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CONTINUOUS SELECTIONS FOR LIPSCHITZ MULTIFUNCTIONS

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Abstract. In [**11**] an example presented a Hausdorff continuous, u.s.c. and l.s.c. multifunction from $\langle -1, 0\rangle$ to $\mathbb{R}$ which had no continuous selection. The multifunction was not locally Lipschitz. In this paper we show that a locally Lipschitz multifunction from $\mathbb{R}$ to a Banach space, which has "locally finitely dimensional" closed values does have a continuous selection.

1. Introduction

The research in the selection theory was started by Michael in 1956 (see for example [**15**], [**16**]) by proving several continuous selection theorems. Then, the problem of the existence of selections of various types – linear e. g. [**7**], measurable [**13**], Carathéodory [**8**], quasicontinuous [**10**], [**14**], Lipschitz [**3**], [**6**] etc. – was studied in many papers. A Lipschitz selection theorem for compact-valued multifunctions defined on a closed interval, with values in a metric space, was proved in [**5**]. Recent results concerning selections are listed in [**18**].

In general, there is no guarantee that a "nice" multifunction will have a continuous selection. Even closed-valued continuous multifunctions defined on compact interval and with values in $\mathbb{R}$ need not have a continuous selection (see[**11**]). In this paper, we show, in particular, that if such a multifunction is locally Lipschitz, it does have a continuous selection. This will be a consequence of a more general assertion, Theorem 3.

2. Notation and terminology

For definiton of basic notions: multifunction, selection, l.s.c. u.s.c. and Hausdorff continuous multifunction, Hausdorff metric etc see e.g. [**12**] and [**17**].

In what follows we denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}$ the real line with its usual topology and by $\mathbb{B}$ an arbitrary Banach space over $\mathbb{R}$. If X is a metric space, $x \in X$ and r is a positive real number, we denote the closed ball with the center x and diameter r by $B(x, r)$. Throughout this paper we consider only multifinctions with nonvoid values.

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If K is a positive real number, and (X,d), (Y,ϱ) are metric spaces, we say that a multifunction F from X to Y is K-Lipschitz if for every x_1, x_2 from X the inequality $H_\varrho(F(x_1), F(x_2)) \leq Kd(x_1, x_2)$ is true. (By H_ϱ we denote a Hausdorff metric on $2^Y - \{\emptyset\}$ derived in a natural way from ϱ).

Before proving our main results we need the following technical lemma:

Lemma 1. *Let Y be a Banach space over $\mathbb{R}$. Let $a \in \mathbb{R}$, let m be a positive real number. Let $I = \langle a, a+m\rangle$ ($I = \langle a-m, a\rangle$) $\subset \mathbb{R}$. Let $F : I \to Y$ be a K-Lipschitz multifunction. Let $r > 0$, $r < K$. Let $b \in F(a)$. Then there exists an M-Lipschitz function $f : I \to Y$ such that $M = (K+r)$, $f(a) = b$ and for each x in I*

$$d(f(x), F(x)) = \inf\{d(f(x), t); t \in F(x)\} < r.$$

Moreover $f(I) \subseteq B(b, 2Km)$ holds.

Proof. Let us consider the case $I = \langle a, a+m\rangle$. The case $I = \langle a-m, a\rangle$ is symmetrical.

Let $n \in N$ be such that $K\frac{m}{n} < \frac{r}{6}$ and $\frac{m}{n} < \frac{1}{3}$. Let us define $x_i = a + \frac{m}{n}i$ for $i = 0, 1, 2, \dots n$. Denote $b = y_0$. Since F is K-Lipschitz, there exists a point $y_1 \in F(x_1)$ such that

$$\begin{aligned} d(y_0, y_1) &\leqq H(F(x_0), F(x_1)) + \frac{rm}{2n} \\ &\leqq Kd(x_0, x_1) + \frac{rm}{2n} \leqq K\frac{m}{n} + \frac{rm}{2n} \leqq \left(K + \frac{r}{2}\right)\frac{m}{n}. \end{aligned}$$

By final induction we can find a set $\{y_0, y_1, \dots, y_n\}$ such that $\forall i = 0, 1, 2, \dots, n$, $y_i \in F(x_i)$ and

$$d(y_i, y_{i+1}) \leqq \left(K + \frac{r}{2}\right)\frac{m}{n} \qquad \text{for} \qquad i \leqq n-1.$$

Let us define a continuous function $f : \langle a, a+m\rangle \to Y$ in this way: $f(x_i) = y_i$, $i = 0, 1, 2, \dots, n$

$$f(x) = \frac{1}{m}\left[n(x - x_i)y_{i+1} + n(x_{i+1} - x)y_i\right] \qquad \text{if} \qquad x \in (x_i, x_{i+1}).$$

We will prove that f is $(K + \frac{r}{2})$-Lipschitz on $\langle a, a+m\rangle$.

(I) Let $x, x' \in \langle x_i, x_{i+1}\rangle$, for some $i \in \{0, 1, \dots, n\}$, $x < x'$. We obtain

$$\begin{aligned} &d(f(x), f(x')) \\ &\quad = \frac{1}{m}\|n(x' - x_i)y_{i+1} + n(x_{i+1} - x')y_i - n(x - x_i)y_{i+1} - n(x_{i+1} - x)y_i\| \\ &\quad = \frac{n}{m}\|(x' - x)y_{i+1} - (x' - x)y_i\| \leqq \frac{n}{m}|x' - x| \cdot \|(y_{i+1} - y_i)\| \\ &\quad \leqq \frac{n}{m}|(x' - x)|\left(K + \frac{r}{2}\right)\frac{m}{n} \leqq \left(K + \frac{r}{2}\right)|x' - x|. \end{aligned}$$

(II) In general, if $x < x_i < x_{i+1} \dots, x_{i+k} < x'$ for some $i, k \in \{0, 1, \dots, n\}$, $i + k < n$ then, because of (I)

$$\begin{aligned} d(f(x), f(x')) &\\ \leqq d(f(x), f(x_i)) &+ d(f(x_i), f(x_{i+1})) + \dots + d(f(x_{i+k-1}), f(x_{i+k})) \\ &+ d(f(x_{i+k}), f(x')) \\ \leqq \Big(\Big(K + \frac{r}{2}\Big)|x_i - x| &+ \Big(K + \frac{r}{2}\Big)|x_{i+1} - x_i| + \dots + \Big(K + \frac{r}{2}\Big)|x' - x_{i+k}|\Big) \\ = \Big(K + \frac{r}{2}\Big)|x' - x|. \end{aligned}$$

Now, let $x \in \langle a, a+m\rangle$, then $x \in \langle x_i, x_{i+1}\rangle$ for some $i \in \{0, 1, \dots, n\}$. So

$$\begin{aligned} d(f(x), F(x)) &= \inf\{d(f(x), t), t \in F(x)\} \\ &= \inf\Big\{\Big\|\frac{n}{m}(x - x_i)y_{i+1} + \frac{n}{m}(x_{i+1} - x)y_i - t\Big\|; t \in F(x)\Big\} \end{aligned}$$

Since F is K-Lipschitz there exists a point p from $F(x)$ such that $d(p, y_{i+1}) \leqq (K + \frac{r}{2})(x_{i+1} - x)$ therefore

$$\begin{aligned} d(f(x), p) &\leqq d(f(x), y_i) + d(y_i, y_{i+1}) + d(y_{i+1}, p) \\ &\leqq \Big(K + \frac{r}{2}\Big)(x - x_i) + \Big(K + \frac{r}{2}\Big)\frac{m}{n} + \Big(K + \frac{r}{2}\Big)(x_{i+1} - x) \\ &\leqq \Big(K + \frac{r}{2}\Big)(x_{i+1} - x_i) + \Big(K + \frac{r}{2}\Big)\frac{m}{n} \leqq 2\Big(K + \frac{r}{2}\Big)\frac{m}{n} \leqq 2\frac{r}{6} + r\frac{m}{n} < r. \end{aligned}$$

so $d(f(x), F(x)) < r$ for each x from $\langle a, a+m\rangle$.

Now, since $f(a) = b$ and f is a $(K+r)$-Lipschitz function, for r such that $r < K$ and for each x from $\langle a, a+m\rangle$ we have

$$d(b, f(x)) = d(f(a), f(x)) \leqq (K + r)|x - a| \leqq 2K|a + m - a| \leqq 2Km$$

so $f(\langle a, a+m\rangle) \subseteq B(b, 2Km)$. □

Theorem 1. *Let $\mathbb{B}$ be a finitely dimensional Banach space. Let $a \in \mathbb{R}$, let l be a positive real number. Let $I = \langle a, a+l\rangle$ ($\langle a-l, a\rangle$). Let $F : I \to \mathbb{B}$ be a K-Lipschitz multifunction with closed values. Then F has a K-Lipschitz selection on I.*

Proof. We will prove the Theorem only for the case $I = \langle a, a+l\rangle$. According to Lemma 1 there exists a sequence $\{f_i\}_{i=1}^{\infty}$ of functions $f_i : \langle a, a+l\rangle \to \mathbb{B}$ such that for each index i from $\mathbb{N}$ and each x from $\langle a, a+l\rangle$ $d(f_i(x), F(x)) < \frac{1}{i}$ is true. Moreover each function f_i is $\big(K + \frac{1}{i}\big)$-Lipschitz and $f_i(\langle a, a+l\rangle) \subset B(b, 2Kl)$. This implies that for every x from X the set $\{f_i(x); i = 1, 2, \dots\}$ is precompact.

Since $\mathbb{B}$ is finitely dimensional, according to Arzela-Ascoli theorem the set $M = \{f_i; i \in 1, 2, \dots\}$ is precompact. So there exists a continuous function $f : \langle a, a+l\rangle \to \mathbb{B}$ such that f is a uniform limit of a sequence $\{f_{i_j}\}_{j=1}^{\infty}$ (a subsequence of $\{f_i\}_{i=1}^{\infty}$) of functions from M.

Let us consider an $\varepsilon > 0$. As we have proved above there exists an index k such that f_{i_j} is $(K + \varepsilon)$-Lipschitz for each $j \geqq k$. That means that the function f is also $(K + \varepsilon)$-Lipschitz. f is proved to be K-Lipschitz.

Now it is simple to realize that f is a selection of F. For each $\varepsilon > 0$ there exists an index m such that for each x from X

$$d(f_{i_m}(x), F(x)) < \varepsilon \qquad \text{and} \qquad \sup_{x\in\langle a,a+l\rangle} |f_{i_m}(x) - f(x)| < \varepsilon.$$

So for every x from X $d(f(x), F(x)) < 2\varepsilon$. Since ε was an arbitrary positive real number, for each x from X $d(f(x), F(x)) = 0$ is true. F has closed values so f is a selection of F. $\square$

3. Main results

Theorem 2. *Let $\mathbb{B}$ be a finitely dimensional Banach space over $\mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{B}$ be a K-Lipschitz multifunction with closed values. Then F has a K-Lipschitz selection on $\mathbb{R}$.*

Proof. This is a simple consequence of Theorem 1 so we will only give an outline of the proof. Let b be an element of the set $F(0)$. Using Theorem 1, we can define by induction K-Lipschitz selections $f_1, f_2, \dots f_{2i}, f_{2i+1}, \dots$ of F such that for each nonnegative integer i the function f_{2i} (f_{2i+1}) is defined on $\langle 2i, 2i+2\rangle$ $(\langle -2i-2, -2i\rangle)$ and $f_{2i}(2i+2) = f_{2(i+1)}(2i+2)$ $(f_{2i+1}(-2i-2) = f_{2(i+1)+1}(-2i-2))$ and such that $f_1(0) = f_2(0) = b$. It is easy to see that a function $f : \mathbb{R} \to \mathbb{B}$ defined by $f(x) = f_{2i}(x)$ if $x \in \langle 2i, 2i+2\rangle$ and $f(x) = f_{2i+1}(x)$ if $x \in \langle -2i-2, -2i\rangle$ is correctly defined and it is a K-Lipschitz selection of F. $\square$

Theorem 2 is true for certain multifunctions with non-convex and non-compact values. It is a generalization of a result, obtained for multifunctions with convex compact values:

Corollary 1. [**6**, Corollary 2] *Let n be a positive integer, let $\mathbb{B} = R^n$. Let $F : \mathbb{R} \to \mathbb{B}$ be a K-Lipschitz multifunction with convex compact (and nonvoid) values. Then F has a K-Lipschitz selection on $\mathbb{R}$.*

In the following lemma we shall use the following assumption concerning a multifunction F from $\mathbb{R}$ to a Banach space $\mathbb{B}$:

Assumption LFD. For every x from $\mathbb{R}$ there exists an open neighborhood $O_x \subset \mathbb{R}$ and a finitely dimensional set $B_x \subset \mathbb{B}$ such that $F(O_x) \subset B_x$.

We say that a multifunction $F : \mathbb{R} \to \mathbb{B}$ is *locally Lipschitz* if for every real x there exists an open interval U_x and a positive real constant K_x such that $x \in U_x$ and F is K_x-Lipschitz on U_x.

Lemma 2. *Let $\mathbb{B}$ be a Banach space. Let $F : \mathbb{R} \to \mathbb{B}$ be a locally Lipschitz mutifunction with closed values. Let F satisfy the assumption LFD. Let $a \in \mathbb{R}$ and $b \in F(a)$. Then for every real c, d, $c < d$ satisfying $c \le a \le d$ there exists a Lipschitz selection $f : \langle c, d\rangle \to \mathbb{B}$ of F such, that $f(a) = b$.*

Proof. It suffices to show that F is Lipschitz on $\langle c, d\rangle$ and that there exists a finitely dimensional subset Z of $\mathbb{B}$ such that $F(\langle c, d\rangle) \subset Z$. After that we can apply Theorem 1.

We proceed by a usual "locally on compact implies globally on compact" procedure. Obviously for every x from $\langle c,d\rangle$ there exists an open interval U_x, a positive real number K_x and a finitely dimensional subset B_x of $\mathbb{B}$ such that $x \in U_x$, $F(U_x) \subset B_x$ and F is K_x-Lipschitz on U_x.

Consider the following open cover C of $\langle c,d\rangle$: $C = \{U_x; x \in \langle c,d\rangle\}$. There exists a finite subcover S of C and a positive integer n such that $S = \{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$. Let us denote $M = \max\{K_{x_1}, K_{x_2}, \ldots, K_{x_n}\}$. Then F is M-Lipschitz on each interval U_{x_i} for $i \in \{1, 2, \ldots, n\}$. The fact $\langle c,d\rangle \subset U := \bigcup_{i=1}^{n} U_{x_i}$ implies F is M-Lipschitz on $\langle c,d\rangle$.

Moreover, $F(\langle c,d\rangle) \subset F(U) \subset Z := \bigcup_{i=1}^{n} B_{x_i}$, and we can see that Z is finitely dimensional.

If $c < a < d$ Theorem 1 implies F has an M-Lipschitz selection h (g) on $\langle c,a\rangle$ ($\langle a,d\rangle$) such that $g(a) = h(a) = b$. So if $c < a < d$ the function $f : \langle c,d\rangle \to \mathbb{B}$ defined by $f(x) = g(x)$ on $\langle c,a\rangle$ and $f(x) = h(x)$ on $\langle a,d\rangle$ is a Lipschitz selection of F on $\langle c,d\rangle$. The proof for the cases $a = c$, $a = d$ is even easier. □

To realize that the assumptions of our final result, Theorem 3, can hardly be weakened let us compare the following three assertions:

(1) There exists a finitely valued Lipschitz multifunction from a unit circle into $\mathbb{R}^2$ that has no continuous selection. (See Example 1. Of course, each multifunction with values in $\mathbb{R}^2$ or $\mathbb{R}$ automatically satisfies the assumption LFD.)

(2) There exists a Hausdorff continuous multifunction from the compact interval $\langle -1,0\rangle$ to $\mathbb{R}$ with closed values, which is locally Lipschitz in every point of $\langle -1,0)$ and has no continuous selection (See Example 2).

(3) Each locally Lipschitz multifunction with closed values from $\mathbb{R}$ to a Banach space, satisfying the assumption LFD has a continuous selection. (See Theorem 3).

The examples presented below are based on ideas, used in examples published in [**4**] and [**11**].

Example 1. Let $K = \cos(t) + \mathrm{i} \cdot \sin(t);\quad t \in \langle 0, 2\pi)$ be the unit circle in the complex plane.

For each t from $\langle 0, 2\pi)$ let us denote

$$a_t = \cos(t) + \mathrm{i} \cdot \sin(t), \qquad b_t = \cos\left(\frac{t}{2}\right) + \mathrm{i} \cdot \sin\left(\frac{t}{2}\right)$$

$$c_t = \cos\left(\pi + \frac{t}{2}\right) + \mathrm{i} \cdot \sin\left(\pi + \frac{t}{2}\right)$$

Let us define a two-valued multifunction $F : K \to K$ by $F(a_t) = \{b_t, c_t\}$ for every t from $\langle 0, 2\pi)$.

This multifunction has compact (even finite) values and is Lipschitz.This can be seen by two ways.

An intuitive way is the easier one. If we draw a picture of our circle, we realize, that with t "moving" from 0 towards 2π the point a_t is moving from the point

$[1,0]$ to $[0,1]$, then $[-1,0]$ and finally to $[1,0]$ again. In this time the two-tuple $[b_t, c_t]$ travels around the circle too, but its speed is the half of the speed of a_t.

Now we show in an exact way that F is 1-Lipschitz. Let t_1, t_2 be from $\langle 0, 2\pi)$, $t_1 > t_2$. We have

$$\begin{aligned}|a_{t_1} - a_{t_2}| &= \sqrt{(\cos(t_1) - \cos(t_2))^2 + (\sin(t_1) - \sin(t_2))^2} \\ &= \sqrt{2 - 2\cos(t_1)\cos(t_2) - 2\sin(t_1)\sin(t_2)} = \sqrt{2(1 - \cos(t_1 - t_2))} \\ &= \sqrt{2}\sqrt{1 - \cos(t_1 - t_2))}.\end{aligned}$$

Similarly

$$|b_{t_1} - b_{t_2}| = \sqrt{2}\sqrt{1 - \cos\left(\frac{t_1 - t_2}{2}\right)}.$$

And, of course,

$$|c_{t_1} - c_{t_2}| = |b_{t_1} - b_{t_2}|.$$

Moreover

$$|b_{t_1} - c_{t_2}| = |c_{t_1} - b_{t_2}| = \sqrt{2}\sqrt{1 - \cos\left(\frac{t_1 - t_2}{2} - \pi\right)} = \sqrt{2}\sqrt{1 + \cos\left(\frac{t_1 - t_2}{2}\right)}.$$

Therefore

$$\begin{aligned}H(F(a_{t_1}), F(a_{t_2})) &= H(\{b_{t_1}, c_{t_1}\}, \{b_{t_2}, c_{t_2}\}) \\ &\leq \min\{|b_{t_1} - b_{t_2}|, |b_{t_1} - c_{t_2}|\} \\ &= \min\left\{\sqrt{2}\sqrt{1 - \cos\left(\frac{t_1 - t_2}{2}\right)}, \sqrt{2}\sqrt{1 + \cos\left(\frac{t_1 - t_2}{2}\right)}\right\}\end{aligned}$$

Now it is sufficient to show that

$$\begin{aligned}\min&\left\{\sqrt{1 - \cos\left(\frac{t_1 - t_2}{2}\right)}, \sqrt{1 + \cos\left(\frac{t_1 - t_2}{2}\right)}\right\} \\ &\leq \sqrt{1 - \cos(t_1 - t_2)} = \frac{1}{\sqrt{2}}|a_{t_1} - a_{t_2}|\end{aligned}$$

for all $t_1, t_2,\ 2\pi > t_1 > t_2 \geq 0$.

So the last thing we need to verify is that for all $l \in \langle 0, 2\pi)$

$$\min\left\{1 - \cos\left(\frac{l}{2}\right), 1 + \cos\left(\frac{l}{2}\right)\right\} \leq 1 - \cos(l)$$

or equivalently $\forall l \in \langle 0, 2\pi)$:

$$(*) \qquad \cos\left(\frac{l}{2}\right) - \cos(l) \geq 0 \quad \text{or} \quad \cos\left(\frac{l}{2}\right) + \cos(l) \leq 0.$$

Since

$$\cos\left(\frac{l}{2}\right) - \cos(l) = 2\sin\left(\frac{3}{4}l\right)\sin\left(\frac{l}{4}\right)$$
$$\cos\left(\frac{l}{2}\right) + \cos(l) = 2\cos\left(\frac{3}{4}l\right)\cos\left(\frac{l}{4}\right)$$

it is easy to verify that

$$\cos\left(\frac{l}{2}\right) - \cos(l) \geq 0 \qquad \forall l \in \left\langle 0, \frac{4}{3}\pi \right\rangle$$
$$\cos\left(\frac{l}{2}\right) + \cos(l) \leq 0 \qquad \forall l \in \left\langle \frac{2}{3}\pi, 2\pi \right\rangle$$

Therefore $(*)$ is verified and for all t_1, t_2 from $\langle 0, 2\pi)$, $t_1 > t_2$,

$$H(F(a_{t_1}), F(a_{t_2})) \leq |a_{t_1} - a_{t_2}|.$$

F is proved to be 1-Lipschitz.

Nevertheless, F has no continuous selection on K. It has two natural continuous selections on each $K_\varepsilon \subset K$ where the set K_ε is defined by $K_\varepsilon = \{a_t; t \in \langle 0, 2\pi - \varepsilon)\}$ for every positive $\varepsilon < 2\pi$. These selections are: $f(a_t) = b_t$ and $g(a_t) = c_t$ for each a_t from K_ε.

However, no of these selections can be prolonged to K, For example $f(a_0) = b_0 = [1, 0]$, but $\lim\limits_{t \to 2\pi^-} f(a_t) = \lim\limits_{t \to 2\pi^-} b_t = [-1, 0]$.

Example 2. [**11**] Let $F : \langle -1, 0\rangle \to \mathbb{R}$ be defined as follows:

$F(0) = \mathbb{R}$

$$F(x) = \left\{\frac{n(n+1)}{2}x + \frac{k}{2^n}; k \in \mathbb{Z}\right\} \cup \left\{n(n+1)\frac{2^n+1}{2^{n+1}}x + \frac{n+1}{2^{n+1}} + \frac{k}{2^n}; k \in \mathbb{Z}\right\}$$

for every positive integer n and every $x \in \left\langle -\frac{1}{n}, -\frac{1}{n+1}\right\rangle$.

In other words: the intersection of the graph of F with the set $\left\langle -\frac{1}{n}, -\frac{1}{n+1}\right\rangle \times \mathbb{R}$ is a system of segments joining the following couples of points: the point $\left[\frac{-1}{n}, \frac{m}{2^n}\right]$ with the point $\left[-\frac{1}{n+1}, \frac{m}{2^n} + \frac{1}{2}\right]$ and $\left[-\frac{1}{n}, \frac{m}{2^n}\right]$ with the point $\left[-\frac{1}{n+1}, \frac{m}{2^n} + \frac{1}{2} + \frac{1}{2^{n+1}}\right]$ where m is an arbitrary integer.

To show that F is locally Lipschitz on $\langle -1, 0)$ it is sufficient to show that it is $n(n+1)$-Lipschitz on $I_n = \left\langle \frac{-1}{n}, \frac{-1}{n+1}\right\rangle$ for every $n \in \mathbb{N}$, $n > 0$.

Let $x_1, x_2 \in I_n$. Let $y_1 \in F(x_1)$. Then there exists an integer k such that

$$y_1 = \frac{n(n+1)}{2}x_1 + \frac{k}{2^n} \qquad \text{or} \qquad y_1 = n(n+1)\frac{2^n+1}{2^{n+1}}x_1 + \frac{n+1}{2^{n+1}} + \frac{k}{2^n}.$$

There exists also y_2 from $F(x_2)$ such that

$$y_2 = \frac{n(n+1)}{2}x_2 + \frac{k}{2^n} \qquad \text{or} \qquad y_2 = n(n+1)\frac{2^n+1}{2^{n+1}}x_2 + \frac{n+1}{2^{n+1}} + \frac{k}{2^n}$$

so $|y_1 - y_2|$ equals

$$\frac{n(n+1)}{2}|x_1 - x_2| \qquad \text{or} \qquad \frac{n(n+1)(2^n+1)}{2^{n+1}}|x_1 - x_2|.$$

In both cases we have

$$(**) \qquad |y_1 - y_2| \leq K_n|x_1 - x_2|, \qquad \text{where} \qquad K_n = n(n+1).$$

In the same way we can pick an y_2 from $F(x_2)$ first and find a y_1 from $F(x_1)$ such that the inequality $(**)$ is true.

This means that for each x_1, x_2 from I_n $H(F(x_1), F(x_2)) \leq K_n|x_1 - x_2|$ is true.

We have just proved that F is locally Lipschitz on $\langle -1, 0)$. The Hausdorff continuity of F on $\langle -1, 0\rangle$ is proved in [**11**].

F has no continuous selection on $\langle -1, 0\rangle$: every continuous selection f of F defined on the set $\langle -1, 0)$ has the property $\lim\limits_{t \to 0^-} f(t) = +\infty$.

Next we will prove our main theorem:

Theorem 3. *Let $\mathbb{B}$ be a Banach space over $\mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{B}$ be a locally Lipschitz mutifunction with closed values. Let F satisfy the assumption LFD. Let $a \in \mathbb{R}$ and $b \in F(a)$. Then F has a continuous selection f on $\mathbb{R}$ such that $f(a) = b$.*

Proof. For $n = 1, 2, 3\ldots$ denote $I_n = \langle -n, n\rangle$. In what follows we procced by induction. Let us suppose, without loss of generality, that $a = 0$.
(1) According to Lemma 2 there exists a Lipschitz selection $f_1 : T_1 \to \mathbb{B}$ of F on the interval I_1 such that $f(a) = b$. Let us denote $f_1(-1) = b_1$ and $f_1(1) = c_1$.
(2) Let us suppose that for n *in* $\mathbb{N}$, $n = 1, 2, \ldots k$ there exist Lipschitz selections f_n of F on I_n such that if $l, m \in \{1, 2, \ldots k\}$, $l > m$ then $f_l(x) = f_m(x)$ for each x from I_m.

For each of the n considered let us denote $f_n(-n) = b_n$ and$f_n(n) = c_n$.

Since $b_k \in F(-k)$ there exists a Lipschitz selection g_k of F on $\langle -k-1, -k\rangle$ such that $g_k(-k) = b_k$. Since $c_k \in F(k)$ there exists a Lipschitz selection h_k of F on $\langle k, k+1\rangle$ such that $h_k(k) = c_k$.

Let us define a function f_k on I_k by

$$\begin{aligned} f_k(x) &= g_k(x) && \text{for } x \text{ from } \langle -k-1, -k\rangle \\ f_k(x) &= f_{k-1}(x) && \text{for } x \text{ from } \langle -k, k\rangle \\ f_k(x) &= h_k(x) && \text{for } x \text{ from } \langle k, k+1\rangle. \end{aligned}$$

We have just constructed by induction a sequence of Lipschitz selections f_k of F on the intervals I_k such that if $k_1 < k_2$ then $f_{k_2}(x) = f_{k_1}(x)$ for all x from I_{k_1}. All functions f_k are continuous selections of F on their domains.

Let us define a function $f : \mathbb{R} \to \mathbb{B}$ by

$$\begin{aligned} f(x) &= f_1(x) && \text{for } x \in \langle -1, 1\rangle, \\ f(x) &= f_k(x) && \text{for } x \in \langle -k-1, -k\rangle \cup \langle k, k+1\rangle,\ k = 1, 2, \ldots \end{aligned}$$

The function f is a selection of F on $\mathbb{R}$. It is continuous because all functions f_k are continuous. □

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