[ \int \int_{R_+^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\mu} |F^A_t(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2}, \quad \mu > 1,
\]
where
\[
F^A_t(f)(x, y) = \int_{R^n} \frac{R_{m+1}(A; x, z)}{|x - z|^m} \psi_t(y - z)f(z)dz,
\]
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\[ R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha, \]

and \( \psi_t(x) = t^{-n} \psi(x/t) \) for \( t > 0 \). We denote by \( F_t(f)(y) = f \ast \psi_t(y) \). We also define

\[
g_\mu(f)(x) = \left( \int \int_{R^n+1} \left( \frac{t}{t + |x - y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{tn+1} \right)^{1/2},
\]

which is the Littlewood-Paley operator (see \([17]\)).

Let \( H \) be the Hilbert space \( H = \left\{ h : ||h|| = \left( \int \int_{R^n+1} |h(t)|^2 \frac{dydt}{tn+1} \right)^{1/2} < \infty \right\} \). Then for each fixed \( x \in R^n \), \( F_t^A(f)(x, y) \) may be viewed as a mapping from \( (0, +\infty) \) to \( H \), and it is clear that

\[
\tilde{g}_\mu^A(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^A(f)(x, y) \right\|,
\]

\[
g_\mu(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_t(f)(y) \right\|.
\]

We also consider the variant of \( g_\mu^A \), which is defined by

\[
\tilde{g}_\mu^A(f)(x) = \left[ \int \int_{R^n+1} \left( \frac{t}{t + |x - y|} \right)^{n\mu} |\tilde{F}_t^A(f)(x, y)|^2 \frac{dydt}{tn+1} \right]^{1/2}, \quad \mu > 1,
\]

where

\[
\tilde{F}_t^A(f)(x, y) = \int_{R^n} Q_{m+1}(A; x, z) \frac{\psi_t(y - z)f(z)dz}{|x - z|^m}
\]
and

\[ Q_{m+1}(A; x, z) = R_m(A; x, z) - \sum_{|\alpha|=m} D^\alpha A(x)(x-z)^\alpha. \]

Note that when \( m = 0 \), \( g^A_\mu \) is just the commutator of Littlewood-Paley operator (see \([1], [14], [15]\)). It is well known that multilinear operators, as an extension of commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see \([4] - [8], [12], [13]\)). In \([11], [16]\), the endpoint boundedness properties of commutators generated by the Calderon-Zygmund operator and BMO functions are obtained. The main purpose of this paper is to study the weighted endpoint boundedness of the multilinear Littlewood-Paley operators. Throughout this paper, \( M(f) \) will denote the Hardy-Littlewood maximal function of \( f \), \( Q \) will denote a cube of \( \mathbb{R}^n \) with side parallel to the axes. For a cube \( Q \) and any locally integral function \( f \) on \( \mathbb{R}^n \), we denote that \( f(Q) = \int_Q f(x)\,dx \), \( f_Q = |Q|^{-1} \int_Q f(x)\,dx \) and \( f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|\,dy \). Moreover, for a weight functions \( w \in A_1 \) (see \([10]\)), \( f \) is said to belong \( \operatorname{BMO}(w) \) if \( f^\# \in L^\infty(w) \) and define \( ||f||_{\operatorname{BMO}(w)} = ||f^\#||_{L^\infty(w)} \); if \( w = 1 \), we denote that \( \operatorname{BMO}(\mathbb{R}^n) = \operatorname{BMO}(w) \). Also, we give the concepts of atomic and weighted \( H^1 \) space. A function \( a \) is called a \( H^1(w) \) atom if there exists a cube \( Q \) such that \( a \) is supported on \( Q \), \( ||a||_{L^\infty(w)} \leq w(Q)^{-1} \) and \( \int a(x)\,dx = 0 \). It is well known that, for \( w \in A_1 \), the weighted Hardy space \( H^1(w) \) has the atomic decomposition characterization (see \([2]\)).

We shall prove the following theorems in Section 3.

**Theorem 1.** Let \( D^\alpha A \in \operatorname{BMO}(\mathbb{R}^n) \) for \( |\alpha| = m \) and \( w \in A_1 \). Then \( g^A_\mu \) is bounded from \( L^\infty(w) \) to \( \operatorname{BMO}(w) \).

**Theorem 2.** Let \( D^\alpha A \in \operatorname{BMO}(\mathbb{R}^n) \) for \( |\alpha| = m \) and \( w \in A_1 \). Then \( \tilde{g}^A_\mu \) is bounded from \( H^1(w) \) to \( L^1(w) \).

**Theorem 3.** Let \( D^\alpha A \in \operatorname{BMO}(\mathbb{R}^n) \) for \( |\alpha| = m \) and \( w \in A_1 \). Then \( g^A_\mu \) is bounded from \( H^1(w) \) to weak \( L^1(w) \).
Theorem 4. Let $D^\alpha A \in \text{BMO}(\mathbb{R}^n)$ for $|\alpha| = m$ and $w \in A_1$.

(i) If for any $H^1(w)$-atom $a$ supported on certain cube $Q$ and $u \in 3Q \setminus 2Q$, there is

$$
\int_{(4Q)^c} \left| \frac{t}{t + |x-y|} \right|^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-u)^\alpha}{|x-u|^m} \int_Q \psi_t(y-z) D^\alpha A(z) a(z) dz \right| w(x) dx \leq C,
$$

then $g^A_\mu$ is bounded from $H^1(w)$ to $L^1(w)$;

(ii) If for any cube $Q$ and $u \in 3Q \setminus 2Q$, there is

$$
\frac{1}{w(Q)} \int_Q \left| \frac{t}{t + |x-y|} \right|^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \cdot \int_{(4Q)^c} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(y-z) f(z) dz \right| w(x) dx
\leq C ||f||_{L^\infty(w)},
$$

then $\tilde{g}^A_\mu$ is bounded from $L^\infty(w)$ to BMO($w$).

Remark. In general, $g^A_\mu$ is not bounded from $H^1(w)$ to $L^1(w)$. 
2. Some Lemmas

We begin with two preliminary lemmas.

**Lemma 1.** (see [7].) Let $A$ be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A : x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

**Lemma 2.** Let $w \in A_1$, $1 < p < \infty$ $1 < r \leq \infty$, $1/q = 1/p + 1/r$ and $D^\alpha A \in \text{BMO}(\mathbb{R}^n)$ for $|\alpha| = m$. Then $g^A_\mu$ is bounded from $L^p(w)$ to $L^q(w)$, that is

$$\|g^A_\mu(f)\|_{L^q(w)} \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} \|f\|_{L^p(w)}.$$

**Proof.** By Minkowski inequality and the condition of $\psi$, we have

$$g^A_\mu(f)(x)$$

$$\leq \int_{\mathbb{R}^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^m} \left( \int_{\mathbb{R}^n+1} |\psi_t(y - z)|^2 \left( \frac{t}{t + |x - y|} \right)^{n\mu} \frac{dydt}{t^{1+n}} \right)^{1/2} dz$$

$$\leq C \int_{\mathbb{R}^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^m} \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{t^{-2n}}{(1 + |y - z|/t)^{2n+2}} \cdot \left( \frac{t}{t + |x - y|} \right)^{n\mu} \frac{dydt}{t^{1+n}} \right)^{1/2} dz$$

$$\leq C \int_{\mathbb{R}^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^m} \left[ \int_0^\infty \left( \int_{\mathbb{R}^n} \left( \frac{t}{t + |x - y|} \right)^{n\mu} \cdot \frac{dy}{(t + |y - z|)^{2n+2}} \right) t dt \right]^{1/2} dz,$$
noting that
\[
t^{-n} \int_{\mathbb{R}^n} \left( \frac{t}{t + |x - y|} \right)^{n\mu} \frac{dy}{(t + |y - z|)^{2n+2}} \leq CM \left( \frac{1}{(t + |\cdot - z|)^{2n+2}} \right)(x) \leq C \frac{1}{(t + |x - z|)^{2n+2}}
\]
and
\[
\int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+2}} = C|x - z|^{-2n},
\]
we obtain
\[
g^A_\mu(f)(x) \leq C \int_{\mathbb{R}^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^{m+n}} \left( \int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+2}} \right)^{1/2} dz
\]
\[
= C \int_{\mathbb{R}^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x - z|^{m+n}} dz,
\]
thus, the lemma follows from [8] [9].
3. Proof of Theorems

Proof of Theorem 1. It is only to prove that there exists a constant $C_Q$ such that

$$
\frac{1}{w(Q)} \int_Q |g_\mu^A(f)(x) - C_Q |w(x)| \leq C ||f||_{L^\infty(w)}
$$

holds for any cube $Q$. Fix a cube $Q = Q(x_0, l)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (D^\alpha A) \tilde{\eta}^\alpha,$
then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A) \tilde{\eta}$ for $|\alpha| = m$. We write $F_t^A(f) = F_t^A(f_1) + F_t^A(f_2)$
for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{\mathbb{R}^n \setminus \tilde{Q}},$ then

$$
\frac{1}{w(Q)} \int_Q |g_\mu^A(f)(x) - g_\mu^A(f_2)(x_0)| \leq C ||f||_{L^\infty(w)}.
$$

Now, let us estimate $I$ and $II$. First, by the $L^\infty$ boundedness of $g_\mu^A$ (Lemma 2), we gain

$$
I \leq ||g_\mu^A(f_1)||_{L^\infty(w)} \leq C ||f||_{L^\infty(w)}.
$$
To estimate $II$, we write

$$
\left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^A(f_2)(x, y) - \left( \frac{t}{t + |x_0 - y|} \right)^{n\mu/2} F_t^A(f_2)(x_0, y)
$$

$$
= \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} \int \left[ \frac{1}{|x - z|^m} - \frac{1}{|x_0 - z|^m} \right] \psi_t(y - z)R_m(\tilde{A}; x, z)f_2(z)dz
$$

$$
+ \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} \int \frac{\psi_t(y - z)f_2(z)}{|x_0 - z|^m} [R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)]dz
$$

$$
+ \int \left[ \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} - \left( \frac{t}{t + |x_0 - y|} \right)^{n\mu/2} \right] \psi_t(y - z)\left. R_m(\tilde{A}; x_0, z)f_2(z) \right|_{x_0 - z^m} dz
$$

$$
= \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \left[ \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} \frac{(x - z)^\alpha}{|x - z|^m} \right] - \left( \frac{t}{t + |x_0 - y|} \right)^{n\mu/2} \frac{(x_0 - z)^\alpha}{|x_0 - z|^m} \psi_t(y - z)\tilde{A}(z)f_2(z)dz
$$

$$
:= II_1^t(x) + II_2^t(x) + II_3^t(x) + II_4^t(x),
$$
Note that $|x - z| \sim |x_0 - z|$ for $x \in \tilde{Q}$ and $z \in R^n \setminus \tilde{Q}$, similar to the proof of Lemma 2 and by Lemma 1, we have

$$\frac{1}{w(Q)} \int_{Q} ||II^t_1(x)||w(x)dx$$

$$\leq \frac{C}{w(Q)} \int_{Q} \left( \int_{R^n \setminus \tilde{Q}} \frac{|x - x_0||f(z)|}{|x - z|^{|n+m+1|}} |R_m(\tilde{A}; x, z)|dz \right) w(x)dx$$

$$\leq \frac{C}{w(Q)} \int_{Q} \left( \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \frac{|x - x_0||f(z)|}{|x - z|^{|n+m+1|}} |R_m(\tilde{A}; x, z)|dz \right) w(x)dx$$

$$\leq C \sum_{k=0}^{\infty} \frac{kl(2^k l)^m}{(2^k l)^{n+m+1}} \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \left( \int_{2^{k+1}\tilde{Q}} |f(z)|dz \right)$$

$$\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^\infty(w)} \sum_{k=0}^{\infty} k2^{-k} \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^\infty(w)};$$

For $II^t_2(x)$, by the formula (see [7]):

$$R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z) = R_m(\tilde{A}; x, x_0) + \sum_{0<|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; x_0, z)(x - x_0)^\beta$$

and Lemma 1, we get

$$|R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)| \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} (|x - x_0|^m + \sum_{0<|\beta|<m} |x_0 - z|^{m-|\beta|}|x - x_0|^{|\beta|}),$$

thus, for $x \in Q$,
\[ ||II_2^t(x)|| \]
\[ \leq C \int_{\mathbb{R}^n} \frac{|f_2(z)|}{|x - z|^{m+n}} |R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)| dz \]
\[ \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\text{BMO}} \int_{\mathbb{R}^n} \frac{|x - x_0|^m + \sum_{0<|\beta|<m} |x_0 - z|^{m-|\beta|} |x - x_0|^{|\beta|}}{|x_0 - z|^{m+n}} |f_2(z)| dz \]
\[ \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\text{BMO}} \sum_{k=0}^{\infty} \frac{k l^m}{(2^k l)^{m+n}} \int_{2^k Q} |f(z)| dz \]
\[ \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\text{BMO}} ||f||_{L^\infty(w)} \sum_{k=1}^{\infty} k 2^{-km} \]
\[ \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\text{BMO}} ||f||_{L^\infty(w)}; \]

For \( II_3^t(x) \), by the inequality: \( a^{1/2} - b^{1/2} \leq (a - b)^{1/2} \) for \( a \geq b > 0 \), we obtain, similar to the estimate of Lemma 2 and \( II_1 \),
\[\|II^t_3(x)\|\]
\[\leq C \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n_+} \left[ \frac{t^{n\mu/2} |x - x_0|^{1/2} |\psi_t(y - z)| |f_2(z)| |R_m(\tilde{A}; x_0, z)|}{(t + |x - y|)(n\mu + 1)|x_0 - z|^{m}} \right]^{2} \frac{dydt}{t^{n+1}} \right)^{1/2} dz \]
\[\leq C \int_{\mathbb{R}^n} \frac{|f_2(z)||x - x_0|^{1/2} |R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^{m}} \cdot \left( \int_{\mathbb{R}^n_+} \left( \frac{t}{t + |x - y|} \right)^{n\mu + 1} \frac{t^{-n} dydt}{(t + |y - z|)^{2n+2}} \right)^{1/2} dz \]
\[\leq C \int_{\mathbb{R}^n} \frac{|f_2(z)||x - x_0|^{1/2} |R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^{m}} \left( \int_{0}^{\infty} \frac{dt}{(t + |x - z|)^{2n+2}} \right)^{1/2} dz \]
\[\leq C \int_{\mathbb{R}^n} \frac{|f_2(z)||x - x_0|^{1/2} |R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^{m+n+1/2}} dz \]
\[\leq C \sum_{k=0}^{\infty} \frac{kl^{1/2}(2k)^m}{(2k)^{n+m+1/2}} \sum_{|\alpha|=m} ||D^\alpha A||_{\text{BMO}} \left( \int_{1}^{\infty} |f(z)| dz \right) \]
\[\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\text{BMO}} ||f||_{L^\infty(w)} \sum_{k=0}^{\infty} k2^{-k/2} \]
\[\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\text{BMO}} ||f||_{L^\infty(w)};\]
For $II_4^t(x)$, similar to the estimates of $II_1^t(x)$ and $II_3^t(x)$, we have

$$||II_4^t(x)|| \leq C \int_{\mathbb{R}^n \setminus \tilde{Q}} \left( \frac{|x - x_0|}{|x - z|^{n+1}} + \frac{|x - x_0|^{1/2}}{|x - z|^{n+1/2}} \right) \sum_{|\alpha| = m} |D^\alpha \tilde{A}(z)||f(z)|dz$$

$$\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{BMO} ||f||_{L^\infty(w)} \sum_{k=0}^\infty k(2^{-k} + 2^{-k/2})$$

$$\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{BMO} ||f||_{L^\infty(w)}.$$

Combining these estimates, we complete the proof of Theorem 1. □

**Proof of Theorem 2.** It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom $a$ (that is that $a$ satisfies: supp $a \subset Q = Q(x_0, r)$, $||a||_{L^\infty(w)} \leq w(Q)^{-1}$ and $\int a(y)dy = 0$ (see [8])), we have

$$||\tilde{g}^A_\mu(a)||_{L^1(w)} \leq C.$$

We write

$$\int_{\mathbb{R}^n} \tilde{g}^A_\mu(a)(x)w(x)dx = \left[ \int_{|x-x_0| \leq 2r} + \int_{|x-x_0| > 2r} \right] \tilde{g}^A_\mu(a)(x)w(x)dx := J + JJ.$$

For $J$, by the following equality

$$Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (x - y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),$$
we have, similar to the proof of Lemma 2,

\[ \tilde{g}_\mu^A(a)(x) \leq g_\mu^A(a)(x) + C \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |a(y)| dy, \]

thus, \( \tilde{g}_\mu^A \) is \( L^\infty \)-bounded by Lemma 2 and [3]. We see that

\[ J \leq C \| \tilde{g}_\mu^A(a) \|_{L^\infty(w)} w(2Q) \leq C \| a \|_{L^\infty(w)} w(Q) \leq C. \]

To obtain the estimate of \( JJ \), we denote that \( \tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{2B} x^\alpha \). Then \( Q_m(A; x, y) = Q_m(\tilde{A}; x, y) \). We write, by the vanishing moment of \( a \) and \( Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (x - y)^\alpha D^\alpha A(x) \), for \( x \in (2Q)^c \),

\[
\begin{align*}
\tilde{F}^A_t(a)(x, y) &= \int \psi_t(y - z) R_m(\tilde{A}; x, z) \frac{1}{|x - z|^m} a(z) dz - \frac{1}{\alpha!} \int \psi_t(y - z) D^\alpha \tilde{A}(z)(x - z)^\alpha \frac{1}{|x - z|^m} a(z) dz \\
&= \int \left[ \psi_t(y - z) R_m(\tilde{A}; x, z) - \frac{1}{|x - x_0|^m} \psi_t(y - x_0) R_m(\tilde{A}; x, x_0) \right] \frac{1}{|x - z|^m} a(z) dz \\
&\quad - \frac{1}{\alpha!} \int \left[ \psi_t(y - z)(x - z)^\alpha \frac{1}{|x - z|^m} - \frac{1}{|x - x_0|^m} \psi_t(y - x_0)(x - x_0)^\alpha \right] D^\alpha \tilde{A}(x) a(z) dz,
\end{align*}
\]

thus, similar to the proof of \( II \) in Theorem 1, we obtain

\[
\| \tilde{F}^A_t(a)(x, y) \| \leq C \frac{|Q|^{1+1/n}}{w(Q)} \left( \sum_{|\alpha|=m} \| D^\alpha A \|_{BMO} |x - x_0|^{-n-1} + |x - x_0|^{-n-1} |D^\alpha \tilde{A}(x)| \right),
\]
note that if $w \in A_1$, then $\frac{w(Q_2)}{|Q_2|} \frac{|Q_1|}{w(Q_1)} \leq C$ for all cubes $Q_1, Q_2$ with $Q_1 \subset Q_2$. Thus, by Holder’ inequality and the reverse of Holder’ inequality for $w \in A_1$, taking $p > 1$ and $1/p + 1/p' = 1$, we obtain

\[
JJ \leq C \sum_{|\alpha| = m} ||D^\alpha A||_{BMO} \sum_{k=1}^{\infty} 2^{-k} \left( \frac{|Q|}{w(Q)} \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right) \\
+ C \sum_{|\alpha| = m} \sum_{k=1}^{\infty} 2^{-k} \frac{|Q|}{w(Q)} \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^\alpha \tilde{A}(x)|^p dx \right)^{1/p} \cdot \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^{p'} dx \right)^{1/p'}
\]

\[
\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{BMO} \sum_{k=1}^{\infty} k2^{-k} \left( \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \frac{|Q|}{w(Q)} \right) \leq C,
\]

which together with the estimate for $J$ yields the desired result. This finishes the proof of Theorem 2.

\[\Box\]

**Proof of Theorem 3.** By the equality

\[ R_{m+1}(A; x, y) = Q_{m+1}(A; x, y) + \sum_{|\alpha| = m} \frac{1}{\alpha!} (x - y)^\alpha (D^\alpha A(x) - D^\alpha A(y)) \]

and similar to the proof of Lemma 2, we get

\[
g^\alpha_\mu(f)(x) \leq \tilde{g}^\alpha_\mu(f)(x) + C \sum_{|\alpha| = m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x - y|^n} |f(y)| dy,
\]

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by Theorem 1 and 2 with [3], we obtain

\[
\begin{align*}
& w(\{x \in \mathbb{R}^n : g^A_\mu(f)(x) > \lambda\}) \\
& \leq w(\{x \in \mathbb{R}^n : \tilde{g}^A_\mu(f)(x) > \lambda/2\}) + w(\{x \in \mathbb{R}^n : \sum_{|\alpha|=m} \int |D^\alpha A(x) - D^\alpha A(y)| |f(y)|dy > C\lambda\}) \\
& \leq C\|f\|_{H^1(w)}/\lambda.
\end{align*}
\]

This completes the proof of Theorem 3. \(\square\)

**Proof of Theorem 4.** (i) It suffices to show that there exists a constant \(C > 0\) such that for every \(H^1(w)\)-atom \(a\) with \(\text{supp} a \subset Q = Q(x_0, d)\), there is

\[\|g^A_\mu(a)\|_{L^1(w)} \leq C.\]

Let \(\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)Qx^\alpha.\) We write, by the vanishing moment of \(a\) and for \(u \in 3Q \setminus 2Q,\)

\[
\begin{align*}
F_t^A(a)(x, y) \\
& = \chi_{4Q}(x)F_t^A(a)(x, y) \\
& \quad + \chi(4Q^c)(x) \int_{\mathbb{R}^n} \left[ \frac{R_m(\tilde{A}; x, z)\psi_t(y - z)}{|x - z|^m} - \frac{R_m(\tilde{A}; x, u)\psi_t(y - u)}{|x - u|^m} \right] a(z)dz \\
& \quad - \chi(4Q^c)(x) \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left[ \frac{(x - z)^\alpha}{|x - z|^m} - \frac{(x - u)^\alpha}{|x - u|^m} \right] \psi_t(y - z)D^\alpha A(z)a(z)dz \\
& \quad - \chi(4Q^c)(x) \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{(x - u)^\alpha}{|x - u|^m} \psi_t(y - z)D^\alpha A(z)a(z)dz,
\end{align*}
\]
then
\[ g_{\mu}^A(a)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^A(a)(x, y) \right\| \]
\[ \leq \chi_{4Q}(x) \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^A(a)(x, y) \right\| \]
\[ + \chi_{(4Q)^c}(x) \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} \int_{\mathbb{R}^n} \left[ \frac{R_m(\tilde{A}; x, z)\psi_t(y - z)}{|x - z|^m} - \frac{R_m(\tilde{A}; x, u)\psi_t(y - u)}{|x - u|^m} \right] a(z)dz \right\| \]
\[ + \chi_{(4Q)^c}(x) \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left[ \frac{(x - z)^\alpha}{|x - z|^m} - \frac{(x - u)^\alpha}{|x - u|^m} \right] \cdot \psi_t(y - z)\psi_t(y - z) D^\alpha A(z)a(z)dz \right\| \]
\[ + \chi_{(4Q)^c}(x) \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{(x - u)^\alpha}{|x - u|^m} \psi_t(y - z)a(z)dz \right\| \]
\[ = L_1(x) + L_2(x) + L_3(x, u) + L_4(x, u). \]

By the \(L^\infty(w)\)-boundedness of \(g_{\mu}^A\), we get
\[ \int_{\mathbb{R}^n} L_1(x)w(x)dx = \int_{\mathbb{R}^n} g_{\mu}^A(a)(x)w(x)dx \leq \|g_{\mu}^A(a)\|_{L^\infty(w)w(4Q)} \leq C \|a\|_{L^\infty(w)w(Q)} \leq C; \]

Similar to the proof of Theorem 1, we obtain
\[ \int_{\mathbb{R}^n} L_2(x)w(x)dx \leq C \]
and

\[ \int_{\mathbb{R}^n} L_3(x, u) w(x) dx \leq C. \]

Thus, using the condition of \( L_4(x, u) \), we obtain

\[ \int_{\mathbb{R}^n} g_\mu^A(a)(x) w(x) dx \leq C. \]

(ii) For any cube \( Q = Q(x_0, d) \), let \( \tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_Q x^\alpha \). We write, for \( f = f \chi_{4Q} + f \chi_{(4Q)^c} = f_1 + f_2 \) and \( u \in 3Q \setminus 2Q \),

\[ \tilde{F}_t^A(f)(x, y) = \tilde{F}_t^A(f_1)(x, y) + \int_{\mathbb{R}^n} \frac{R_m(\tilde{A}; x, z)}{|x - z|^m} \psi_t(y - z) f_2(z) dz \]

\[ - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{\mathbb{R}^n} \left[ \frac{(x - z)^\alpha}{|x - z|^m} - \frac{(u - z)^\alpha}{|u - z|^m} \right] \psi_t(y - z) f_2(z) dz \]

\[ - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{\mathbb{R}^n} \frac{(u - z)^\alpha}{|u - z|^m} \psi_t(y - z) f_2(z) dz, \]
then
\[
\left| \tilde{g}_\mu^A(f)(x) - g_\mu \left( \frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|} f_2 \right)(x_0) \right|
\]
\[
= \left| \left( \frac{t}{t + |x - y|} \right)^{\mu/2} \tilde{F}_t^A(f)(x, y) - \left( \frac{t}{t + |x_0 - y|} \right)^{\mu/2} F_t \left( \frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|} f_2 \right)(x_0) \right|
\]
\[
\leq \left| \left( \frac{t}{t + |x - y|} \right)^{\mu/2} \tilde{F}_t^A(f)(x, y) - \left( \frac{t}{t + |x_0 - y|} \right)^{\mu/2} F_t \left( \frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|} f_2 \right)(x_0) \right|
\]
\[
\leq \left| \left( \frac{t}{t + |x - y|} \right)^{\mu/2} \tilde{F}_t^A(f_1)(x, y) \right|
\]
\[
+ \left| \int_{\mathbb{R}^n} \left[ \left( \frac{t}{t + |x - y|} \right)^{\mu/2} \frac{R_m(\tilde{A}; x, z)}{|x - z|^m} \psi_t(y - z) - \left( \frac{t}{t + |x_0 - y|} \right)^{\mu/2} \frac{R_m(\tilde{A}; x_0, z)}{|x_0 - z|^m} \psi_t(x_0 - z) \right] f_2(z) dz \right|
\]
\[
+ \left| \left( \frac{t}{t + |x - y|} \right)^{\mu/2} \cdot \sum_{|\alpha| = m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)Q) \int_{\mathbb{R}^n} \left[ \frac{(y - z)^\alpha}{|y - z|^m} - \frac{(u - z)^\alpha}{|u - z|^m} \right] \psi_t(y - z) f_2(z) dz \right|
\]
\[
+ \left| \left( \frac{t}{t + |x - y|} \right)^{\mu/2} \cdot \sum_{|\alpha| = m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)Q) \int_{\mathbb{R}^n} \frac{(u - z)^\alpha}{|u - z|^m} \psi_t(y - z) f_2(z) dz \right|
\]
\[
= M_1(x) + M_2(x) + M_3(x, u) + M_4(x, u).
\]
By the $L^\infty(w)$-boundedness of $\tilde{g}_\mu^A$, we get
\[
\frac{1}{w(Q)} \int_Q M_1(x)w(x)dx \leq \|\tilde{g}_\mu^A(f_1)\|_{L^\infty(w)} \leq C\|f\|_{L^\infty(w)};
\]
Similar to the proof of Theorem 1, we obtain
\[
\frac{1}{w(Q)} \int_Q M_2(x)w(x)dx \leq C\|f\|_{L^\infty(w)}
\]
and
\[
\frac{1}{w(Q)} \int_Q M_3(x,u)w(x)dx \leq C\|f\|_{L^\infty(w)}.
\]
Thus, using the condition of $M_4(x,u)$, we obtain
\[
\frac{1}{w(Q)} \int_Q \left| \tilde{g}_\mu^A(f)(x) - g_\mu \left( R_m(\tilde{A}; x_0, \cdot) \right) f_2(x_0) \right| w(x)dx \leq C\|f\|_{L^\infty(w)}.
\]
This completes the proof of Theorem 4. \qed

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