Acta Math. Univ. Comenianae
Vol. LXXVI, 2(2007), pp. 143–147

ON PREŠIĆ TYPE GENERALIZATION OF THE BANACH CONTRACTION MAPPING PRINCIPLE

L. B. ĆIRIĆ AND S. B. PREŠIĆ

ABSTRACT. Let (X,d) be a metric space, k a positive integer and T a mapping of X^k into X. In this paper we proved that if T satisfies conditions (2.1) and (2.2) below, then there exists a unique x in X such that $T(x,x,\ldots,x)=x$. This result generalizes the corresponding theorems of the second author [**4**], [**5**] and the theorem of Dhage [**3**].

1. Introduction

The well known Banach contraction mapping principle states that if (X,d) is a complete metric space and $T: X\to X$ is a self mapping such that

$$d(Tx,Ty)\le \lambda d(x,y)$$

for all $x,y\in X$, where $0\le\lambda<1$, then there exists a unique $x\in X$ such that $T(x)=x$. In recent years many generalizations of this principle have appeared ([**1**], [**2**], [**6**]). A special type generalization was introduced by the second author [**4**], [**5**].

Considering the convergence of ceratin sequences Prešić proved the following result.

Theorem 1. *Let (X,d) be a complete metric space, k a positive integer and $T: X^k\to X$ a mapping satisfying the following contractive type condition*

$$\begin{aligned}&d(T(x_1,x_2,x_3,\ldots,x_k),T(x_2,x_3,\ldots,x_k,x_{k+1}))\\ &\quad\le q_1d(x_1,x_2)+q_2d(x_2,x_3)+\ldots+q_kd(x_k,x_{k+1}),\end{aligned}\tag{1.1}$$

for every $x_1,\ldots,x_{k+1}$ in X, where $q_1,q_2,\ldots,q_k$ are non-negative constants such that $q_1+q_2+\ldots+q_k<1$. Then there exists a unique point x in X such that $T(x,x,\ldots,x)=x$. Moreover, if $x_1,x_2,x_3,\ldots,x_k$ are arbitrary points in X and for $n\in\mathbb{N}$,

$$x_{n+k}=T(x_n,x_{n+1},\ldots,x_{n+k-1}),$$

then the sequence $\{x_n\}_{n=1}^\infty$ is convergent and

$$\lim x_n = T(\lim x_n,\lim x_n,\ldots,\lim x_n).$$

Received April 11, 2005.
2000 *Mathematics Subject Classification.* Primary 54H25.
Key words and phrases. Fixed point, Cauchy sequence, Complete metric space.

Remark that condition (1.1) in the case $k = 1$ reduces to the well-known Banach contraction mapping principle. So, Theorem 1 is a generalization of the Banach fixed point theorem.

2. Main theorem

Inspired with the results in Theorem 1 we shall prove the following theorem.

Theorem 2. *Let (X, d) be a complete metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition*

$$d(T(x_1, x_2, \dots, x_k), T(x_2, \dots, x_k, x_{k+1})) \le \lambda \max\{d(x_i, x_{i+1}) : 1 \le i \le k\}, \tag{2.1}$$

where $\lambda \in (0, 1)$ is constant and $x_1, \dots, x_{k+1}$ are arbitrary elements in X. Then there exists a point x in X such that $T(x, \dots, x) = x$. Moreover, if $x_1, x_2, x_3, \dots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$,

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1}),$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n).$$

If in addition we suppose that on diagonal $\Delta \subset X^k$,

$$d(T(u, \dots, u), T(v, \dots, v)) < d(u, v) \tag{2.2}$$

holds for all $u, v \in X$, with $u \ne v$, then x is the unique point in X with $T(x, x, \dots, x) = x$.

Proof. Let $x_1, \dots, x_k$ be k arbitrary points in X. Using these points define a sequence $\{x_n\}$ as follows:

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1}) \qquad (n = 1, 2, \dots).$$

For simplicity set $\alpha_n = d(x_n, x_{n+1})$. We shall prove by induction that for each $n \in \mathbb{N}$:

$$\alpha_n \le K\theta^n \qquad (\text{where } \theta = \lambda^{1/k},\ K = \max\{\alpha_1/\theta, \alpha_2/\theta^2, \dots, \alpha_k/\theta^k\}). \tag{2.3}$$

According to the definition of K we see that (2.3) is true for $n = 1, \dots, k$. Now let the following k inequalities:

$$\alpha_n \le K\theta^n,\ \alpha_{n+1} \le K\theta^{n+1},\ \dots,\ \alpha_{n+k-1} \le K\theta^{n+k-1}$$

be the induction hypotheses. Then we have:

$$\begin{aligned}
\alpha_{n+k} &= d(x_{n+k}, x_{n+k+1}) \\
&= d(T(x_n, x_{n+1}, \dots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \dots, x_{n+k})) \\
&\le \lambda \max\{\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k-1}\} \qquad (\text{by (2.1) and the definition of } \alpha_i) \\
&\le \lambda \max\{K\theta^n, K\theta^{n+1}, \dots, K\theta^{n+k-1}\} \qquad (\text{by the induction hypotheses}) \\
&= \lambda K\theta^n \qquad (\text{as } 0 \le \theta < 1) \\
&= K\theta^{n+k} \qquad (\text{as } \theta = \lambda^{1/k})
\end{aligned}$$

and the inductive proof of (2.3) is complete. Next using (2.3) for any $n,\, p \in \mathbb{N}$ we have the following argument:

$$\begin{aligned} d(x_n, x_{n+p}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\le K\theta^n + K\theta^{n+1} + \ldots + K\theta^{n+p-1} \\ &\le K\theta^n(1 + \theta + \theta^2 + \ldots) \\ &= K\theta^n/(1-\theta) \end{aligned}$$

by which we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is a complete space, there exists x in X such that

$$x = \lim_{n\to\infty} x_n.$$

Then for any integer n we have

$$\begin{aligned} d(x, T(x,\ldots,x) &\le d(x, x_{n+k}) + d(x_{n+k}, T(x.\ldots,x)) \\ &= d(x, x_{n+k}) + d(T(x_n,\ldots,x_{n+k-1}), T(x,\ldots,x)) \\ &\le d(x, x_{n+k}) + d(T(x,\ldots,x,x), T(x,\ldots,x,x_n)) + \\ &\quad d(T(x,\ldots,x,x_n), T(x,\ldots,x_n,x_{n+1})) + \ldots \\ &\quad + d(T(x,x_n,x_{n+1},x_{n+k-2}), T(x_n,x_{n+1},\ldots\cdot x_{n+k-1})) \\ &\le d(x,x_{n+k}) + \lambda d(x,x_n) + \lambda \max\{d(x,x_n), d(x_n,x_{n+1})\} + \ldots \\ &\quad + \lambda \max\{d(x,x_n), d(x_n,x_{n+1}\ ), \ldots, d(x_{n+k-2}, x_{n+k-1})\}. \end{aligned}$$

Taking the limit when n tends to infinity we obtain $d(x, T(x,\ldots,x)) \le 0$, which implies $T(x,\ldots,x) = x$. Thus we proved that

$$\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n).$$

Now suppose that (2.2) holds. To prove the uniqueness of the fixed point, let us assume that for some $y \in X$, $y \ne x$, we have $T(y,\ldots,y) = y$. Then by (2.2), $d(x,y) = d(T(x,\ldots,x), T(y,\ldots,y)) < d(x,y)$, which is a contradiction. So, x is the unique point in X such that $T(x,x,\ldots,x) = x$. □

Remark 1. Theorem 2 is a generalization of Theorem 1, as the condition (1.1) implies the conditions (2.1) and (2.2). Indeed, since

$$\begin{aligned} &q_1 d(x_1,x_2) + q_2 d(x_2,x_3) + \ldots + q_k d(x_k, x_{k+1}) \\ \le\ &(q_1 + q_2 + \ldots + q_k) \max\{d(x_1,x_2), d(x_2,x_3), \ldots, d(x_k,x_{k+1})\} \end{aligned}$$

and $q_1 + q_2 + \ldots + q_k < 1$, we conclude the implication (1.1) ⇒ (2.2). Next, for any $u, v \in X$ with $u \ne v$, from (1.1) we have

$$\begin{aligned} &d(T(u,u,\ldots,u), T(v,v,\ldots,v)) \\ \le\ &d(T(u,\ldots,u), T(u,\ldots,u,v)) + d(T(u,\ldots,u,v), T(u,\ldots,u,v,v)) + \ldots \\ &+ d(T(u,v,\ldots,v), T(v,v,\ldots,v)) \\ \le\ &q_k d(u,v) + q_{k-1} d(u,v) + \ldots + q_1 d(u,v) \\ =\ &(q_k + q_{k-1} + \ldots + q_1) d(u,v) < d(u,v), \end{aligned}$$

and consequently we conclude the implication (1.1) ⇒ (2.2).

The following example shows that the condition (2.2) in Theorem 2 can not be omitted.

Example 1. Let $X = [0,1] \cup [2,3]$ and let $T : X^2 \to X$ be a mapping defined by

$$\begin{aligned} T(x,y) &= \frac{x+y}{4}, && \text{if}(x,y) \in [0,1] \times [0,1], \\ T(x,y) &= 1 + \frac{x+y}{4}, && \text{if}(x,y) \in [2,3] \times [2,3], \\ T(x,y) &= \frac{x+y}{4} - \frac{1}{2}, && \text{if } (x,y) \in [0,1] \times [2,3], \quad \text{or} \quad (x,y) \in [2,3] \times [0,1]. \end{aligned}$$

Then for any $x, y \in [0,1]$ we have $T(x,y) = z \in [0,1]$ and for $x, y \in [2,3]$ we have $T(x,y) = z \in [2,3]$. Thus, for $x, y \in [0,1]$, or $x, y \in [2,3]$, we have

$$\begin{aligned} d(T(x,y), T(y,z)) &= \left|\frac{x+y}{4} - \frac{y+z}{4}\right| = \left|\frac{x-y}{4} + \frac{y-z}{4}\right| \\ &\le \left|\frac{x-y}{4}\right| + \left|\frac{y-z}{4}\right| \le \frac{1}{2} \max\{d(x,y),\ d(y,z)\}. \end{aligned}$$

For $(x,y) \in [0,1] \times [2,3]$, or $(x,y) \in [2,3] \times [0,1]$ we have $T(x,y) = z \in [0,1]$. Therefore, if $y \in [2,3]$, then

$$d(T(x,y), T(y,z)) = \left|\frac{x+y}{4} - \frac{y+z}{4}\right| \le \frac{1}{2} \max\{d(x,y),\ d(y,z)\}.$$

If $y \in [0,1]$, then

$$\begin{aligned} d(T(x,y), T(y,z)) &= \left|\frac{x+y}{4} - \frac{1}{2} - \frac{y+z}{4}\right| = \left|\frac{x-y}{4} - \frac{1}{2} + \frac{y-z}{4}\right| \\ &\le \left|\frac{x-y}{4} - \frac{1}{2}\right| + \left|\frac{y-z}{4}\right| < \left|\frac{x-y}{4}\right| + \left|\frac{y-z}{4}\right| \\ &\le \frac{1}{2} \max\{d(x,y),\ d(y,z)\}. \end{aligned}$$

Thus, T satisfies (2.1) with $\lambda = 1/2$, but for $x = 0$ and $y = 2$ we have $T(0,0) = 0$ and $T(2,2) = 2$.

3. Applications

We shall illustrate an application of Theorem 2 to the convergence problem of real sequences.

Let $\{x_n\}_1^\infty$ be a real sequence, $x_1, \ldots, x_k$ be a given its k members and let x_n, for $n \ge k+1$, be defined by a recursive relation:

$$x_n = \rho(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}).$$

To investigate the convergence of $\{x_n\}_1^\infty$, it suffices to substitute T for ρ in a recursive relation assuming earlier that $T : \mathbb{R}^k \to \mathbb{R}$. If we find that T satisfies (2.1), then the convergence of $\{x_n\}_1^\infty$ will follow from Theorem 2.

References

1. Ćirić Lj. B., *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.
2. Ćirić Lj. B., *A generalization of Caristi's fixed point theorem*, Math. Pannonica **3/2** (1992), 51–57.
3. Dhage B. C., *Generalization of Banach contraction principle,* J. Indian Acad. Math. **9** (1987), 75–86.
4. Prešić S. B., *Sur la convergence des suites*, Comptes Rendus de l'Acad. des Sci. de Paris, **260** (1965), 3828–3830.
5. Prešić S. B., Sur une classe d'inéquations aux différences finite et sur la convergence de certain*es suites*, Publ. de l'Inst. Math. Belgrade, **5**(19) (1965), 75-78.
6. Rhoades B. E., *A comparison of various definitions of contractive mappings*, an application-Trans. Amer. Math. Soc. **226** (1977), 257–290.

L. B. Ćirić, Faculty of Mechanical Engineering, Al. Rudara 12-35, 11 070 Belgrade, Serbia, *e-mail*: `lciric@afrodita.rcub.bg.ac.yu`

S. B. Prešić, Mathematical Faculty, ul. Braće Jugovića 16, 11 000 Belgrade, Serbia