AN INTEGRAL UNIVALENT OPERATOR

D. BREAZ AND N. BREAZ

ABSTRACT. In this paper we consider the class of univalent functions defined by the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, z \in U$, where $f(z) = z + a_2 z^2 + \ldots$, is an analytic function in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. We present univalence conditions for the operator

$$G_{\alpha,n}(z) = \left((n(\alpha-1)+1) \int_0^z g_1^{\alpha-1}(t) \ldots g_n^{\alpha-1}(t)\, \mathrm{d}t \right)^{\frac{1}{n(\alpha-1)+1}}.$$

1. INTRODUCTION

Let U be the unit disc, $U = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $H(U)$ the class of holomorphic functions on U and consider the set of analytic functions

$$A_n = \left\{ f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots \right\} \tag{1}$$

If $n = 1$ then $A_1 = A$. Let $H_u(U)$ be the class of univalent functions on the unit disc U and S the class of regular and univalent functions $f(z) = z + a_2 z^2 + \ldots$ in U, which satisfy the condition $f(0) = f'(0) - 1 = 0$.

In their paper [3], Ozaki and Nunokawa proved the following results:

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Theorem 1. *If we assumethat $g \in A$ satisfies the condition*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, \quad z \in U \tag{2}$$

then f is univalent in U.

Lemma 1 (The Schwartz Lemma). *Let the analytic function g be regular in the unit disc U and $g(0) = 0$. If $|g(z)| \leq 1, \forall z \in U$, then*

$$|g(z)| \leq |z|, \qquad \forall z \in U \tag{3}$$

and equality holds only if $g(z) = \varepsilon z$, where $|\varepsilon| = 1$.

Theorem 2. *Let α be a complex number with $\operatorname{Re}\alpha > 0$, and let $f = z + a_2 z^2 + \ \ldots$ be a regular function on U. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{z f''(z)}{f'(z)}\right| \leq 1, \qquad \forall z \in U \tag{4}$$

then for any complex number β with $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ the function

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)\, \mathrm{d}t\right)^{\frac{1}{\beta}} = z + \ldots \tag{5}$$

is regular and univalent in U.

Theorem 3. *Assume that $g \in A$ satisfies condition (2), and let α be a complex number with*

$$|\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{3}. \tag{6}$$

If

(7) $$|g(z)| \leq 1, \qquad \forall z \in U$$

then the function

(8) $$G_\alpha(z) = \left(\alpha \int_0^z g^{(\alpha-1)}(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}}$$

is of class S.

2. Main results

Theorem 4. *Let $g_i \in A$, $\forall i = 1, \ldots, n$, $n \in \mathbb{N}^*$, satisfy the properties*

(9) $$\left|\frac{z^2 g_i'(z)}{g_i^2(z)} - 1\right| < 1, \quad \forall z \in U, \quad \forall i = 1, \ldots, n$$

and $\alpha \in \mathbb{C}$, with

(10) $$|\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{3n}.$$

If $|g_i(z)| \leq 1$, $\forall z \in U$, $\forall i = 1, \ldots, n$, then the function

(11) $$G_{\alpha,n}(z) = \left((n(\alpha-1)+1) \int_0^z g_1^{\alpha-1}(t) \ldots g_n^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}}$$

is univalent.

Proof. From (11) we have:

$$G_{\alpha,n}(z) = \left((n(\alpha-1)+1) \int_0^z t^{n(\alpha-1)} \left(\frac{g_1(t)}{t} \right)^{\alpha-1} \cdot \ldots \cdot \left(\frac{g_n(t)}{t} \right)^{\alpha-1} \mathrm{d}t \right)^{\frac{1}{n(\alpha-1)+1}} \tag{12}$$

We consider the function

$$f(z) = \int_0^z \left(\frac{g_1(t)}{t} \right)^{\alpha-1} \ldots \left(\frac{g_n(t)}{t} \right)^{\alpha-1} \mathrm{d}t. \tag{13}$$

The function f is regular in U, and from (13) we obtain

$$f'(z) = \left(\frac{g_1(z)}{z} \right)^{\alpha-1} \ldots \left(\frac{g_n(z)}{z} \right)^{\alpha-1} \tag{14}$$

and

$$\begin{aligned} f''(z) &= (\alpha-1) \left(\frac{g_1(z)}{z} \right)^{\alpha-2} \frac{zg_1'(z) - g_1(z)}{z^2} \left(\frac{g_2(z)}{z} \right)^{\alpha-1} \ldots \left(\frac{g_n(z)}{z} \right)^{\alpha-1} + \ldots \\ &+ \left(\frac{g_1(z)}{z} \right)^{\alpha-1} \ldots \left(\frac{g_{n-1}(z)}{z} \right)^{\alpha-1} (\alpha-1) \left(\frac{g_n(z)}{z} \right)^{\alpha-2} \frac{zg_n'(z) - g_n(z)}{z^2}. \end{aligned} \tag{15}$$

Next we calculate the expresion $\frac{zf''}{f'}$.

$$\frac{zf''(z)}{f'(z)} = \frac{z\left((\alpha-1)\left(\frac{g_1(z)}{z}\right)^{\alpha-2}\frac{zg_1'(z)-g_1(z)}{z^2}\left(\frac{g_2(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1}\right)}{\left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1}} + \dots$$

$$+ \frac{z\left(\left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_{n-1}(z)}{z}\right)^{\alpha-1}(\alpha-1)\left(\frac{g_n(z)}{z}\right)^{\alpha-2}\frac{zg_n'(z)-g_n(z)}{z^2}\right)}{\left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1}} \tag{16}$$

$$= (\alpha-1)\frac{zg_1'(z)-1}{g_1(z)} + \dots + (\alpha-1)\frac{zg_n'(z)-1}{g_n(z)}.$$

The modulus

$$\left|\frac{zf''}{f'}\right| \tag{17}$$

can then be evaluated as

$$\begin{aligned}\left|\frac{zf''(z)}{f'(z)}\right| &= \left|(\alpha-1)\frac{zg_1'(z)-1}{g_1(z)} + \dots + (\alpha-1)\frac{zg_n'(z)-1}{g_n(z)}\right| \\ &\leq \left|(\alpha-1)\frac{zg_1'(z)-1}{g_1(z)}\right| + \dots + \left|(\alpha-1)\frac{zg_n'(z)-1}{g_n(z)}\right| \\ &= |\alpha-1|\left|\frac{zg_1'(z)-1}{g_1(z)}\right| + \dots + |\alpha-1|\left|\frac{zg_n'(z)-1}{g_n(z)}\right|.\end{aligned} \tag{18}$$

By multiplying the first and the last terms of (18) with $\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} > 0$, we obtain

$$
\begin{aligned}
\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| &\le \frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}|\alpha-1|\left(\left|\frac{zg_1'(z)}{g_1(z)}\right|+1\right)+\dots \\
&\quad+\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}|\alpha-1|\left(\left|\frac{zg_n'(z)}{g_n(z)}\right|+1\right) \\
&\le |\alpha-1|\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left(\left|\frac{z^2g_1'(z)}{g_1^2(z)}\right|\frac{|g_1(z)|}{|z|}+1\right)+\dots \\
&\quad+|\alpha-1|\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left(\left|\frac{z^2g_n'(z)}{g_n^2(z)}\right|\frac{|g_n(z)|}{|z|}+1\right).
\end{aligned}
\tag{19}
$$

By applying the Schwartz Lemma and using (19), we obtain

$$
\begin{aligned}
\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| &\le |\alpha-1|\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left(\left|\frac{z^2g_1'(z)}{g_1^2(z)}-1\right|+2\right)+\dots \\
&\quad+|\alpha-1|\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\alpha}\left(\left|\frac{z^2g_n'(z)}{g_n^2(z)}-1\right|+2\right).
\end{aligned}
\tag{20}
$$

Since g_i satisfies the condition (2) $\forall i=1,\dots,n$, then from (20) we obtain:

$$
\begin{aligned}
\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| &\le 3|\alpha-1|\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}+\dots+3|\alpha-1|\frac{1-|z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \\
&\le \frac{3|\alpha-1|}{\mathrm{Re}\,\alpha}+\dots+\frac{3|\alpha-1|}{\mathrm{Re}\,\alpha}=\frac{3n|\alpha-1|}{\mathrm{Re}\alpha}.
\end{aligned}
\tag{21}
$$

But $|\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{3n}$ so from (7) we obtain that

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{22}$$

for all $z \in U$ and the Theorem 2 implies that the function $G_{\alpha,n}$ is in the class S. □

Corollary 1. *Let $g \in A$ satisfy (2) and α a complex number such that*

$$|\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{3k}, \qquad k \in \mathbb{N}^* \tag{23}$$

If $|g(z)| \leq 1, \forall z \in U$, then the function

$$G_{\alpha}^{k}(z) = \left((k(\alpha-1)+1)\int_0^z g^{k(\alpha-1)}(t)\,\mathrm{d}t\right)^{\frac{1}{k(\alpha-1)+1}} \tag{24}$$

is univalent.

Proof. We consider the functions

$$G_{\alpha}^{k}(z) = \left((k(\alpha-1)+1)\int_0^z t^{k(\alpha-1)}\left(\frac{g(t)}{t}\right)^{k(\alpha-1)}\mathrm{d}t\right)^{\frac{1}{k(\alpha-1)+1}} \tag{25}$$

and

$$f(z) = \int_0^z \left(\frac{g(t)}{t}\right)^{k(\alpha-1)}\mathrm{d}t. \tag{26}$$

The function f is regular in U. From (26) we obtain

$$f'(z) = \left(\frac{g(z)}{z}\right)^{k(\alpha-1)}$$

and

$$f''(z) = k(\alpha-1)\left(\frac{g(z)}{z}\right)^{k(\alpha-1)-1} \frac{zg'(z)-g(z)}{z^2}.$$

Next we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} k|\alpha-1|\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right), \quad \forall z \in U. \tag{27}$$

Applying the Schwartz Lemma and using (27) we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le |\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(\left|\frac{z^2g'(z)}{g^2(z)}-1\right|+2\right). \tag{28}$$

Since g satisfies conditions (2) then from (28) and (23) we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{3k|\alpha-1|}{\operatorname{Re}\alpha}\left(1-|z|^{2\operatorname{Re}\alpha}\right) \le \frac{3k|\alpha-1|}{\operatorname{Re}\alpha} \le 1. \tag{29}$$

Now Theorem 2 and (29) imply that $G_\alpha^k \in S$. □

Corollary 2. *Let $f, g \in A$, satisfy (1) and α the complex number with the property*

$$|\alpha-1| \le \frac{\operatorname{Re}\alpha}{6}. \tag{30}$$

If $|f(z)| \leq 1, \forall z \in U$ *and* $|g(z)| \leq 1, \forall z \in U$*, then the function*

$$G_\alpha(z) = \left((2\alpha - 1) \int_0^z f^{(\alpha-1)}(t)\, g^{(\alpha-1)}(t)\, \mathrm{d}t \right)^{\frac{1}{2\alpha-1}} \tag{31}$$

is univalent.

Proof. In Theorem 1 we set $n = 2$, $g_1 = f$, $g_2 = g$. □

Remark 1. *Theorem 1 is a generalization of Theorem 3.*

Remark 2. *From Corollary 1, for* $k = 1$*, we obtain Theorem 3.*

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D. Breaz, Department of Mathematics, "1 Decembrie 1918" University, Alba Iulia, Romania, *e-mail*: `dbreaz@uab.ro`

N. Breaz, Department of Mathematics, "1 Decembrie 1918" University, Alba Iulia, Romania, *e-mail*: `nbreaz@uab.ro`