RANKS AND INDEPENDENCE OF SOLUTIONS OF THE MATRIX EQUATION $AXB + CYD = M$

YONGGE TIAN

Abstract. Suppose $AXB + CYD = M$ is a consistent matrix equation. In this paper, we give some formulas for the maximal and minimal ranks of two solutions X and Y to the equation. In addition, we investigate the independence of solutions X and Y to this equation.

1. Introduction

Throughout this paper, the notation A^T, A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the transpose, conjugate transpose, rank and range (column space) of a matrix A over the field $\mathbb{C}$ of complex numbers, respectively. A matrix X is called a generalized inverse of A, denoted by A^-, if it satisfies $AXA = A$. In addition, E_A and F_A stand for the two oblique projectors $E_A = I - AA^-$ and $F_A = I - A^-A$ induced by A and A^-.

Linear matrix equations have been the objects of many studies in matrix theory and its applications. The primary work in the investigation of a matrix equation is to give its solvability conditions and general solutions. In additions to these two problems, many other topics can be investigated for a matrix equation. For example, the uniqueness of solution, minimal norm solutions, least-squares solutions, Hermitian solutions, and skew-Hermitian solutions to the equation. For some simplest matrix equations, it is easy to characterize the solvability and to give general solutions by generalized inverses. For instance, the matrix equation $AXB = C$, where A, B and C

Received October 3, 2004.

2000 *Mathematics Subject Classification.* Primary 15A03, 15A09, 15A24.

Key words and phrases. Generalized inverse, matrix equation, rank equality, rank formulas for partitioned matrices, general solution, independence of solutions.

are $m \times p$, $q \times n$ and $m \times n$ matrices, respectively, is consistent if and only if $AA^{-}CB^{-}B = C$. In this case, the general solution of $AXB = C$ can be written as $X = A^{-}CB^{-} + (\, I_p - A^{-}A)U + V(\, I_q - BB^{-}\,)$, where U and V are arbitrary. Many problems can be considered for solutions of $AXB = C$, one of which is to determine the maximal and minimal possible ranks of solutions. The present author has shown in [10] that

$$\max_{AXB=C} r(X) = \min\{\, p, \quad q, \quad p + q + r(C) - r(A) - r(B)\,\},$$
$$\min_{AXB=C} r(X) = r(C).$$

Write complex solution of $AXB = C$ as $X = X_0 + X_1 i$, where X_0 and X_1 are both real. The present author also gives in [10] the maximal and minimal ranks of X_0 and X_1. In addition to $AXB = C$, another well-known matrix equation is

$$AXB + CYD = M, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times n}$, $C \in \mathbb{C}^{m\times s}$, $D \in \mathbb{C}^{t\times n}$, $M \in \mathbb{C}^{m\times n}$. Equation (1.1) and its applications have been investigated extensively, see, e.g., [1, 3, 4, 6, 7, 13, 15]. A regression model related to (1.1) is

$$M = AXB + CYD + \varepsilon,$$

where both X and Y are unknown parameter matrices and ε is a random error matrix. This model is also called the nested growth curve model in the literature, see, e.g., [5, 14].

The rank of a matrix A, a key concept in linear algebra, is the dimension of the vector space generated by the columns or rows of A, that is, the maximum number of linearly independent columns or rows of A. Equivalently, the rank of a matrix A is the largest order of square submatrix of A which determinant is nonzero. If a matrix has some variant entries, the rank of the matrix is also variant with respect to the entries.

A general method for solving linear matrix equations is the vec operation of a matrix $Z = (z_{ij}) \in \mathbb{C}^{m\times n}$ defined by

$$\mathrm{vec}\, Z = [\, z_{11}, \ldots, z_{m1},\, z_{12}, \ldots, z_{m2}, \ldots, z_{1n}, \ldots, z_{mn}\,]^T.$$

Applying the well-known formula $\mathrm{vec}\,(AXB) = (B^T \otimes A)\mathrm{vec}\,X$, where $B^T \otimes A$ is the Kronecker product of B^T and A, to (1.1) gives

$$[\, B^T \otimes A,\ D^T \otimes C\,]\begin{bmatrix} \mathrm{vec}X \\ \mathrm{vec}Y \end{bmatrix} = \mathrm{vec}M, \tag{1.2}$$

where $[\, B^T \otimes A,\ D^T \otimes C\,]$ is a row block matrix. Hence (1.2) is solvable if and only if $[\, B^T \otimes A,\ D^T \otimes C\,][\, B^T \otimes A,\ D^T \otimes C\,]^{-}\mathrm{vec}M = \mathrm{vec}M$. In such a case, the general solution of (1.2) can be written as

$$\begin{aligned}\begin{bmatrix} \mathrm{vec}X \\ \mathrm{vec}Y \end{bmatrix} =& [\, B^T \otimes A,\ D^T \otimes C\,]^{-}\mathrm{vec}M \\ &+ \ (I - [\, B^T \otimes A,\ D^T \otimes C\,]^{-}[\, B^T \otimes A,\ D^T \otimes C\,])V,\end{aligned} \tag{1.3}$$

where V is an arbitrary column vector. Result (1.3) implies that the general solutions X and Y of (1.1) are in fact two linear matrix expressions involving variant entries.

Since the two matrices X and Y satisfying (1.1) are not necessarily unique, it is of interest to find the maximal and minimal possible ranks of X, Y, AXB and CYD in (1.1). Another problem on a pair solutions X and Y to (1.1) is concerned with their independence, where the independence means that for any two pairs of solutions X_1, Y_1 and X_2, Y_2 of (1.1), the two new pairs X_1, Y_2 and X_2, Y_1 are also solutions to (1.1). This problem can also be solved through some rank formulas associated with (1.1).

Some useful rank formulas for partitioned matrices are given in the following lemma.

Lemma 1.1 ([2]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then:*

(a) $r[\, A,\ B\,] = r(A) + r(E_AB) = r(B) + r(E_BA)$.

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C)$.

(c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C)$.

The formulas in Lemma 1.1 can be used to simplify various matrix expressions involving generalized inverses of matrices. For example,

$$r\begin{bmatrix} E_{A_1}B_1 \\ E_{A_2}B_2 \end{bmatrix} = r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \end{bmatrix} - r(A_1) - r(A_2), \tag{1.4}$$

$$r[\, D_1F_{C_1},\, D_2F_{C_2} \,] = r\begin{bmatrix} D_1 & D_2 \\ C_1 & 0 \\ 0 & C_2 \end{bmatrix} - r(C_1) - r(C_2), \tag{1.5}$$

$$r\begin{bmatrix} A & BF_{B_1} \\ E_{C_1}C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & C_1 \\ 0 & B_1 & 0 \end{bmatrix} - r(B_1) - r(C_1). \tag{1.6}$$

Lemma 1.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, $B_1 \in \mathbb{C}^{m\times p}$ and $C_1 \in \mathbb{C}^{q\times n}$ be given, $X \in \mathbb{C}^{k\times l}$, $Y \in \mathbb{C}^{k\times n}$, $Z \in \mathbb{C}^{m\times l}$ and $U \in \mathbb{C}^{p\times q}$ be variant matrices. Then*

$$\max_X r(\, A - BXC \,) = \min\left\{ r[\, A,\, B \,],\quad r\begin{bmatrix} A \\ C \end{bmatrix} \right\}, \tag{1.7}$$

$$\min_X r(\, A - BXC \,) = r[\, A,\, B \,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1.8}$$

$$\max_{Y,\, Z} r(\, A - BY - ZC \,) = \min\left\{ m,\quad n,\quad r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right\}, \tag{1.9}$$

$$\min_{Y,\, Z} r(\, A - BY - ZC \,) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C), \tag{1.10}$$

$$\max_{Y,Z,U} r(A - BY - ZC - B_1UC_1) = \min\left\{ m, \quad n, \quad r\begin{bmatrix} A & B & B_1 \\ C & 0 & 0 \end{bmatrix}, \quad r\begin{bmatrix} A & B \\ C & 0 \\ C_1 & 0 \end{bmatrix} \right\}, \tag{1.11}$$

$$\min_{Y,Z,U} r(A - BY - ZC - B_1UC_1) = r\begin{bmatrix} A & B & B_1 \\ C & 0 & 0 \end{bmatrix} + r\begin{bmatrix} A & B \\ C & 0 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B & B_1 \\ C & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix} - r(B) - r(C). \tag{1.12}$$

Results (1.7) and (1.8) are shown in [12]; (1.9) and (1.10) are shown in [8, 9]. The general expressions of X and Y satisfying (1.7)–(1.10) are given in [8, 9, 12]. Combining (1.7) and (1.9), (1.8) and (1.10) yields (1.11) and (1.12), respectively.

2. Ranks of solutions to $AXB + CYD = M$

Concerning the solvability conditions and general solutions of (1.1), the following results have been shown.

Lemma 2.1.

(a) [3] *There are* X *and* Y *that satisfy* (1.1) *if and only if*

$$r[\, A, C, M \,] = r[\, A, C \,], \quad r\begin{bmatrix} B \\ D \\ M \end{bmatrix} = r\begin{bmatrix} B \\ D \end{bmatrix}, \tag{2.1}$$

$$r\begin{bmatrix} M & A \\ D & 0 \end{bmatrix} = r(A) + r(D), \quad r\begin{bmatrix} M & C \\ B & 0 \end{bmatrix} = r(B) + r(C), \tag{2.2}$$

or equivalently,

$$[A, C][A, C]^{-}M = M, \quad M\begin{bmatrix} B \\ D \end{bmatrix}^{-}\begin{bmatrix} B \\ D \end{bmatrix} = M,$$

$$(I_m - AA^{-})M(I_n - D^{-}D) = 0, \quad (I_m - CC^{-})M(I_n - B^{-}B) = 0.$$

(b) [6, 7] *Under* (2.1) *and* (2.2), *the general solutions of* X *and* Y *to* (1.1) *can be decomposed as*

$$X = X_0 + X_1X_2 + X_3, \qquad Y = Y_0 + Y_1Y_2 + Y_3,$$

where X_0 *and* Y_0 *are a pair of special solutions of* (1.1), X_1, X_2, X_3 *and* Y_1, Y_2, Y_3 *are the general solutions of the following four homogeneous matrix equations*

$$AX_1 = -CY_1, \quad X_2B = Y_2D, \quad AX_3B = 0, \quad CY_3D = 0,$$

or explicitly,

$$X = X_0 + S_1F_GUE_HT_1 + F_AV_1 + V_2E_B, \tag{2.3}$$

$$Y = Y_0 + S_2F_GUE_HT_2 + F_CW_1 + W_2E_D, \tag{2.4}$$

where

$$S_1 = [I_p, 0], \; S_2 = [0, I_s], \; T_1 = \begin{bmatrix} I_q \\ 0 \end{bmatrix}, \; T_2 = \begin{bmatrix} 0 \\ I_t \end{bmatrix}, \; G = [A, C], \; H = \begin{bmatrix} B \\ -D \end{bmatrix};$$

the matrices U, V_1, V_2, W_1 *and* W_2 *are arbitrary.*

For convenience, we adopt the following notation

$$J_1 = \{ X \in \mathbb{C}^{p\times q} \mid AXB + CYD = M \}, \tag{2.5}$$

$$J_2 = \{ Y \in \mathbb{C}^{s\times t} \mid AXB + CYD = M \}. \tag{2.6}$$

Results (2.3) and (2.4) imply that the general solutions of (1.1) are in fact two linear matrix expressions, each of them involves three independent variant matrices. Applying Lemma 1.2 to (2.3) and (2.4) gives the following result.

Theorem 2.2. *Suppose that the matrix equation (1.1) is solvable, and let J_1 and J_2 be defined in (2.5) and (2.6). Then:*

(a) *The maximal and minimal ranks of solution X of (1.1) are given by*

$$\max_{X \in J_1} r(X) = \min\left\{ p, \quad q, \quad p+q+r[\,M,\, C\,] - r[\,A,\, C\,] - r(B), \right.$$
$$\left. p+q+r\begin{bmatrix} M \\ D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix} - r(A) \right\},$$

$$\min_{X \in J_1} r(X) = r[\,M,\, C\,] + r\begin{bmatrix} M \\ D \end{bmatrix} - r\begin{bmatrix} M & C \\ D & 0 \end{bmatrix}.$$

(b) *The maximal and minimal ranks of solution Y of (1.1) are given by*

$$\max_{Y \in J_2} r(Y) \;=\; \min\left\{ s, \quad t, \quad s+t+r[\,M,\, A\,] - r[\,C,\, A\,] - r(D), \right.$$
$$\left. s+t+r\begin{bmatrix} M \\ B \end{bmatrix} - r\begin{bmatrix} D \\ B \end{bmatrix} - r(C) \right\},$$

$$\min_{Y \in J_2} r(Y) \;=\; r[\,M,\, A\,] + r\begin{bmatrix} M \\ B \end{bmatrix} - r\begin{bmatrix} M & A \\ B & 0 \end{bmatrix}.$$

Proof. Applying (1.7) and (1.8) to (2.3) yields

$$\max_{X\in J_1} r(X) = \max_{U,\,V_1,\,V_2} r(\,X_0 + S_1F_GUE_HT_1 + F_AV_1 + V_2E_B\,) = \min\left\{ p, \quad q, \quad r\begin{bmatrix} X_0 & F_A & S_1F_G \\ E_B & 0 & 0 \end{bmatrix}, \quad r\begin{bmatrix} X_0 & F_A \\ E_B & 0 \\ E_HT_1 & 0 \end{bmatrix} \right\},$$

$$\begin{aligned}\min_{X\in J_1} r(X) &= \min_{U,\,V_1,\,V_2} r(\,X_0 + S_1F_GUE_HT_1 + F_AV_1 + V_2E_B\,)\\ &= r\begin{bmatrix} X_0 & F_A & S_1F_G \\ E_B & 0 & 0 \end{bmatrix} + r\begin{bmatrix} X_0 & F_A \\ E_B & 0 \\ E_HT_1 & 0 \end{bmatrix} - r\begin{bmatrix} X_0 & F_A & S_1F_G \\ E_B & 0 & 0 \\ E_HT_1 & 0 & 0 \end{bmatrix} - r(F_A) - r(E_B),\end{aligned}$$

where $r(F_A) = p - r(A)$ and $r(E_B) = q - r(B)$. As shown in (1.4), (1.5) and (1.6), the ranks of the block matrices in these two formulas can be simplified further by Lemma 1.1, as well as the equality $AX_0B + CY_0D = M$ and elementary block matrix operations

$$\begin{aligned} r&\begin{bmatrix} X_0 & F_A & S_1F_G \\ E_B & 0 & 0 \end{bmatrix}\\ &= r\begin{bmatrix} X_0 & I_p & S_1 & 0 \\ I_q & 0 & 0 & B \\ 0 & A & 0 & 0 \\ 0 & 0 & G & 0 \end{bmatrix} - r(A) - r(B) - r(G) = r\begin{bmatrix} 0 & I_p & 0 & 0 \\ I_q & 0 & 0 & 0 \\ 0 & 0 & -AS_1 & AX_0B \\ 0 & 0 & G & 0 \end{bmatrix} - r(A) - r(B) - r(G)\\ &= r\begin{bmatrix} -A & 0 & AX_0B \\ A & C & 0 \end{bmatrix} + p + q - r(A) - r(B) - r(G) = r[\,C,\ AX_0B\,] + p + q - r(B) - r(G)\\ &= r[\,C,\ M\,] + p + q - r(B) - r(G),\end{aligned}$$

$$
r\begin{bmatrix} X_0 & F_A \\ E_B & 0 \\ E_HT_1 & 0 \end{bmatrix}
$$

$$
= r\begin{bmatrix} X_0 & I_p & 0 & 0 \\ I_q & 0 & B & 0 \\ T_1 & 0 & 0 & H \\ 0 & A & 0 & 0 \end{bmatrix} - r(A) - r(B) - r(H) = r\begin{bmatrix} 0 & I_p & 0 & 0 \\ I_q & 0 & 0 & 0 \\ 0 & 0 & -T_1B & H \\ 0 & 0 & AX_0B & 0 \end{bmatrix} - r(A) - r(B) - r(H)
$$

$$
= r\begin{bmatrix} B & B \\ 0 & D \\ AX_0B & 0 \end{bmatrix} + p + q - r(A) - r(B) - r(H) = r\begin{bmatrix} D \\ AX_0B \end{bmatrix} + p + q - r(A) - r(H)
$$

$$
= r\begin{bmatrix} D \\ M \end{bmatrix} + p + q - r(A) - r(H),
$$

$$
r\begin{bmatrix} X_0 & F_A & S_1F_G \\ E_B & 0 & 0 \\ E_HT_1 & 0 & 0 \end{bmatrix}
$$

$$
= r\begin{bmatrix} X_0 & I_p & S_1 & 0 & 0 \\ I_q & 0 & 0 & B & 0 \\ T_1 & 0 & 0 & 0 & H \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \end{bmatrix} - r(A) - r(B) - r(G) - r(H)
$$

$$
= r\begin{bmatrix} 0 & I_p & 0 & 0 & 0 \\ I_q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -T_1B & H \\ 0 & 0 & -AS_1 & AX_0B & 0 \\ 0 & 0 & G & 0 & 0 \end{bmatrix} - r(A) - r(B) - r(G) - r(H)
$$

$$= r\begin{bmatrix} 0 & 0 & -B & B \\ 0 & 0 & 0 & -D \\ -A & 0 & AX_0B & 0 \\ A & C & 0 & 0 \end{bmatrix} + p + q - r(A) - r(B) - r(G) - r(H)$$

$$= r\begin{bmatrix} 0 & 0 & -B & 0 \\ 0 & 0 & 0 & D \\ -A & 0 & 0 & 0 \\ 0 & C & 0 & M \end{bmatrix} + p + q - r(A) - r(B) - r(G) - r(H)$$

$$= r\begin{bmatrix} M & C \\ D & 0 \end{bmatrix} + p + q - r(G) - r(H).$$

Thus, we have (a). Similarly, we can show (b). □

Furthermore, we can give the formulas for the maximal and minimal ranks of AXB and CYD in (1.1) when it is solvable.

Theorem 2.3. *Suppose that there are X and Y that satisfy (1.1), and let J_1 and J_2 be defined in (2.5) and (2.6). Then*

$$\max_{X\in J_1} r(AXB) = \min\left\{ r[\,M,\,C\,] - r[\,A,\,C\,] + r(A),\ \ r\begin{bmatrix} M \\ D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix} + r(B) \right\}, \tag{2.7}$$

$$\min_{X\in J_1} r(AXB) = r[\,M,\,C\,] + r\begin{bmatrix} M \\ D \end{bmatrix} - r\begin{bmatrix} M & C \\ D & 0 \end{bmatrix}, \tag{2.8}$$

(2.9) $$\max_{Y\in J_2} r(CYD) = \min\left\{ r[\,M,\ A\,] - r[\,C,\ A\,] + r(C), r\begin{bmatrix} M \\ B \end{bmatrix} - r\begin{bmatrix} D \\ B \end{bmatrix} + r(D)\right\},$$

(2.10) $$\min_{Y\in J_2} r(CYD) = r[\,M,\ A\,] + r\begin{bmatrix} M \\ B \end{bmatrix} - r\begin{bmatrix} M & A \\ B & 0 \end{bmatrix}.$$

Proof. Applying (1.7) and (1.8) to $AXB = AX_0B + AS_1F_GUE_HT_1B$ yields

$$\begin{aligned}\max_{X\in J_1} r(AXB) &= \max_{U} r(\,AX_0B + AS_1F_GUE_HT_1B\,)\\ &= \min\left\{ r[\,AX_0B,\ AS_1F_G\,],\quad r\begin{bmatrix} AX_0B \\ E_HT_1B \end{bmatrix}\right\},\\ \min_{X\in J_1} r(AXB) &= \min_{U} r(\,AX_0B + AS_1F_GUE_HT_1B\,)\\ &= r[\,AX_0B,\ AS_1F_G\,] + r\begin{bmatrix} AX_0B \\ E_HT_1B \end{bmatrix} - r\begin{bmatrix} AX_0B & AS_1F_G \\ E_HT_1B & 0 \end{bmatrix}.\end{aligned}$$

Also find by Lemma 1.1, $AX_0B + CY_0D = M$ and elementary block matrix operations that

$$\begin{aligned} r[\,AX_0B,\ AS_1F_G\,] &= r\begin{bmatrix} AX_0B & AS_1 \\ 0 & G \end{bmatrix} - r(G) = r\begin{bmatrix} AX_0B & A & 0 \\ 0 & A & C \end{bmatrix} - r(G)\\ &= r[\,AX_0B,\ C\,] + r(A) - r(G) = r[\,M,\ C\,] + r(A) - r(G),\end{aligned}$$

$$\begin{aligned} r\begin{bmatrix} AX_0B \\ P_HT_1B \end{bmatrix} &= r\begin{bmatrix} AX_0B & 0 \\ T_1B & H \end{bmatrix} - r(H) = r\begin{bmatrix} AX_0B & 0 \\ B & B \\ 0 & -D \end{bmatrix} - r(H)\\ &= r\begin{bmatrix} AX_0B \\ D \end{bmatrix} + r(B) - r(H) = r\begin{bmatrix} M \\ D \end{bmatrix} + r(B) - r(H),\end{aligned}$$

$$r\begin{bmatrix} AX_0B & AS_1F_G \\ E_HT_1B & 0 \end{bmatrix} = r\begin{bmatrix} AX_0B & AS_1 & 0 \\ T_1B & 0 & H \\ 0 & G & 0 \end{bmatrix} - r(G) - r(H) = r\begin{bmatrix} AX_0B & A & 0 & 0 \\ B & 0 & 0 & B \\ 0 & 0 & 0 & -D \\ 0 & A & C & 0 \end{bmatrix} - r(G) - r(H)$$

$$= r\begin{bmatrix} 0 & A & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & D \\ 0 & 0 & C & AX_0B \end{bmatrix} - r(G) - r(H) = r\begin{bmatrix} M & C \\ D & 0 \end{bmatrix} + r(A) + r(B) - r(G) - r(H).$$

Therefore, we have (2.7) and (2.8). In the same manner, one can show (2.9) and (2.10). □

3. Independence of solutions X and Y to $AXB + CYD = M$

The independence of the two matrices X_1 and X_2 that satisfy the matrix equation $A_1X_1 + A_2X_2 = B$ is investigated in the author's recent paper [11]. In this section, we consider the independence of X and Y that satisfy (1.1).

Consider J_1 and J_2 in (2.5) and (2.6) as two independent matrix sets. If for any given $X \in J_1$ and $Y \in J_2$, the pair satisfy (1.1), X and Y for (1.1) are said to be independent. The independence of solutions X and Y for (1.1) can also be examined through the rank formulas in Lemma 1.2.

Theorem 3.1. *Suppose that the matrix equation (1.1) is solvable. Moreover, let J_1 and J_2 in (2.5) and (2.6) as two independent matrix sets. Then*

$$\max_{X\in J_1,\, Y\in J_2} r(\, M - AXB - CYD\,) = \min\left\{ r(A) + r(C) - r[\, A,\, C\,],\;\; r(B) + r(D) - r\begin{bmatrix} B \\ D \end{bmatrix} \right\}. \tag{3.1}$$

In particular,

(a) *Solutions X and Y of (1.1) are independent if and only if*

$$\mathscr{R}(A) \cap \mathscr{R}(C) = \{0\} \quad \text{or} \quad \mathscr{R}(B^*) \cap \mathscr{R}(D^*) = \{0\}. \tag{3.2}$$

(b) *If* (3.2) *holds, the general solution of* (1.1) *can be written as the two independent forms*

$$X = X_0 + S_1F_GU_1E_HT_1 + F_AV_1 + V_2E_B, \tag{3.3}$$

$$Y = Y_0 + S_2F_GU_2E_HT_2 + F_CW_1 + W_2E_D, \tag{3.4}$$

where X_0 and Y_0 are a pair of special solutions to (1.1), U_1, U_2, V_1, V_2, W_1 *and* W_2 *are arbitrary.*

Proof. Writing (2.3) and (2.4) as two independent matrix expressions, substituting them into $M - AXB - CYD$ and observing $AS_1F_G = -CS_2F_G$ and $E_HT_1B = E_HT_2D$ gives

$$\begin{aligned} & M - AXB - CYD \\ & = M - AX_0B - CY_0D - AS_1F_GU_1E_HT_1B - CS_2F_GU_2E_HT_2D \\ & = -AS_1F_GU_1E_HT_1B - CS_2F_GU_2E_HT_2D \\ & = -AS_1F_GU_1E_HT_1B + AS_1F_GU_2E_HT_1B \\ & = AS_1F_G(\,-U_1 + U_2\,)E_HT_1B, \end{aligned}$$

where U_1 and U_2 are arbitrary. Then it follows by (1.3) that

$$\begin{aligned} \max_{X \in J_1,\, Y \in J_2} r(\,M - AXB - CYD\,) &= \max_{U_1,\, U_2} r[\,AS_1F_G(\,-U_1 + U_2\,)E_HT_1B\,] \\ &= \min\{\,r(AS_1F_G), \quad r(E_HT_1B)\,\}, \end{aligned}$$

where by Lemma 1.1

$$r(AS_1F_G) = r\begin{bmatrix} AS_1 \\ G \end{bmatrix} - r(G) = r\begin{bmatrix} A & 0 \\ A & C \end{bmatrix} - r(G) = r(A) + r(C) - r(G),$$

$$r(E_HT_1B) = r[\,T_1B,\, H\,] - r(H) = r\begin{bmatrix} B & B \\ 0 & -D \end{bmatrix} - r(H) = r(B) + r(D) - r(H).$$

Therefore, (3.1) follows. Result (3.2) follows from (3.1); the solutions in (3.3) and (3.4) follow from (2.3) and (2.4). □

Remark 3.2. The matrix equation (1.1) is one of the basic linear matrix equations. Many other types of matrix equations can be solved through (1.1). For example, From Lemma 1.2, one can derive necessary and sufficient conditions for the matrix equation $AXA^* + BYB^* = C$ to have Hermitian and skew-Hermitian solutions. From Lemma 2.1, one can also give necessary and sufficient conditions for the two matrix equations $AXB+(AXB)^* = C$ and $AXB - (AXB)^* = C$ to be solvable.

1. Baksalary J. K., Kala R., *The matrix equation $AXB + CYD = E$*, Linear Algebra Appl. bf 30 (1980), 141–147.
2. G. Marsaglia G. and Styan G. P. H., *Equalities and inequalities for ranks of matrices*, Linear and Multilinear Algebra **2** (1974), 269–292.
3. Özgüler A. B., *The matrix equation $AXB + CYD = E$ over a principal ideal domain*, SIAM J. Matrix. Anal. Appl. **12** (1991), 581–591.
4. Shim S-Y., Y. Chen Y., *Least squares solution of matrix equation $AXB^* + CYD^* = E$*, SIAM J. Matrix. Anal. Appl. **24** (2003), 802–808.
5. Srivastava M. S., *Nested growth curve models*, Sankhyā, Ser A **64** (2002), 379–408.
6. Tian Y., *The general solution of the matrix equation $AXB = CYD$*, Math. Practice Theory **1** (1988), 61–63.
7. ______, *Solvability of two linear matrix equations*, Linear and Multilinear Algebra **48** (2000), 123–147.
8. ______, *The minimal rank of the matrix expression $A - BX - YC$*, Missouri J. Math. Sci. **14** (2002), 40–48.
9. ______, *Upper and lower bounds for ranks of matrix expressions using generalized inverses*, Linear Algebra Appl. **355** (2002), 187–214.
10. ______, *Ranks of solutions of the matrix equation $AXB = C$*, Linear and Multilinear Algebra **51** (2003), 111–125.
11. ______, *Uniqueness and independence of submatrices in solutions of matrix equations*, Acta Math. Univ. Comenianae **72** (2003), 159–163.
12. Tian Y. and Cheng S., *The maximal and minimal ranks of $A - BXC$ with applications*, New York J. Math. **9** (2003), 345–362.
13. von Rosen D., *Some results on homogeneous matrix equations*, SIAM J. Matrix Anal. Appl. **14** (1993), 137–145.
14. ______, *Homogeneous matrix equations and multivarite models*, Linear Algebra Appl. **193** (1993), 19–33.

15. Xu G., Wei M. and Zheng D., *On solution of matrix equation $AXB + CYD = F$*, Linear Algebra Appl. **279** (1998), 93–109.

Yongge Tian, School of Economics, Shanghai University of Finance and Economics, Shanghai, 200433, China,
e-mail: yongge@mail.shufe.edu.cn