# ASCOLI'S THEOREM
# IN ALMOST QUIET QUASI-UNIFORM SPACE

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ABSTRACT. In this paper we have generalized Ascoli's theorem on almost quiet quasi-uniform space. We have also discussed some properties of the collection of all $\delta$-continuous functions and the collection of all $\delta$-equicontinuous functions.

## 1. Introduction

In [1] it is shown that Doitchinov's concept of quietness is sufficient to extend some classical results regarding uniform spaces to the much broader setting of quasi-uniform spaces. In [2] almost quiet quasi-uniform space has been introduced and it has been shown that a topological space is almost quiet quasi uniformizable if and only if it is almost regular.

In this paper, endeavour has been made to generalize Ascoli's theorem in almost quiet quasi-uniform spaces.

Throughout this paper, for int(cl $(A)$) where $A \subset X$ (where $X$ is a topological space), we shall use the notation $\dot{\overline{A}}$.

A quasi-uniformity on a set $X$ is a filter $\mathcal{U}$ on $X \times X$ such that $(a)$ each member of $\mathcal{U}$ contains the diagonal of $X \times X$ and $(b)$ if $U \in \mathcal{U}$, then $V_\circ V \subset U$ for some $V \in \mathcal{U}$. The pair $(X,\mathcal{U})$ is called a quasi-uniform space. $\mathcal{U}$

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generates a topology $\tau(\mathcal{U})$ containing all subsets $G$ of $X$ such that for each $x \in G$, there exists $U \in \mathcal{U}$ such that $U[x] \subset G$.

**Definition 1.1.** [2] A topological space $(X, \tau)$ is said to be almost quiet quasi-uniformizable iff there exists a compatible quasi-uniformity $\mathcal{U}$ with the following properties: for $U \in \mathcal{U}$ and $x \in X$, there exists $V_x \in \mathcal{U}$ for which the following conditions hold: if $\{x_\alpha : \alpha \in A\}$ & $\{y_\beta : \beta \in B\}$ be two nets such that $(x, x_\alpha) \in V_x$ for $\alpha \in A$, $(y_\beta, y) \in V_x$ (for some $y \in X$), for $\beta \in B$, and $(y_\beta, x_\alpha) \to 0$ ( i.e., for any $V \in \mathcal{U}$, $\exists\ \beta_V$ & $\alpha_V$ belonging to $B$ and $A$ respectively such that $(y_\beta, x_\alpha) \in V$ for $\beta \geq \beta_V$ & $\alpha \geq \alpha_V$), then $y \in \dot{\overline{U[x]}}$, where the closure and the interior of $U[x]$ and $\overline{U[x]}$ respectively are taken under the topology $\tau$; we call $V_x$ subordinated to $U$ with respect to $x$.

**Definition 1.2.** [6] A topological space $(X, \tau)$ is almost regular if for every point $x \in X$ and each neighbourhood $M$ of $x$, there exists an open set $U$ such that $x \in U \subset \overline{U} \subset \dot{\overline{M}}$, where $\overline{M} = \mathrm{cl}\,(M)$ and $\dot{\overline{M}} = \mathrm{int}(\mathrm{cl}\,M)$.

**Definition 1.3.** [5] Let $X$ be a topological space. A subset $S \subset X$ is said to be regular open (respectively, regular closed) if $\mathrm{int}\,(\mathrm{cl}\,S) = S$ (respectively, $\mathrm{cl}\,(\mathrm{int}\,S) = S$). A point $x \in S$ is said to be a $\delta$-cluster point of $S$ if $S \cap U \neq \emptyset$, for every regular open set $U$ containing $x$. The set of all $\delta$-cluster points of $S$ is called the $\delta$-closure of $S$ and is denoted by $[S]_\delta$. If $[S]_\delta = S$, then $S$ is said to be $\delta$-closed. The complement of a $\delta$-closed set is called a $\delta$-open set.

For every topological space $(X, \tau)$, the collection of all $\delta$-open sets forms a topology for $X$, which is weaker than $\tau$. This topology $\tau^*$ has a base consisting of all regular open sets in $(X, \tau)$.

**Definition 1.4.** [5] A function $f : X \to Y$ is said to be $\delta$-continuous at a point $x \in X$, if for every regular open neighbourhood $V$ of $f(x)$ in $Y$, $\exists$ a $\delta$-open neighbourhood $U$ of $x$ such that $f(U) \subseteq V$.

The collection of all $\delta$-continuous functions from $X$ to $Y$ is denoted by $D(X, Y)$.

**Definition 1.5.** [2] Let $\mathcal{F}$ be a family of functions from a topological space $X$ to a quasi-uniform space $(Y, \mathcal{U})$. Then $\mathcal{F}$ is called $\delta$-equicontinuous at $x \in X$, if for $V \in \mathcal{U}$, there exists a regular open neighbourhood $N$ of $x$ such that $f(N) \subset \dot{\overline{V[f(x)]}}$, for every $f \in F$.

**Definition 1.6.** [7] A set $A \subset (X, \tau)$ is said to be N-closed in $X$ or simply N-closed, if for any cover of $A$ by $\tau$-open sets, there exists a finite subcollection the interiors of the closures of which cover $A$; interiors and closures are of course w.r.t. $\tau$.

A set $(X, \tau)$ is said to be nearly compact iff it is N-closed in $X$.

**Definition 1.7.** [3] The N-R topology on $Y^X$ denoted by $N_{\Re}$ is generated by the sets of the form $\{T(C,U) :$ $C$ is N-closed in $X$ and $U$ is regular open in $Y\}$, where $T(C,U) = \{f \in Y^X : f(C) \subseteq U\}$.

**Theorem 1.8.** [3] *Let $Z \subset Y^X$ be endowed with the N-R topology $N_{\Re}$. Then $T(x,U)$ is $\delta$-open in $(Z, N_{\Re})$, where $U$ is regular open in $Y$ and $Y$ is almost regular.*

**Definition 1.9.** [3] Let $Z \subset Y^X$; if $\tau$ is such a topology on $Z$ such that $P : Z \times X \to Y : (f,x) \to f(x)$ is $\delta$-continuous, then we say that $\tau$ is $\delta$-admissible.

For a topological space $X$ and a quasi-uniform space $(Y,\mathcal{U})$, the quasi-uniformity $Q$ of quasi-uniform convergence on $Y^X$ is defined by the collection $\{L_V : V \in \mathcal{U}\}$ where $L_V = \{(f,g) \in Y^X \times Y^X : (f(x), g(x)) \in V,$ for each $x \in X\}$; the topology $\tau(Q)$ generated by $Q$ is called the topology of quasi-uniform convergence. The basic $\tau(Q)$ neighbourhood of an arbitrary $f \in Y^X$ is of the form $L_V[f] = \{g \in Y^X : (f,g) \in L_V\}$.

Another quasi-uniformity on $Y^X$ can be constructed by considering quasi-uniform convergence on each member of a family $\wp$ of subsets of the domain space. Explicitly, if $F$ is a family of functions on a set $X$ to a quasi-uniform space $(Y,\mathcal{U})$ and $\wp$ is a family of subsets of $X$, then the quasi-uniformity of quasi-uniform convergence on members of $\wp$ abbreviated as $\mathcal{U}|_{\wp}$ has for a subbase, the family of all sets of the form $\{(f,g) : (f(x), g(x)) \in V$ for all $x \in A; V \in \mathcal{U}, A \in \wp\}$. We denote it by $L_V^A$.

**Lemma 1.10.** [3] If $\mathcal{F} \subset Y^X$ be endowed with a topology $\wp$ where the subbase for $\wp$ is $\{T(x,U) : x \in X, U$ is regular open in $Y\}$, then each $T(x,U)$ is $\delta$-open in $\wp$ if $Y$ is almost regular.

**Note 1.11** ([2])**.** If $W \in \mathcal{U}$ is a regular open surrounding in a uniform space $(X,\mathcal{U})$ then $W[x]$ is a regular open subset of $X$.

## 2. Main Results

**Proposition 2.1.** [3] *Let $X$ be a topological space and let $(Y,\mathcal{U})$ be an almost quiet quasi-uniform space. If $\mathcal{H}$ is a $\delta$-equicontinuous collection of functions, then its closure $\overline{\mathcal{H}}^{\wp}$ relative to the topology $\wp$ is also $\delta$-equicontinuous.*

**Lemma 2.2.** [4] *Let $H$ be an N-closed subset of an almost quiet quasi-uniform space $(X,\mathcal{U})$. Then for some regular open set $U$ of $X$, $\exists$ a surrounding $D \in \mathcal{U}$ such that $D[H] \subset U$.*

**Theorem 2.3.** [5] *The image of an N-closed set under a $\delta$-continuous map is N-closed.*

**Proposition 2.4.** *Let $X$ be a topological space and let $(Y,\mathcal{U})$ be an almost quiet quasi-uniform space. Then the topology of quasi-uniform convergence on N-closed sets coincides with the N-R topology on $D(X,Y)$.*

*Proof.* Let $\tau$ denotes the topology of quasi-uniform convergence on N-closed sets and $\sigma$ denotes the N-R topology on $D(X,Y)$. Consider $T(K,U) \in \sigma$ and let $f \in T(K,U)$, then $f(K) \subset U$. Since $f(K)$ is N-closed and $U$ is regular open in $(Y,\mathcal{U})$, by Lemma 2.2 there exists a surrounding $V \in \mathcal{U}$ such that $V[f(K)] \subset U$. Choose

$$L_V^K = \{(f,g) : (f(x),g(x)) \in V,\ \forall x \in K\}.$$

Then $L_V^K \in \mathcal{U}|_\infty$ (where $\infty$ is the collection of all N-closed sets in $X$). We show that for any $f \in T(K,U)$, $f \in L_V^K[f] \subset T(K,U)$ showing that $T(K,U) \in \tau$, i.e., $\sigma \subset \tau$: in fact, let $g \in L_V^K[f]$; then $(f,g) \in L_V^K$, i.e., $(f(x),g(x)) \in V$ for all $x$ in $K$ which implies that $g(x) \in V[f(x)]$ for all $x$ in $K$, i.e., $g(K) \subset V[f(K)] \subset U$. Thus $g \in T(K,U)$.

Now, let $S \in \tau$ and let $f \in S$ where $f \in D(X,Y)$. Then there is a $L_V^K \in \mathcal{U}|_\infty$ ($K \in \infty$) such that $f \in L_V^K[f] \subset S$. We show that $\bigcap_{i=1}^{n} T(K_i,U_i)$, for N-closed sets $K_i \subset X$; $i = 1,2,\dots,n$ and regular open sets $U_i$, $i = 1,2,\dots,n$ in $Y$ contains $f$ and is contained in $L_V^K[f]$. Choose a regular open symmetric $W \in \mathcal{U}$ such that $W \circ W \circ W \circ W \subset V$, $K \subset X$ being N-closed, $f(K)$ is N-closed in $Y$ and $\{W[f(x)] : x \in K\}$ is a cover of $f(K)$ and has a finite subcover say,

$$\{W[f(x_i)] : i = 1, 2, \ldots, n\}, \qquad x_i \in K. \tag{1}$$

Obviously, $W[f(x_i)]$ are regular open neighbourhoods of $f(x_i)$, $i = 1, 2, \ldots, n$ (by Note 1.11); $f : X \to Y$ being $\delta$-continuous, $f^{-1}[W[f(x_i)]]$, $i = 1, 2, \ldots, n$ are regular open neighbourhoods of $x_i$ in $X$, $i = 1, 2, \ldots, n$. Choose, $K_i = K \cap f^{-1}[W[f(x_i)]]$. Then $K_i$'s are N-closed in $X$.

Now,

$$W \subset W \circ W \circ W \quad \text{implies} \quad W \circ W \circ W \in \mathcal{U}.$$

Choose, $U_i = (W \circ W \circ W)[f(x_i)]$. We show that, for regular open $W$, $(W \circ W \circ W)[x]$ is regular open. Let $y \in \overline{(W \circ W \circ W)[x]}$ and we show that $\overline{W[x]} \times \overline{W[x]} \subset \overline{W}$. Let $(x, y) \notin \overline{(W \circ W \circ W)}$. Since $\overline{W} \subset \overline{(W \circ W \circ W)}$, $(x, y) \notin \overline{W}$. Then there exists neighbourhoods $U_x$ and $U_y$ of $x$ and $y$ respectively such that

$$(U_x \times U_y) \cap W = \phi.$$

If $t \in U_y$, then $(x, t) \notin W$ implies $t \notin W[x]$, i.e., $U_y \cap W[x] = \phi$, i.e., $y \notin \overline{W[x]}$. Hence, $(x, y) \notin \overline{W[x]} \times \overline{W[x]}$. Thus,

$$\overline{W[x]} \times \overline{W[x]} \subset \overline{W},$$

i.e.,

$$(x, y) \in \operatorname{int}(\overline{W}) = W \subset W \circ W \circ W,$$

i.e.,

$$y \in (W \circ W \circ W)[x].$$

Therefore,

$$\overline{(W \circ W \circ W)[x]} \subset (W \circ W \circ W)[x].$$

Hence, $(W \circ W \circ W)[x]$ is regular open. Thus, $U_i$'s are regular open in $Y$ for $i = 1, 2, \ldots, n$. Let $g \in \bigcap_{i=1}^{n} T(K_i, U_i)$, let $x \in K$. Then $f(x) \in W[f(x_i)]$ for some $i : 1 \leq i \leq n$ by (1), i.e., $x \in f^{-1}[W[f(x_i)]]$, i.e., $x \in K_i$.

Now

$$\begin{aligned} g \in T(K_i, U_i) &\Rightarrow g(K_i) \subset U_i \Rightarrow g(x) \in U_i \\ &\Rightarrow g(x) \in (W \circ W \circ W)[f(x_i)] \\ &\Rightarrow (f(x_i), g(x)) \in W \circ W \circ W. \qquad (2) \end{aligned}$$

Also,

$$f(x) \in W[f(x_i)] \;\Rightarrow\; (f(x_i), f(x)) \in W. \qquad (3)$$

By (2) and (3),

$$(f(x), g(x)) \in W \circ W \circ W \circ W \subset V.$$

Since $x$ is any point of $K$, $(f(x), g(x)) \in V$ for all

$$x \in K \Rightarrow (f, g) \in L_V^K \Rightarrow g \in L_V^K[f] \Rightarrow \bigcap_{i=1}^{n} T(K_i, U_i) \subset L_V^K[f].$$

We now show that $f \in T(K_i, U_i)$ for each $i = 1, 2, \ldots, n$, i.e., $f(K_i) \subset U_i$ for each $i = 1, 2, \ldots, n$.

Now, $f(K_i) \subset W[f(x_i)]$, $i = 1, 2, \ldots, n$ implies

$$f(K_i) \subset (W \circ W \circ W)[f(x_i)] = U_i, \qquad i = 1, 2, \ldots, n.$$

Hence

$$f \in T(K_i, U_i), \qquad \text{for } i = 1, 2, \ldots, n.$$

Thus the proposition is proved. □

**Lemma 2.5.** *Each jointly $\delta$-continuous topology on N-closed sets is larger than the N-R topology.*

*Proof.* Suppose that a topology $\tau$ for $Z \subset Y^X$ is jointly $\delta$-continuous on N-closed sets, $U$ is a regular open subset of $Y$, $K$ is an N-closed subset of $X$ and $P$ is the map such that $P(f,x) = f(x)$. It must be shown that $T(K,U)$ is open to show that $\tau \supset N_{\Re}$.

The set $V = (Z \times K) \cap P^{-1}(U)$ is regular open in $Z \times K$ because $P|_{Z\times K}$ is $\delta$-continuous for any N-closed $K \subset X$. If $f \in T(K,U)$, then

$$f(K) \subset U, \quad \text{i.e.,} \quad \{f\} \times K \subset P^{-1}(U) \quad \text{i.e.,} \quad \{f\} \times K \subset V.$$

Now $\{f\}$ is N-closed in $Z$ and $K$ is so in $X$. Cover $\{f\}\times K$ by basis elements $U\times W$ lying in $V$. The space $\{f\}\times K$ is N-closed, since it is $\delta$-homeomorphic to $K$. Therefore we can choose finitely many $U_i$, $W_i$, $i = 1, 2, \ldots, n$ such that

$$\{f\} \times K \subset \dot{\overline{U_i}} \times \dot{\overline{W_i}}.$$

Then $\operatorname{int}(\operatorname{cl}(U_i))$, $i = 1, 2, \ldots, n$ are open sets. Let $N = \bigcap_{i=1}^{n} \dot{\overline{U_i}}$. Thus $N$ is open and contains $f$. We assert that the sets $\dot{\overline{U_i}} \times \dot{\overline{W_i}}$, which were chosen to cover $\{f\} \times K$ actually cover $N \times K$. Let $(g,y) \in N \times K$. Consider $(f,y) \in \{f\} \times K$. Then $(f,y) \in \dot{\overline{U_i}} \times \dot{\overline{W_i}}$ for some $i$, i.e., $y \in \dot{\overline{W_i}}$. Because $g \in N$, $g \in \dot{\overline{U_i}}$, for each $i = 1, 2, \ldots, n$. Therefore, $(g,y) \in \dot{\overline{U_i}} \times \dot{\overline{W_i}}$. Since all the sets $\dot{\overline{U_i}} \times \dot{\overline{W_i}}$ lie in $V$ and cover $N\times K$, $N\times K \subset V$. Hence there exists a $\tau$-neighbourhood $N$ of $f$ such that $N \times K \subset P^{-1}(U)$. For each $f \in N$, $f(K) \subset U$, i.e., $N \subset T(K,U)$. Thus

$$f \in N \subset T(K,U)$$

gives $T(K,U)$ is open in $\tau$ and hence $\tau \supset N_{\Re}$. □

**Proposition 2.6.** *Let $X$ be a topological space and $(Y,\mathcal{U})$ be an almost quiet quasi-uniform space. If $F$ is a $\delta$-equicontinuous collection of functions, then the N-R topology coincides with the topology $\wp$.*

*Proof.* We consider $P : F \times X \to Y : (f, x) \to f(x)$. We show that if $F$ has the topology $\wp$, then $P$ is $\delta$-continuous. Let $W \in \mathcal{U}$ be regular open. Choose $V \in \mathcal{U}$ such that $V \circ V \subset U$. Consider the set

$$T = \{h : h(x) \in V[f(x)]\}. \tag{4}$$

By Lemma 1.10 $T$ is a neighbourhood of $f$ in $(F, \wp)$. $F$ being $\delta$-equicontinuous, there exists a regular open neighbourhood $U$ of $x$ such that

$$f^*(U) \subset V[f^*(x)] \qquad \text{for all } f^* \in F. \tag{5}$$

Consider the neighbourhood $T \times U$ of $(f, x)$ and let $(g, y) \in T \times U$. Then $g(x) \in V[f(x)]$ by (4) and $g(y) \in V[g(x)]$ by (5). Hence, $(f(x), g(x)) \in V$ and $(g(x), g(y)) \in V$ giving $(f(x), g(y)) \in V \circ V \subset W$, i.e., $g(y) \in W[f(x)]$, i.e., $P(g, y) \in W[f(x)]$, i.e.,

$$P(T \times U) \subset W[f(x)].$$

Hence $P$ is $\delta$-continuous and thus joint $\delta$-continuity of $\wp$ follows. Now each jointly $\delta$-continuous topology is larger than the N-R topology and the N-R topology coincides with the topology of quasi-uniform convergence on N-closed sets since $F \subset D(X, Y)$.

Now we show that $\tau_\wp \subset N_\Re$. For each $x \in X$, $\{x\}$ is N-closed in $X$ and thus

$$\begin{aligned} &\{T(x, U) : x \in X, U \text{ is regular open in } Y\} \\ &\qquad \subset \{T(C, U) : C \text{ is N-closed in } X \text{ and } U \text{ is regular open in } Y\} \end{aligned}$$

and thus $\tau_\wp \subset N_\Re$ in $Z \subset Y^X$. Thus we can conclude that if $F$ is a $\delta$-equicontinuous collection of functions, then the N-R topology coincides with the topology $\wp$. □

## 3. Ascoli's theorem in almost quiet quasi-uniform space

In this section we generalize Ascoli's theorem in almost quiet quasi-uniform space.

**Theorem 3.1.** *Let $X$ be a nearly compact topological space and $(Y,\mathcal{U})$ be an almost quiet quasi-uniform $T_2$ space. Let $\tau_N$ denote the topology of quasi-uniform convergence on N-closed sets. Then a subset $H \subset D(X,Y)$ is $\tau_N$-compact iff*

(a) *$H$ is $\tau_N$-closed.*

(b) *$\overline{\Pi_x(H)}$ is compact for each $x \in X$ and*

(c) *$H$ is $\delta$-equicontinuous.*

*Proof.* Since $H$ is $\delta$-equicontinuous by Proposition 2.1, its $\tau_\wp$ closure $\overline{H}^\wp$ is also $\delta$-equicontinuous. But $\overline{H}^\wp$ is a $\tau_\wp$-closed subset of the $\tau_\wp$-compact product set $\Pi\{\overline{\Pi_x(H)} : x \in X\}$ and thus $\overline{H}^\wp$ is itself $\tau_\wp$-compact. Using Proposition 2.4 and Proposition 2.6 above we conclude that $\overline{H}^\wp$ is $\tau_N$-compact. Now, the $\tau_N$-closed subset $H$ of the $\tau_N$-compact subset $\overline{H}^\wp$ is also $\tau_N$-compact. Hence $H$ is $\tau_N$-compact.

Conversely, let $H \subset D(X,Y)$ be $\tau_N$-compact. Since $Y$ is $T_2$, we first show that $(D(X,Y), \tau_N)$ is also so. Let $f, g \in D(X,Y)$ be such that $f \neq g$. Then $\exists\ x \in X$ such that $f(x) \neq g(x)$. Since $Y$ is $T_2$, there exists disjoint open neighbourhoods $U$ and $V$ such that $f(x) \in U$, $g(x) \in V$. Hence, $f(x) \in U = \operatorname{int} U \subset \operatorname{int}(\overline{U}) = \dot{\overline{U}}$. Now,

$$U \cap V = \phi \Rightarrow \overline{U} \cap V = \phi \Rightarrow \dot{\overline{U}} \cap V = \phi \Rightarrow V \subseteq Y \setminus \dot{\overline{U}},$$

i.e.,

$$V = \operatorname{int} V \subseteq \operatorname{int}(Y \setminus \dot{\overline{U}}) = M.$$

Then $M$ is regular open and $g(x) \in M$ with $\dot{\overline{U}} \cap M = \phi$. Now, $\{x\}$ is N-closed in $X$ and $f \in L^{\{x\}}_{\dot{\overline{U}}}$, $g \in L^{\{x\}}_M$ with $L^{\{x\}}_{\dot{\overline{U}}} \cap L^{\{x\}}_M = \phi$. Hence $(D(X,Y), \tau_N)$ is $T_2$. If $H$ is $\tau_N$-compact, then $H$ is $\tau_N$-closed and $\Pi_x(H)$ is compact for each $x \in X$ and hence closed in $Y$. Thus $\overline{\Pi_x(H)}$ is compact in $Y$ for each $x \in X$.

Now if $Z \subset Y^X$ and $P \subset X$, then $Z|_P = \{f|_P : f \in Z\}$. Let $\mathcal{C}$ denote the collection of all N-closed sets in $X$ and let $P \in \mathcal{C}$. We show that $H|_P$ is $\delta$-equicontinuous on $P$. Let $x_0 \in P$ and $W \in \mathcal{U}$ be regular open. Choose regular open symmetric $V \in \mathcal{U}$ such that $V \circ V \circ V \subset W$. Then $\{L_V^P[f] : f \in H\}$ is a cover of $H$ by neighbourhoods of members of $H$ in the topology of quasi-uniform convergence on N-closed sets and by the given condition of $\tau_N$-compactness, there exists $f_i$, $i = 1, 2, \ldots, n$ (belonging to $H$) such that $H \subset \bigcup_{i=1}^{n} L_V^P[f_i]$. Let $f \in H$. Then

$$f \in L_V^P[f_i] \qquad \text{for some } i. \tag{6}$$

Since each $f_i|_P$ is $\delta$-continuous at $x_0$, there is a regular open neighbourhood $U_i$ of $x_0$ in $P$ such that $f_i|_P(U_i) \subset V[f_i(x_0)]$, i.e.

$$x \in U_i \Rightarrow (f_i(x_0), f_i(x)) \in V. \tag{7}$$

Let $U = \bigcap_{i=1}^{n} U_i$. Obviously $U$ is a regular open neighbourhood of $x_0$ in $P$. We show that $f|_P(U) \subset W(f(x_0))$ for all $f \in H$. Let $f \in H$. By (6), $f \in L_V^P[f_i]$ for some $i$, i.e., $(f_i(x), f(x)) \in V$, for all $x \in P$ and hence

$$(f_i(x_0), f(x_0)) \in V \quad [\text{since } x_0 \in P]. \tag{8}$$

Again

$$x \in U \Rightarrow x \in P \Rightarrow (f_i(x), f(x)) \in V. \tag{9}$$

From (7), (8) and (9) we get, $x \in U \Rightarrow (f(x_0), f(x)) \in V \circ V \circ V \subset W$ for each $f \in H \Rightarrow f(x) \in W[f(x_0)]$ for all $f \in H$, i.e.,

$$f(U) \subset W[f(x_0)] \qquad \text{for all } f \in H,$$

i.e.,

$$f|_P(U) \subset W[f(x_0)] \qquad \text{for each } f \in H.$$

Since $X$ is nearly compact, $X$ is N-closed in $X$. Hence $f(U) \subset W[f(x_0)]$ for all $f \in H$, i.e., $H$ is $\delta$-equicontinuous. □

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