A REMARK ON THE LARGE TIME BEHAVIOR OF SOLUTIONS OF VISCOUS HAMILTON-JACOBI EQUATIONS

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1. INTRODUCTION AND MAIN RESULT

Consider the viscous Hamilton-Jacobi equation

$$(1) \quad \begin{cases} u_t - \Delta u = |\nabla u|^q, & t > 0, \quad x \in \mathbb{R}^N \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $q > 0$ and $u_0 \in C_b(\mathbb{R}^N)$. It is known [6] that (1) admits a unique classical solution, global for $t > 0$.

The large time behavior of solutions of problem (1) has been studied recently by several authors, see [1]–[5], [7, 8] and the references therein. In particular it was shown by Gilding [5] that the large time limits

$$\underline{\omega} := \liminf_{t \to \infty} v(x, t) \le \overline{\omega} := \limsup_{t \to \infty} v(x, t)$$

are independent of $x \in \mathbb{R}^N$. One of the main results of [5] is the following.

Theorem A. *Assume $0 < q < 2$ and $u_0 \in C_b(\mathbb{R}^N)$. Then $\underline{\omega} = \overline{\omega}$.*

It was known that Theorem A fails for the linear heat equation and, moreover, Gilding observed that it fails for $q = 2$. The aim of this short note is to show that the assumption $q < 2$ in Theorem A is actually necessary.

Theorem 1. *Assume $q \ge 2$. Then there exists $u_0 \in C_b(\mathbb{R}^N)$ such that $\underline{\omega} < \overline{\omega}$.*

Proof. It is known (see e. g. [5, Proposition H1]) that there exists $v_0 \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that the solution v of the heat equation

$$(2) \quad \begin{cases} v_t - \Delta v = 0, & t > 0, \quad x \in \mathbb{R}^N \\ v(0, x) = v_0(x), & x \in \mathbb{R}^N \end{cases}$$

satisfies

$$(3) \quad \underline{\omega}^* := \liminf_{t \to \infty} v(x, t) < \overline{\omega}^* := \limsup_{t \to \infty} v(x, t), \quad x \in \mathbb{R}^N.$$

Moreover, upon replacing v_0 by $\lambda v_0 + \mu$ for suitable constants λ, μ, one can assume that

$$\underline{\omega}^* = 0 \tag{4}$$

and

$$\|v_0\|_\infty \le 1/2, \qquad \|\nabla v_0\|_\infty \le 1/2.$$

Now, set

$$u_0(x) := e^{v_0(x)} - 1. \tag{5}$$

The function $w := e^v - 1$ satisfies

$$\begin{cases} w_t - \Delta w = |\nabla w|^2, & t > 0, \quad x \in \mathbb{R}^N \\ w(0, x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{6}$$

Let u be the solution of (1) with initial data u_0 defined by (5). We note that

$$\|\nabla u_0\|_\infty \le \|\nabla v_0\|_\infty \| e^{v_0} \|_\infty \le (1/2) e^{1/2} < 1.$$

Since it is known (see e.g. [**5**, Lemma 2]) that $|\nabla u|$ satisfies a maximum principle, it follows that

$$|\nabla u| \le \|\nabla u_0\|_\infty < 1 \quad \text{in } Q := (0, \infty) \times \mathbb{R}^N.$$

Due to $q \ge 2$, we deduce that

$$u_t - \Delta u = |\nabla u|^q \le |\nabla u|^2 \quad \text{in } Q.$$

In view of (6), it follows from the comparison principle that

$$u \le w = e^v - 1 \quad \text{in } Q.$$

In particular, there holds

$$\underline{\omega} \le e^{\underline{\omega}^*} - 1 = 0. \tag{7}$$

But on the other hand, we have $u_0 \ge v_0$ due to (5). In view of (2), the maximum principle implies that $u \ge v$, hence

$$\overline{\omega} \ge \overline{\omega}^*. \tag{8}$$

Combining (3), (4), (7) and (8), we conclude that

$$\overline{\omega} \ge \overline{\omega}^* > \underline{\omega}^* = 0 \ge \underline{\omega}$$

and the proof of Theorem 1 is complete. $\square$

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