

ON BLOW-UP AT SPACE INFINITY FOR SEMILINEAR HEAT EQUATIONS

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We are interested in solutions of semilinear heat equations which blow up at space infinity.

In [7], we considered a nonnegative blowing up solution of

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^n, \quad t > 0$$

with initial data u_0 satisfying

$$0 \leq u_0(x) \leq M, \quad u_0 \not\equiv M \quad \text{and} \quad \lim_{|x| \rightarrow \infty} u_0(x) = M,$$

where $p > 1$ and $M > 0$ is a constant. We proved in [7] that the solution u blows up exactly at the blow-up time for the spatially constant solution with initial data M . We moreover proved that u blows up only at the space infinity. In this paper we would like to generalize this result in the following directions.

- (i) (Initial data) We consider more general initial data u_0 which may not converge to M for all directions of x , for example $u_0 \rightarrow M$ as $|x| \rightarrow \infty$ only for x in some sector. It is convenient to introduce a notion of blow up direction at the space infinity. We are able to give necessary and sufficient condition so that a particular direction is a blow-up direction.
- (ii) (Nonlinear term) We extend the class of nonlinearities. It includes e^u and $u^p + u^q$ for $p, q > 1$.

In [8] we consider solutions of the initial value problem for the equation

$$(1) \quad \begin{cases} u_t = \Delta u + f(u), & x \in \mathbb{R}^n, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

The nonlinear term f is assumed to be nonnegative and locally Lipschitz in \mathbb{R} with the property that

$$(2) \quad \liminf_{b \geq b_0, \delta \in (\delta_0, 1)} \frac{\delta^p f(b)}{f(\delta b)} > 0 \quad \text{for} \quad b_0 > 0, \quad \delta_0 \in (0, 1), \quad p > 1.$$

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We take two constants M and N satisfying $M + N > 0$ and

$$(3) \quad f(M) > 0.$$

The initial data u_0 is assumed to be a measurable function in \mathbb{R}^n satisfying

$$(4) \quad -N \leq u_0 \leq M \text{ a.e.} \quad \text{and} \quad u_0 \not\equiv M \text{ a.e.}$$

We are interested in initial data such that $u_0 \rightarrow M$ as $|x| \rightarrow \infty$ for x in some sector of \mathbb{R}^n . We assume that

$$(5) \quad \operatorname{ess\,inf}_{x \in \tilde{B}_m} (u_0(x) - M_m(x - x_m)) \geq 0 \quad \text{for } m = 1, 2, \dots,$$

where

$$(6) \quad \tilde{B}_m = B_{r_m}(x_m)$$

with a sequence $\{r_m\}$ and a sequence of vectors $\{x_m\}_{m=1}^\infty$ and a sequence of functions $\{M_m(x)\}$ satisfying

$$\lim_{m \rightarrow \infty} r_m = \infty, \quad M_m(x) \leq M_{m+1}(x) \quad \text{for } m \geq 1$$

$$\lim_{m \rightarrow \infty} \inf_{s \in [1, r_m]} \frac{1}{|B_s|} \int_{B_s(0)} M_m(x) \, dx = M.$$

Here $B_r(x)$ denotes the closed ball of radius r centered at x . (In fact, it follows from (4) that $|x_m| \rightarrow \infty$ as $m \rightarrow \infty$.)

Problem (1) has a unique bounded solution at least locally in time. However, the solution may blow up in finite time. For a given initial value u_0 and nonlinear term f let $T^* = T^*(u_0, f)$ be the maximal existence time of the solution. If $T^* = \infty$, the solution exists globally in time. If $T^* < \infty$, we say that the solution blows up in finite time. It is well known that

$$(7) \quad \limsup_{t \rightarrow T^*} \|u(\cdot, t)\|_\infty = \infty,$$

where $\|\cdot\|_\infty$ denotes the L^∞ -norm in space variables.

In this paper, we are interested in the behavior of a blowing up solution near space infinity as well as the location of blow-up points defined below. A point $x_{BU} \in \mathbb{R}^n$ is called a *blow-up point* (with value $\pm\infty$) if there exists a sequence $\{(x_m, t_m)\}_{m=1}^\infty$ such that

$$t_m \uparrow T^*, \quad x_m \rightarrow x_{BU} \quad \text{and} \quad u(x_m, t_m) \rightarrow \pm\infty \quad \text{as } m \rightarrow \infty.$$

If there exists a sequence $\{(x_m, t_m)\}_{m=1}^\infty$ such that

$$t_m \uparrow T^*, \quad |x_m| \rightarrow \infty \quad \text{and} \quad u(x_m, t_m) \rightarrow \pm\infty \quad \text{as } m \rightarrow \infty,$$

then we say that the solution blows up to $\pm\infty$ at space infinity.

A direction $\psi \in S^{n-1}$ is called a *blow-up direction for the value $\pm\infty$* if there exists a sequence $\{(x_m, t_m)\}_{m=1}^\infty$ with $x_m \in \mathbb{R}^n$ and $t_m \in (0, T^*)$ such that $u(x_m, t_m) \rightarrow \pm\infty$ (as $m \rightarrow \infty$) and

$$(8) \quad \frac{x_m}{|x_m|} \rightarrow \psi \quad \text{as } m \rightarrow \infty.$$

We consider the solution $v(t)$ of an ordinary differential equation

$$(9) \quad \begin{cases} v_t = f(v), & t > 0, \\ v(0) = M. \end{cases}$$

Let $T_v = T^*(M, f)$ be the maximal existence time of the solution of (9), i. e.,

$$T_v = \int_M^\infty \frac{ds}{f(s)}.$$

We are now in position to state our main results.

Theorem 1. *Assume that f is locally Lipschitz in \mathbb{R} and satisfies (2) and (3). Let u_0 be a continuous function satisfying (4) and (5), and $T_v \leq T^*(-N, f)$. Then there exists a subsequence of $\{x_m\}_{m=1}^\infty$ (still denoted by $\{x_m\}$, independent of t) such that*

$$\lim_{m \rightarrow \infty} u(x_m, t) = v(t).$$

The convergence is uniform in every compact subset of $\{t : 0 \leq t < T_v\}$. Moreover, the solution blows up at T_v .

Remark. Our assumption $T_v \leq T^*(-N, f)$ says that the solution does not blow up to minus infinity before it blows up to plus infinity. From the condition (4), it follows that $\lim_{m \rightarrow \infty} |x_m| = \infty$.

This result in particular implies that

$$(10) \quad \sup_{0 < t < T^*} v^{-1}(t) \|u(\cdot, t)\|_\infty < \infty.$$

When we set $f(u) = |u|^{p-1}u$, such a blow-up rate estimate is known for subcritical p ; see e.g. [3], [5], [6] for general bounded initial data without assuming (4) and (5). However, for supercritical p such a blow-up rate estimate (10) may not hold in general; see e.g. [1], [9]. If one considers only radial solutions of (1) for supercritical p less than $1 + 4/(n - 4 - 2(n - 1)^{1/2})$ or $n \leq 10$, then the estimate (10) holds [11]. We would like to emphasize that Theorem 1 does not require any restriction on p .

Our second main result is on the location of blow-up points.

Theorem 2. *Assume the same hypotheses as in Theorem 1. Then the solution of (1) has no blow-up points with $+\infty$ in \mathbb{R}^n . (It blows up only at space infinity.)*

There is a huge literature on location of blow-up points since the work of Weissler [13] and Friedman-McLeod [2]. (We do not intend to list references exhaustively in this paper.) However, most results consider either bounded domains or solutions decaying at space infinity; such a solution does not blow up at space infinity [4].

As far as the authors know, before the result of [7] the only paper discussing blow-up at space infinity is the work of Lacey [10]. He considered the Dirichlet problem in a half line. He studied various nonlinear terms and proved that a solution blows up only at space infinity.

In particular, his result implies that the solution of

$$\begin{cases} u_t = u_{xx} + f(u), & x > 0, \quad t > 0, \\ u(0, t) = 1, & t > 0, \\ u(x, 0) = u_0(x) \geq 1, & x > 0 \end{cases}$$

blows up only at space infinity, where u_0 satisfies $0 \leq u_0 \leq M$ with $M > 1$, and $f(s) = s^p$ and e^s .

His method is based on construction of suitable subsolutions and supersolutions. However, the construction heavily depends on the Dirichlet condition at $x = 0$ and does not apply to the Cauchy problem even for the case $n = 1$.

As previously described, the authors [7] proved the statement of Theorems 1 and 2 assuming that $u_0(x) \leq M$ for sufficiently large M for positive solutions of $u_t = \Delta u + u^p$. Later, Shimojyo [12] had the same results as in [7] by relaxing the assumptions of initial data $u_0 \geq 0$ which is similar to that in the present paper. His approach is a construction of a suitable supersolution which implies that $a \in \mathbb{R}^n$ is not a blow-up point. Although he restricted himself to $f(s) = s^p$, his idea works for our f under slightly stronger assumption on u_0 . Here we give a different approach.

From Shimojyo's results [12], there arises a problem of "blow-up direction" defined in (8). We next study this "blow-up direction" for the value $+\infty$. Our third result is on this blow-up direction. It is convenient to introduce the function A_m defined by

$$(11) \quad A_m(s) = \frac{1}{|B_s(y_m)|} \int_{B_s(y_m)} u_0(z) \, dz$$

for a given sequence $\{y_m\}_{m=1}^{\infty}$. This $A_m(s)$ represents the mean value of u_0 over the ball $B_s(y_m)$.

Theorem 3. *Assume the same hypotheses as in Theorem 1 and let $\{s_m\}_{m=1}^{\infty}$ be a sequence diverging to ∞ in \mathbb{R} . For a given direction $\psi \in S^{n-1}$, the following alternatives hold.*

- (i) *If there exists a sequence $\{y_m\}_{m=1}^{\infty}$ satisfying $\lim_{m \rightarrow \infty} y_m/|y_m| = \psi$ it holds that*

$$\limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m(s) = M,$$

then ψ is a blow-up direction.

- (ii) *If for any sequence $\{y_m\}_{m=1}^{\infty}$ satisfying $\lim_{m \rightarrow \infty} y_m/|y_m| = \psi$ there exists a constant $c \in (1/(M + N), \infty)$ such that*

$$\limsup_{m \rightarrow \infty} \inf_{s \in (1, c)} A_m(s) \leq M - \frac{1}{c},$$

then ψ is not a blow-up direction.

This characterizes blow up directions by profiles of initial data. This is a new result even if $f(u) = |u|^{p-1}u$ or $n = 1$.

Here are the main ideas of the proofs. To prove Theorem 1 we construct a suitable subsolution. To prove Theorem 2 we derive a non blow-up criterion. We do not appeal any energy arguments for rescaled function as is done in our previous paper [7]. Our argument consists of two parts. First we observe that

$$u(x, t) \leq \delta v(t)$$

near a point $a \in \mathbb{R}^n$ with some $\delta \in (0, 1)$ when t is close to the blow-up time. By a bootstrap argument we derive that u is actually bounded near a when t is close to the blow-up time. To prove Theorem 3 we use a comparison argument as in Theorems 1 and 2 and a non blow-up criterion as in the proof of Theorem 2. Moreover, we give conditions on the direction $\psi \in S^{n-1}$ for being the blow-up direction or not cover all of S^{n-1} exclusively.

The detailed proofs will be discussed in paper [8].

REFERENCES

1. Filippas S., Herrero M. A. and Velázquez J. J. L., *Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity*, Proc R. Soc. Lond. Ser. A **456** (2000), 2957–2982.
2. Friedman A. and McLeod B., *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34**(2) (1985), 425–447.
3. Giga Y. and Kohn R. V., *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. **36** (1987), 1–40.
4. ———, *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math. **42**(6) (1989), 845–884.
5. Giga Y., Matsui S. and Sasayama S. *Blow up rate for semilinear heat equations with subcritical nonlinearity*. Indiana Univ. Math. J. **53**(2) (2004), 483–514.
6. ———, *On blow-up rate for sign-changing solutions in a convex domain*, Math. Meth. Appl. Sci. **27** (2004), 1771–1782.
7. Giga Y., Umeda N., *On Blow-up at Space Infinity for Semilinear Heat Equations*, J. Math. Anal. Appl., **316** (2006), 538–555.
8. ———, *Blow-up directions at space infinity for solutions of semilinear heat equations*, Bol. Soc. Paran. Mat., **23** (2005), 9–28.
9. Herrero M. A. and Velázquez J. J. L., *Explosion de solutions d'equations paraboliques semilinaires supercritiques*, C. R. Acad. Sci. Paris, S r. I **319** (1994), 141–145.
10. Lacey A. A., *The form of blow-up for nonlinear parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A **98**(1–2) (1984), 183–202.
11. Matano H. and Merle F., *On nonexistence of type II blowup for a supercritical nonlinear heat equation*, Comm. Pure Appl. Math. **57**(11) (2004), 1494–1541.
12. Shimojyo M., *On blow-up phenomenon at space infinity and its locality for semilinear heat equations*, (In Japanese), Master's Thesis, The University of Tokyo (2005).
13. Weissler F. B., *Single point blow-up for a semilinear initial value problem*, J. Differential Equation **55**, 1984 204–224.

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