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THE FROBENIUS THEOREM ON N-DIMENSIONAL QUANTUM HYPERPLANE

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Abstract. In this paper we introduce the universal ρ-differential calculus on a ρ-algebra and we prove the universality of the construction. We also present submanifolds, distributions, linear connections and two different ρ-differential calculi on S_N^q: the algebra of forms and the algebra of universal differential forms of S_N^q. Finally we prove the Frobenius theorem on the N-dimensional quantum hyperplane S_N^q in a general case, for any ρ-differential calculus on S_N^q.

1. Introduction

The basic idea of noncommutative geometry is to replace an algebra of smooth functions defined on a smooth manifold by an abstract associative algebra A which is not necessarily commutative. In the context of noncommutative geometry the basic role is the generalization of the notion of differential forms (on a manifold). A (noncommutative) differential calculus over the associative algebra A is a $\mathbb{Z}$-graded algebra $\Omega(A) = \oplus_{n\geq 0}\Omega^n(A)$ (where $\Omega^n(A)$ are A-bimodules and $\Omega^0(A) = A$) together with a linear operator $d : \Omega^n(A) \to \Omega^{n+1}(A)$ satisfying $d^2 = 0$ and $d(\omega\omega') = (d\omega)\omega' + (-1)^n\omega d\omega'$ where $\omega \in \Omega^n(A)$.

There are studied some differential calculi associated to A, here we recall two of them: the algebra of forms of A in [**8**] and the algebra of universal differential forms of A in [**2**]. These two differential calculi are studied even in the case when A is a superalgebra in [**10**] and in [**11**], but when A is a ρ-algebra there is studied only the algebra of forms of A in [**1**].

In this paper we introduce the notion of universal derivation of the order (α, β), we construct the algebra of universal differential forms of a ρ-algebra A and we prove the universality of the construction. We apply this to the particular case of the N-dimensional quantum hyperplane S_N^q and thus we obtain a new differential calculus $\Omega_\alpha\left(S_N^q\right)$ on S_N^q, denoted the algebra of universal differential forms of S_N^q.

On the other hand we present submanifolds algebra on S_N^q in the context of noncommutative geometry, we also define linear connections, distributions on S_N^q

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and we prove the Frobenius theorem on S_N^q, for any ρ-differential calculus ΩS_N^q on S_N^q.

2. Differential calculus on ρ-algebras

In this section we define the differential calculus on a ρ-algebra A and we give two examples of such differential calculus on A: the algebra of forms $\Omega(A)$ of a ρ-commutative algebra A from [**1**] and the algebra of universal differential forms $\Omega_\alpha(A)$ of a ρ-algebra A. We briefly review the basic notions of ρ-algebras and ρ-modules on ρ-algebras. (see [**1**], [**3**], [**5**] and [**6**] for details).

First we review the basic notions about ρ-algebras, ρ-bimodules over a ρ-algebra and the derivations on ρ-bimodules.

Let G be an abelian group, additively written, and let A be a G-graded algebra. A is a ρ-algebra if there is a map $\rho : G \times G \to k$ which satisfies the relations

$$\rho(a,b) = \rho(b,a)^{-1} \quad \text{and} \quad \rho(a+b,c) = \rho(a,c)\rho(b,c), \quad \text{for any } a,b,c \in G. \tag{1}$$

The G-degree of a (nonzero) homogeneous element f of A is denoted by $|f|$. The ρ-algebra A is ρ-commutative if $fg = \rho(|f|,|g|)\, gf$ for any $f \in A_{|f|}$ and $g \in A_{|g|}$.

The morphism $f : M \to N$ between the ρ-bimodules M and N over the ρ-algebra A is of degree $\beta \in G$ if $f : M_\alpha \to N_{\alpha+\beta}$ for any $\alpha \in \dot{G}$, $f(am) = \rho(\alpha,|a|)af(m)$ and $f(ma) = f(m)a$ for any $a \in A_{|a|}$ and $m \in M$.

Definition 1. Let $\alpha, \beta \in G$ and M a ρ-bimodule over the ρ-algebra A. A *ρ-derivation of order (α,β) on M (ρ-derivation of degree α and of G-degree β)* is a linear map $X : A \to M$, which fulfils the properties:

1. $X : A_* \to M_{*+\beta}$,
2. $X(fg) = (Xf)g + \rho(\alpha,|f|)f(Xg)$,for any $f \in A_{|f|}$ and $g \in A$.

The left product $fX : A \to M$ between the element $f \in A$ and a ρ-derivation X of the order (α,β) is defined in a natural way: by $(fX)(g) = fX(g)$, for any $g \in A$.

Proposition 1. *If is X a ρ-derivation of order (α,β) on M and $f \in A_{|f|}$, then fX is a derivation of the order $(|f|+\alpha, |f|+\beta)$ on M if and only if the algebra A is ρ-commutative.*

Proof. We have to show that $(fX)(gh) = ((fX)g)h + \rho(|f|+\alpha,|g|)g(fX)h$ and $fX : A_* \to M_{*+|f|+|X|}$, for any $h \in A_{|h|}$ and $f \in A_{|f|}$.

$$\begin{aligned}(fX)(gh) &= f(X(gh)) = f(X(g)h + \rho(\alpha,|g|)gX(h)) \\ &= (fX)(g)h + \rho(\alpha,|g|)fgX(h) = (fX)(g)h + \rho(\alpha,|g|)\rho(|f|,|g|\, gfX(h) \\ &= (fX)(g)h + \rho(\alpha,|f|+|g|)g(fX)(h).\end{aligned}$$

The second relation is obvious. □

The linear map $X : A \to A$ is a *ρ-derivation* on A if it is a ρ-derivation of the order $(|X|,|X|)$. Next we denote by ρ-Der(A) the ρ-bimodule over A of the all ρ-derivations on A.

Definition 2. Let A be a ρ-algebra. We say that $\Omega(A) = \bigoplus_{(n,\beta)\in\mathbb{Z}\times G} \Omega^n_\beta(A)$ is a *ρ-differential calculus on* A if $\Omega(A)$ is a $\mathbb{Z}\times G$-graded algebra, $\Omega(A)$ is a ρ-bimodule over A, $\Omega^0(A) = \bigoplus_{\beta\in G} \Omega^0_\beta(A) = A$ and there is an element $\alpha \in G$ and ρ-derivation $d : \Omega(A) \to \Omega(A)$ of the order $((1,\alpha),(1,0))$ such that $d^2 = 0$.

The first example of a ρ-differential calculus over the ρ-commutative algebra A is the algebra of forms $(\Omega(A), d)$ of A from [**1**].

The second example of a ρ-differential calculus over a ρ-algebra is the universal differential calculus of A from the next subsection.

2.1. The algebra of universal differential forms of a ρ-algebra

In this subsection we define the algebra of universal differential forms of the ρ-algebra A and we prove the universality of the construction. First we introduce the universal derivation of the order (α,β), $\alpha,\beta\in G$.

Definition 3. Let M be a ρ-bimodule over A. The ρ-derivation $D : A \to M$ of the order (α,β) is *universal* if for ρ-derivation $D' : A \to N$ of the order $(\alpha, 0)$, there is an unique morphism $\Phi : M \to N$ of ρ-bimodules of degree β such that $D' = \Phi \circ D$.

Next we will construct an universal derivation over the ρ-algebra A.

Let $\alpha \in G$, $\mu : A\otimes A \to A$ be the map $\mu(x\otimes y) = \rho(\alpha, |y|)\, xy$ and $\Omega^1_\alpha A = \ker(\mu)$.
We define the map

$$d : A \to \Omega^1_\alpha A \tag{2}$$

by

$$dx = 1\otimes x - \rho(\alpha,|x|)\, x\otimes 1, \tag{3}$$

for any $x\in A$.

Proposition 2. *$\Omega^1_\alpha A$ is a ρ-bimodule over A and the map $d : A \to \Omega^1_\alpha A$ is an universal derivation of the order $(\alpha, 0)$.*

Proof. We have to show that $a(b\otimes c) \in \Omega^1_\alpha A$, $(b\otimes c)a \in \Omega^1_\alpha A$ and $(a(b\otimes c))d = a((b\otimes c)d)$, for any $a, d \in A$ and $b\otimes c \in \Omega^1_\alpha A$. Indeed

$$\mu(a(b\otimes c)) = \mu(ab\otimes c) = \rho(\alpha, |ab|)abc = \rho(\alpha,|a|)a(\rho(\alpha,|b|)bc) = 0.$$

The other relations are obvious.
It is easy to see that $\Omega^1_\alpha A$ is the space $\{xdy : x, y \in A \text{ and } d \text{ from } (3)\}$.
Now we show that $d : A \to \Omega^1_\alpha A$ is a ρ-derivation of the order $(\alpha, 0)$:

$$\begin{aligned}(da)b + \rho(\alpha,|a|)adb &= (1\otimes a - \rho(\alpha,|a|)a\otimes 1)b + \rho(\alpha,|a|)a(1\otimes b - \rho(\alpha,|b|)b\otimes 1)\\ &= 1\otimes ab - \rho(\alpha,|a|)a\otimes b + \rho(\alpha,|a|)a\otimes b - \rho(\alpha,|a|+|b|)ab\otimes 1\\ &= 1\otimes(ab) - \rho(\alpha,|ab|)(ab)\otimes 1 = d(ab).\end{aligned}$$

Let M be a ρ-bimodule over A and $D{:}A \to M$ a ρ-derivation of the order (α, β). We define the map $\Phi : \Omega^1_\alpha A \to M$ in the following way:

$$\Phi(a \otimes b) = \rho(\alpha, |a|)aD(b),$$

for any $a \otimes b \in \Omega^1_\alpha A$.

Now we show that Φ is a morphism of ρ-bimodules over A:

$$\begin{aligned}\Phi(a(b \otimes c)) &= \phi((ab) \otimes c) = \rho(\alpha, |ab|)abD(c)\\ &= \rho(\alpha, |a|)a[\rho(\alpha, |b|)bD(c)] = \rho(\alpha, |a|)a\Phi(b \otimes c).\end{aligned}$$

On the other hand

$$\begin{aligned}\Phi((a \otimes b)c) &= \Phi(a \otimes (bc)) = \rho(\alpha, |a|)aD(bc)\\ &= \rho(\alpha, |a|)a[D(b)c + \rho(\alpha, |b|)bD(c)]\\ &= \rho(\alpha, |a|)aD(b)c + \rho(\alpha, |b|)[\rho(\alpha, |a|)ab]D(c)\\ &= \rho(\alpha, |a|)aD(b)c + \rho(\alpha, |b|)\mu(a \otimes b)D(c)\\ &\overset{a\otimes b\in\ker(\mu)}{=} \rho(\alpha, |a|)aD(b)c = \Phi(a \otimes b)c.\end{aligned}$$

Finally we get

$$\begin{aligned}(\Phi \circ d)(a) &= \Phi(1 \otimes a - \rho(\alpha, |a|)a \otimes 1)\\ &= \rho(\alpha, |1|)D(a) - \rho(\alpha, |a|)\rho(\alpha, |a|)aD(1) = D(a)\end{aligned}$$

so we have proved that $\Phi \circ d = D$. □

Let

$$\Omega^n_\alpha A = \underbrace{\Omega^1_\alpha A \underset{A}{\otimes} \ldots \underset{A}{\otimes} \Omega^1_\alpha A}_{n \text{ times}}, \quad \Omega^0_\alpha A = A \quad \text{and} \quad \Omega_\alpha A = \underset{n\geq 0}{\oplus} \Omega^n_\alpha A.$$

Naturally $\Omega_\alpha A$ is a ρ-bimodule over A and a algebra with the multiplication $(\omega, \theta) \mapsto \omega \underset{A}{\otimes} \theta$, for any $\omega,\ \theta \in \Omega_\alpha A$.

Remark 1. The algebra $\Omega_\alpha A$ may be identified with the algebra generated by the algebra A and the derivations da, $a \in A$ which satisfies the following relations:

1. da is linear in a
2. the ρ-Leibniz rule: $d(ab) = d(a)b + \rho(\alpha, |a|)adb$
3. $d(1) = 0$.

$\Omega^n_\alpha A$ is the space of n-forms $a_0da_1...da_n$, $a_i \in A$ for any $0\leq i \leq n$. $\Omega^n_\alpha A$ is a ρ-bimodule over A with the left multiplication

$$a(a_0da_1 \ldots da_n) = aa_0da_1 \ldots da_n, \tag{4}$$

and with the right multiplication given by:

$$\begin{aligned}(a_0da_1 \ldots da_n)a_{n+1} =& \sum_{i=1}^{n}(-1)^{n-i}\rho(\alpha, \sum_{j=i+1}^{n}|a_j|)(a_0da_1 \ldots d(a_ia_{i+1}) \ldots da_{n+1})\\ &+ (-1)^n\rho(\alpha, \sum_{i=1}^{n}|a_j|)a_0a_1da_2 \ldots da_{n+1}.\end{aligned}$$

The multiplication in the algebra $\Omega_\alpha A$ is:

$$(a_0da_1\dots da_n)(a_{n+1}da_{n+2}\dots da_{m+n}) = ((a_0da_1\dots da_n)a_{n+1})da_{n+2}\dots da_{m+n}),$$

for any $a_i \in A$, $0\leq i \leq n+m$, $n, m \in \mathbb{N}$.

We define the G-degree of the n-form $a_0da_1\dots da_n$ in the following way

$$|a_0da_1\dots da_n| = \sum_{i=0}^{n} |a_i|\,.$$

It is obvious that $|\omega_n \cdot \omega_m| = |\omega_n| + |\omega_m|$ for any homogeneous forms $\omega_n \in \Omega_\alpha^n A$ and $\omega_m \in \Omega_\alpha^m A$.

$\Omega_\alpha A$ is a $G' = \mathbb{Z}\times G$-graded algebra with the G' degree of the n-form $a_0da_1\dots da_n$ defined by

$$|a_0da_1\dots da_n|' = \left(n, \sum_{i=0}^{n} |a_i|\right).$$

We may define the cocycle $\rho' : G'\times G' \to k$ on the algebra $\Omega_\alpha A$ thus:

$$\rho'(|a_0da_1\dots da_n|', |b_0db_1\dots db_m|') = (-1)^{nm}\rho\left(\sum_{i=0}^{n}|a_i|, \sum_{i=0}^{m}|b_i|\right) \tag{5}$$

and thus we have that $\Omega_\alpha A$ is a ρ'-algebra. It may be proved that the map $d : \Omega_\alpha A \to \Omega_\alpha A$ is a derivation of the order $((1,\alpha),(1,0))$. Concluding we have the following result:

Theorem 1. *$(\Omega_\alpha A, d)$ is a ρ-differential calculus over A.*

Example 1. In the case when the group G is trivial then A is the usual associative algebra and $\Omega_\alpha A$ is the algebra of universal differential forms of A.

Example 2. If the group G is $\mathbb{Z}_2$ and the cocycle ρ is defined by $\rho(a,b) = (-1)^{ab}$ then A is a superalgebra. In the case when $\alpha = 1$ $\Omega_\alpha A$ is the superalgebra of universal differential forms of A from [**11**].

3. The Frobenius theorem on quantum hyperplane

In this section we give the Frobenius theorem on the N-dimensional quantum hyperplane S_N^q and we give the equations of any globally integrable distributions and parallel with respect to a connection ∇. These results are valid for any ρ-differential calculus on S_N^q. In the first subsection we present two different ρ-differential calculi on S_N^q: the first one is the algebra of forms $\Omega(S_N^q)$ on S_N^q form [**1**] and the second one is the algebra of universal differential forms $\Omega_\alpha(S_N^q)$ on S_N^q. Both of these ρ-differential calculi on S_N^q are different by differential calculus on S_N^q of Wess and Zumino [**15**]. In the second subsection we present submanifolds on S_N^q, in the third subsection we review the basic notions about linear connections on S_N^q and finally we will give the main results of this paper.

3.1. Differential calculi on S_N^q

First we review the basic notions about the N-dimensional quantum hyperplane S_N^q. For more details see [**1**] and [**3**].

The N-dimensional quantum hyperplane S_N^q is the k-algebra generated by the unit element and N linearly independent elements $x_1, \ldots, x_N$ satisfying the relations: $x_i x_j = q x_j x_i$, $i < j$ for some fixed $q \in k$, $q \neq 0$.

S_N^q is a $\mathbb{Z}^N$-graded algebra

$$S_N^q = \bigoplus_{n_1, \ldots, n_N}^{\infty} (S_N^q)_{n_1 \ldots, n_N},$$

with $(S_N^q)_{n_1, \ldots, n_N}$ the one-dimensional subspace spanned by products $x^{n_1} \cdot \ldots \cdot x^{n_N}$. The $\mathbb{Z}^N$-degree of $x^{n_1} \cdot \ldots \cdot x^{n_N}$ is $n = (n_1, \ldots, n_N)$. The cocycle $\rho : \mathbb{Z}^N \times \mathbb{Z}^N \to k$ is

$$\rho(n, n') = q^{\sum\limits_{j,k=1}^{N} n_j n'_k \alpha_{jk}},$$

with $\alpha_{jk} = 1$ for $j < k$, 0 for $j = k$ and -1 for $j > k$.

It may be proved that the N-dimensional quantum hyperplane S_N^q is a ρ-commutative algebra.

Remark that the space of ρ-derivations ρ-Der(S_N^q) is a free S_N^q-module of rank N with $\partial/\partial x_1, \ldots, \partial/\partial x_N$ as the basis, where $\partial/\partial x_i(x_j) = \delta_{ij}$.

Remark 2. Let $(\Omega S_N^q, d)$ a ρ-differential calculus on S_N^q, with $d : \Omega S_N^q \to \Omega S_N^q$ a ρ-derivation of the order $((1, \alpha), (1, 0))$, where $\alpha, .0 \in \mathbb{Z}^N$ It is easy to see that ΩS_N^q is generated by $\{x_1, \ldots, x_N\}$ and there differentials $\{y_1 = dx_1, \ldots, y_N = dx_N\}$ with some relations between them.

Next we give some examples of ρ-differential calculi on S_N^q.

3.1.1. The algebra of forms $\Omega(S_N^q)$ of S_N^q. $\Omega(S_N^q)$ ([**1**], [**3**]) is the algebra determined by the elements $x_1, \ldots, x_N$ and $y_1 = dx_1, \ldots, y_N = dx_N$ with the relations

$$x_j x_k = q^{\alpha_{jk}} x_k x_j, \qquad y_j y_k = -q^{\alpha_{jk}} y_k y_j, \qquad x_j y_k = q^{\alpha_{jk}} y_k x_j. \tag{6}$$

3.1.2. The algebra of universal differential forms $\Omega_\alpha(A)$ of S_N^q. Next we will apply the construction of the algebra of the universal differential forms of a ρ-algebra from the remark 1 to the ρ-algebra S_N^q and, thus, we will give a new differential calculus on S_N^q denoted by $\Omega_\alpha(S_N^q)$.

Let $\alpha = (n_1, \ldots, n_N)$ be an arbitrary element from $\mathbb{Z}^N$. $\Omega_\alpha(S_N^q)$ is the algebra generated by $a \in S_N^q$ and the symbols da ,which satisfies the following relations:

1. da is linear in a.
2. the ρ-Leibniz rule: $d(ab) = (da)b + \rho(n, |a|)adb$.
3. $d(1) = 0$.

Next we present the structure of the algebra $\Omega_\alpha(S_N^q)$.

We use the following notations $y_i = dx_i$, for any $i \in \{1, \ldots, N\}$. By an easy computation we get the following lemmas:

Lemma 1. *$y_i x_j = \rho(\alpha + |x_i|, |x_j|) x_j y_i$, for any $i, j \in \{1, \ldots, N\}$.*

Lemma 2. *$y_j y_i = \rho(\alpha, |x_i|)\rho(n + |x_i|, |x_j|) y_i y_j$, for any $i, j \in \{1, \ldots, N\}$.*

Lemma 3. *$d(x_i^m) = m\rho^{m-1}(\alpha, |x_i|) x_i^{m-1} y_i$, for any $m \in \mathbb{N}$ and $i \in \{1, \ldots, N\}$.*

Putting together the previous lemmas we obtain the following theorem which gives the structures of the algebra $\Omega_\alpha(S_N^q)$:

Theorem 2. *$\Omega_\alpha(S_N^q)$ is the algebra spanned by the elements x_i and y_i with $i \in \{1, \ldots, N\}$ which satisfies the following relations:*

1. *$x_i x_j = \rho(|x_i|, |x_j|) x_j x_i$,*
2. *$y_i x_j = \rho(\alpha + |x_i|, |x_j|) x_j y_i$,*
3. *$y_j y_i = \rho(, |x_i|)\rho(\alpha + |x_i|, |x_j|) y_i y_j$, for any $i, j \in \{1, \ldots, N\}$.*

3.2. Submanifolds in S_N^q

In this section we use the definition of submanifolds algebra in noncommutative geometry from [**11**] to introduce submanifolds in the quantum hyperplane.

Let C be an ideal in S_N^q. We denote by $Q = S_N^q/C$ the quotient algebra and by $p : S_N^q \to Q$ the quotient map. We consider the following two Lie ρ-subalgebras of ρ-DerS_N^q :

$$G_C = \{X \in \rho\text{-Der}S_N^q / XC \subset C\}$$

and

$$G_A = \{X \in \rho\text{-Der}S_N^q / X(S_N^q) \subset C\}$$

We define the map $\pi : G_C \to \rho$-DerS_N^q by $\pi(X)p(a) = p(Xa)$ for any $a \in S_N^q$ and $X \in G_C$.

Definition 4. The quotient algebra $Q = S_N^q/C$ is a *submanifold algebra* of S_N^q if the map π is sujective.

In this situation we obtain the following the short exact sequence of ρ-Lie algebras.

$$0 \to G_A \to G_C \to \rho\text{-Der}Q \to 0 \tag{7}$$

Let C be the ideal from S_N^q generated by the elements $x_1, x_2, \ldots, x_p$ with $p \in \{1, \ldots, N\}$. Then the algebra Q generated by the elements $x_{p+1}, \ldots, x_N$ and we have that $A = C \oplus Q$.

Theorem 3. *$G_C = (\rho\text{-}\operatorname{Der} Q) \oplus G_A$.*

Proof. Let X be a colour derivation from ρ-DerS_N^q, then

$$X = \sum_{i=1}^{N} X_i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad X_i = X_i^C + X_i^Q$$

with $X_i^C \in C$ and $X_i^Q \in Q$ for any $i \in \{1, \ldots, N\}$.

If $X \in G_C$ then $X(c) \in C$ for any $c \in C$ so

$$\begin{aligned} X(c) &= \sum_{i=1}^{N} X_i^C \frac{\partial c}{\partial x_i} + \sum_{i=1}^{N} X_i^Q \frac{\partial c}{\partial x_i} \\ &= \underbrace{\sum_{i=1}^{N} X_i^C \frac{\partial c}{\partial x_i}}_{\in C} + \sum_{i=1}^{p} X_i^Q \frac{\partial c}{\partial x_i} + \underbrace{\sum_{i=p+1}^{N} X_i^Q \frac{\partial c}{\partial x_i}}_{\in C} \in C \end{aligned}$$

Results that $X_i^Q = 0$ for any $i \in \{1, \ldots, p\}$.

So any element $X = \sum\limits_{i=1}^{N} X_i \, \partial/\partial x_i$ from G_C may be written is the following way

$$X = X^{G_A} + X^Q$$

where

$$X^{G_A} = \sum_{i=1}^{N} X_i^C \frac{\partial}{\partial x_i} \in G_A \qquad \text{and} \qquad X^Q = \sum_{i=1}^{N} X_i^Q \frac{\partial}{\partial x_i} \in \rho\text{-Der}Q$$

with $X_i^Q = 0$ for $i \in \{1, \ldots, p\}$. □

Corollary 1. *Q is submanifolds algebra of S_N^q.*

3.3. Linear connections on S_N^q

In this subsection we use the definition of linear connections on ρ-algebras from the paper [**3**]. Let $(\Omega S_N^q, d)$ be a ρ-differential calculus on S_N^q and $n \in \mathbb{Z}^N$. A linear connection along the field $X = \sum\limits_{i=1}^{N} X_i \frac{\partial}{\partial x_i}$ on the ρ-bimodule $\Omega^n S_N^q$ over S_N^q is a linear map

$$\nabla : \rho\text{-Der}\,(S_N^q) \to \text{End}\,(\Omega^n S_N^q)$$

of degree $|X|$ such that

$$\nabla(X)(a\omega) = \nabla_X(a\omega) = \rho(|X|\,, |\omega|)X(a)\omega + a\nabla_X\omega$$

for any $X \in \rho\text{-Der}S_N^q$, $a \in S_N^q$ and $\omega \in \Omega^n S_N^q$.

Using the structure of the free bimodule $\Omega^n S_N^q$ we deduce that any such connection ∇ is well defined by the *connections coefficients* $\Gamma_{i,i_1,\ldots,i_n}^{j_1,\ldots,j_n} \in S_N^q$ defined by

$$\nabla_{\frac{\partial}{\partial x_i}}(y_{i_1} \ldots y_{i_n}) = \Gamma_{i,i_1,\ldots,i_n}^{j_1,\ldots,j_n} y_{j_1} \ldots y_{j_n}. \tag{8}$$

Remark 3. The connection coefficients $\Gamma_{i,i_1,\ldots,i_n}^{j_1,\ldots,j_n}$ satisfy the some properties which depend on the choice of the ρ-differential calculus $(\Omega S_N^q, d)$.

Example 3. If $(\Omega S_N^q, d)$ is the algebra of forms $\Omega\,(S_N^q)$ of S_N^q we obtain that

$$\Gamma_{i,i_1,\ldots,i_k,i_{k+1},\ldots,i_n}^{j_1,\ldots,j_n} = -q^{\alpha_{i_k,i_{k+1}}} \Gamma_{i,i_1,\ldots,i_{k+1},i_k,\ldots,i_n}^{j_1,\ldots,j_n} \tag{9}$$

for any $i, i_1, \ldots, i_n, j_1, \ldots, j_n \in \{1, \ldots, n\}$.

Example 4. If $(\Omega S_N^q, d)$ is the algebra of universal differential forms $\Omega_\alpha\left(S_N^q\right)$ of S_N^q using an easy computation we obtain that

$$\Gamma_{i,i_1,\ldots,i_k,i_{k+1},\ldots,i_m}^{j_1,\ldots,j_m} = \rho(\alpha, |y_{i_k}|)\rho(\alpha + |y_{i_k}|, |y_{i_{k+1}}|)\Gamma_{i,i_1,\ldots,i_{k+1},i_k,\ldots,i_m}^{j_1,\ldots,j_m}. \tag{10}$$

3.4. Distributions in S_N^q

Next we introduce distributions on S_N^q. Let $(\Omega S_N^q, d)$ be a ρ-differential calculus on S_N^q.

Definition 5. A *distribution* $\mathcal{D}$ on S_N^q is a S_N^q-subbimodule of $\Omega^1 S_N^q$. The distribution $\mathcal{D}$ is *globally integrable* if there is a subset B of S_N^q such that $\mathcal{D}$ is the subspace generated by $S_N^q d(B)$ and by $d(B)S_N^q$.

Definition 6. We say that the distribution $\mathcal{D}$ is *parallel* with respect to the connection $\nabla : \rho\text{-Der}S_N^q \to \mathrm{End}(\Omega^1 S_N^q)$ if

$$\nabla_X(m) = 0, \text{ for any } X \in \rho\text{-Der}S_N^q \text{ and for any } m \in \mathcal{D}.$$

Using the structure of ΩS_N^q we obtain the following structure theorem of globally integrable distributions.

Theorem 4. *Any globally integrable distributions $\mathcal{D}$ determined by $S_N^q d(B)$ and $d(B)S_N^q$ where B is the subset $\{x_1, \ldots, x_p\}$. In this situation we say that the distribution $\mathcal{D}$ has the dimension p.*

3.4.1. The Frobenius theorem for quantum hyperplane. In this section we will give a Frobenius theorem for N-dimensional quantum hyperplane which is obvious from the previous results.

Theorem 5. The Frobenius theorem for quantum hyperplane. *Any globally integrable distribution from S_N^q is given by a maximal submanifold algebra of S_N^q and conversely any submanifold algebra of S_N^q give a globally integrable distribution with the same dimension.*

Proof. Let $\mathcal{D}$ be a global integrable distribution from S_N^q. Then from the theorem 10, $\mathcal{D}$ is given by $S_N^q dQ$, where Q is a subset with p elements from $\{x_1, \ldots x_N\}$. If we denote by C the ideal of S_N^q generated by $\{x_1, \ldots x_N\} \setminus Q$ and using the Corollary 9 we obtain that $Q{=}S_N^q/C$ is a submanifold algebra of S_N^q of the dimension p.

Conversely, if Q is a submanifold algebra of the dimension p of S_N^q results that there is an ideal C of the dimension $N-p$ of S_N^q such that $Q = S_N^q/C$. If we denote by $\{x_1, \ldots, x_{N-p}\}$ is the subset of $\{x_1, \ldots, x_N\}$ which generates the ideal C then the set $Q = \{x_{N-p+1}, \ldots, x_N\}$ generates a distribution of the dimension p of S_N^q. □

We may find the equations of an globally integrable distributions and parallel with respect to a connection ∇.

Theorem 6. *Any globally integrable and parallel distribution $\mathcal{D}$ with respect to a connection $\nabla : \rho\text{-Der}S_N^q \to \text{End}(\Omega^1 S_N^q)$ of dimension p is given by the following equations:*

$$\Gamma_{i,j}^k = 0 \tag{11}$$

for a subset I of $\{1, \ldots, N\}$ with p elements and for any $i \in \{1, \ldots, N\}$, $j \in I$, $k \in \{1, \ldots, N\} \setminus I$.

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