# DIAMETER IN WALK GRAPHS

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Abstract. A walk $W$ of length $k$ is admissible if every two consecutive edges of $W$ are distinct. If $G$ is a graph, then its walk graph $W_k(G)$ has vertex set identical with the set of admissible walks of length $k$ in $G$. Two vertices are adjacent in $W_k(G)$ if and only if one of the corresponding walks can be obtained from the other by deleting an edge from one end and adding an edge to the other end. We show that if the degree of every vertex in $G$ is more than one, then $W_k(G)$ is connected and we find bounds for the diameter of $W_k(G)$.

## 1. Introduction and results

All graphs considered in this paper are finite, connected, without loops and multiple edges. By $\delta(G)$ we denote the minimum degree of $G$ and by $d_G(u,v)$ we denote the distance between two vertices, $u$ and $v$, in $G$. Let $P_k$ be the set of paths of length $k$ in $G$; and let $W_k$ be the set of walks of length $k$ in $G$ in which no two consecutive edges are equal. The vertex set of the path graph $P_k(G)$ (of the walk graph $W_k(G)$) is the set $P_k$ ($W_k$). Two vertices of $P_k(G)$ ($W_k(G)$) are joined by an edge if and only if one can be obtained from the other by "shifting" the corresponding paths (walks) in $G$.

Path graphs were investigated by Broersma and Hoede [2] as a natural generalization of line graphs (observe that $P_1(G)$ is the line graph of $G$, i.e., $P_1(G) = L(G)$). Walk graphs were investigated by Knor and Niepel [3] as a generalization of iterated line graphs. We have $P_1(G) = W_1(G)$, $P_2(G) = W_2(G)$ and for $k \geq 3$ the graph $P_k(G)$ is an induced subgraph of $W_k(G)$.

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Using analogous methods as Belan and Jurica for path graphs in [3], it is easy to find the lower bound for the diameter of walk graphs:

$$\operatorname{diam}(W_k(G)) \geq \operatorname{diam}(G) - k.$$

Since $P_k(G) = W_k(G)$ if the graph $G$ is a tree, analogously as for path graphs in [3], for arbitrary component $H$ of walk graph $W_k(G)$ it can be proved

$$\operatorname{diam}(H) \leq \operatorname{diam}(G) + k(k-2),$$

providing that $\operatorname{diam}(G) \geq k$.

In this paper we improve these results for graphs which do not contain vertices of degree one.

**Theorem 1.** *Let $G$ be a graph with diameter $d \geq 1$ and $\delta(G) > 1$.*

A. *If $d \geq k-1$, then* $\operatorname{diam}(W_k(G)) \leq d + k$.
B. *If $d \leq k-2$, then* $\operatorname{diam}(W_k(G)) \leq 2k - 2$.

**Corollary 2.** *Let $G$ be a connected graph with $\delta(G) > 1$. Then $W_k(G)$ is connected.*

We remark that an analogy of Corollary 2 is not true for path graphs. By [4], we know that there exists a graph $G$ with $\delta(G) = k-1$, such that $P_k(G)$ is disconnected.

**Assertion 3.** *For every $d$, for which $2 \leq d$ and $k-1 \leq d$, there exists a graph $G$ with diameter $d$ and $\delta(G) > 1$, such that* $\operatorname{diam}(W_k(G)) = d + k$.

**Problem 4.** *Let $G$ be a graph with $\delta(G) > 1$ and let* $\operatorname{diam}(G) < k$. *Are the bounds for* $\operatorname{diam}(W_k(G))$ *of part B of Theorem 1 best possible?*

**Theorem 5.** *Let $G$ be a graph with $\delta(G) > 1$.*

I. *If $k$ is even, then* $\operatorname{diam}(W_k(G)) \geq \operatorname{diam}(G)$.
II. *If $k$ is odd, then* $\operatorname{diam}(W_k(G)) \geq \operatorname{diam}(G) - 1$.

Observe that if $G$ is a cycle, then $W_k(G)$ is isomorphic to $G$. Hence the lower bound for $\operatorname{diam}(W_k(G))$ is the best possible if $k$ is even.

**Assertion 6.** *For every odd number $k$ and $d \geq 2k$, there exists a graph $G$ with diameter $d$ and $\delta(G) > 1$, such that* $\operatorname{diam}(W_k(G)) = d - 1$.

It is easy to show that if $k$ is odd and $\operatorname{diam}(G) = 2$, then $W_k(G)$ can not be complete and hence $\operatorname{diam}(W_k(G)) \geq 2$.

We do not know if the lower bound of part II of Theorem 5 is best possible for $3 \leq d < 2k$. The value for the lower bound is equal either to $d - 1$ or $d$.

**Problem 7.** *Let $G$ be a graph with $\delta(G) > 1$ and let* $\operatorname{diam}(G) < 2k$*. Are the bounds for* $\operatorname{diam}(W_k(G))$ *of part II of Theorem 5 best possible?*

## 2. Proofs

We remark that throughout the paper we use $k$ only for the length of walks for walk graph $W_k(G)$. We denote the vertices of $W_k(G)$ by small letters $a, b, \ldots,$ while the corresponding walks of length $k$ in $G$ we denote by capital letters $A, B, \ldots$ It means that if $A$ is a walk of length $k$ in $G$ and $a$ is a vertex in $W_k(G)$, then $a$ is necessarily the vertex corresponding to the walk $A$.

Let $A$ be a walk of length $k$ in $G$. By $A(i)$, $0 \leq i \leq k$, we denote the $i$-th vertex of $A$. If $A$ and $B$ are walks in $G$ such that $A = B$, then either $A(i) = B(k - i)$, $0 \leq i \leq k$; or $A(i) = B(i)$, $0 \leq i \leq k$.

**Lemma 8.** *Let $A_0$ and $A_0'$ be two admissible walks in $G$. If $A_0(0) = A_0'(0)$ and $A_0(1) \neq A_0'(1)$, then $d_{W_k(G)}(a_0, a_0') \leq k$.*

*Proof.* We define $k$ walks $A_1, A_2, \ldots, A_k$ by "shifting forwards". Let $A_i(0) = A'_0(i)$ and $A_i(1) = A_{i-1}(0)$, $A_i(2) = A_{i-1}(1), \ldots, A_i(k) = A_{i-1}(k-1)$, where $i = 1, 2, \ldots, k$. We have

$$A_1(0) = A'_0(1) \neq A_0(1) = A_1(2) \qquad \text{and}$$
$$A_i(0) = A'_0(i) \neq A'_0(i-2) = A_{i-2}(0) = A_i(2), \qquad i = 2, 3, \ldots, k.$$

Hence $A_i$ are admissible walks and $a_{i-1}$ and $a_i$ are adjacent in $W_k(G)$. Therefore $d_{W_k(G)}(a_0, a_k) \leq k$. Since

$$\begin{aligned} A_k(0) &= A'_0(k), \\ A_k(1) &= A_{k-1}(0) = A'_0(k-1), \\ &\vdots \\ A_k(k) &= A_{k-1}(k-1) = \ldots = A_{k-k}(0) = A'_0(0), \end{aligned}$$

we have $A_k = A'_0$ and hence $d_{W_k(G)}(a_0, a'_0) \leq k$. □

**Lemma 9.** *Let $G$ be a graph with $\delta(G) > 1$ and let $A_0$ and $A'_0$ be two admissible walks in $G$. Let $A_0(p) = A'_0(r)$ and $A_0(p+1) = A'_0(r+1)$. Then*

$$d_{W_k(G)}(a_0, a'_0) \leq 2k - 2.$$

*Proof.* Let $A_0(p) = A'_0(r)$, $A_0(p+1) = A'_0(r+1)$ and let $p \leq r$. (The case $r < p$ can be solved analogously.) We define the walks $A'_1, A'_2, \ldots, A'_{k-r-1}$ by "shifting forwards". Let $A'_i(0)$ be an arbitrary vertex adjacent to $A'_{i-1}(0)$ distinct from $A'_{i-1}(1)$; and

$$A'_i(1) = A'_{i-1}(0), \quad A'_i(2) = A'_{i-1}(1), \quad \ldots, \quad A'_i(k) = A'_{i-1}(k-1),$$

where $i = 1, 2, \ldots, k-r-1$. Since $A'_i(0) \neq A'_{i-1}(1) = A'_i(2)$, $i = 1, 2, \ldots, k-r-1$, the walks $A'_i$ are admissible and the vertices $a'_{i-1}$ and $a'_i$, $i = 1, 2, \ldots, k-r-1$, are adjacent in $W_k(G)$. Therefore

$$d_{W_k(G)}(a'_0, a'_{k-r-1}) \leq k - r - 1.$$

In a similar way we define $p$ walks $A_1, A_2, \dots, A_p$ by "shifting backwards". Let

$$A_i(0) = A_{i-1}(1), \quad A_i(1) = A_{i-1}(2), \quad \dots, \quad A_i(k-1) = A_{i-1}(k)$$

and let $A_i(k)$ be an arbitrary vertex adjacent to $A_i(k-1)$ distinct from $A_i(k-2)$, where $i = 1, 2, \dots, p$. The walks $A_i$ are admissible and the vertices $a_{i-1}$ and $a_i$, $i = 1, 2, \dots, p$, are adjacent in $W_k(G)$. Therefore

$$d_{W_k(G)}(a_0, a_p) \le p.$$

We have

$$\begin{aligned} A_p(0) &= A'_{k-r-1}(k-1) && (\text{since } A_p(0) = A_{p-1}(1) = \dots = A_0(p), \\ & && A'_{k-r-1}(k-1) = A'_{k-r-2}(k-2) = \dots = A'_0(r)) \\ A_p(1) &= A'_{k-r-1}(k) && (\text{as } A_p(1) = A_{p-1}(2) = \dots = A_0(p+1), \\ & && A'_{k-r-1}(k) = A'_{k-r-2}(k-1) = \dots = A'_0(r+1)). \end{aligned}$$

Now we define the walks $A_{p+1}, A_{p+2}, \dots, A_{p+k-1}$ as follows. Let

$$\begin{aligned} &A_{p+i}(0) = A'_{k-r-1}(k-1-i) \qquad \text{and} \\ &A_{p+i}(1) = A_{p+i-1}(0), \quad A_{p+i}(2) = A_{p+i-1}(1), \quad \dots, \quad A_{p+i}(k) = A_{p+i-1}(k-1), \end{aligned}$$

$i = 1, 2, \dots, k-1$. The walks $A_i$ are admissible and thevertices $a_{i-1}$ and $a_i$, $i = 1, 2, \dots, k-1$, are adjacent in $W_k(G)$. Therefore

$$d_{W_k(G)}(a_p, a_{p+k-1}) \le k-1.$$

We have

$$\begin{aligned} &A_{p+k-1}(0) = A'_{k-r-1}(0), \quad A_{p+k-1}(1) = A_{p+k-2}(0) = A'_{k-r-1}(1), \quad \dots, \\ &A_{p+k-1}(k) = A_{p+k-2}(k-1) = \dots = A_p(1) = A'_{k-r-1}(k). \end{aligned}$$

Hence $a_{p+k-1} = a'_{k-r-1}$ and the distance

$$\begin{aligned} d_{W_k(G)}(a_0, a'_0) &\le d_{W_k(G)}(a_0, a_p) + d_{W_k(G)}(a_p, a_{p+k-1}) + d_{W_k(G)}(a'_{k-r-1}, a'_0) \\ &\le p + (k-1) + (k-r-1) = 2k - 2 + p - r. \end{aligned}$$

Since $p \le r$,

$$d_{W_k(G)}(a_0, a'_0) \le 2k - 2.$$

□

*Proof of Theorem 1.* Let $\operatorname{diam}(W_k(G)) = d_{W_k(G)}(a_0, a'_0)$. Since the diameter of $G$ is equal to $d$, we have $d_G(A_0(0), A'_0(0)) = d' \le d$.

Observe that if $d \ge k-1$, then $2k-2 < d+k$; and if $d \le k-2$, then $d+k \le 2k-2$.

If some edge from $A_0$ is equal to some edge from $A'_0$, then $d_{W_k(G)}(a_0, a'_0) \le 2k-2$, by Lemma 9.

Assume that $A_0$ and $A'_0$ are edge disjoint. Let $V = (v_0, v_1, \ldots, v_{d'})$ be a path of length $d'$ in $G$ such that $v_0 = A_0(0)$ and $v_{d'} = A'_0(0)$. If the length of $V$ is zero, we have $A_0(0) = A'_0(0)$ and $A_0(1) \ne A'_0(1)$. Then $d_{W_k(G)}(a_0, a'_0) \le k$, by Lemma 8. Let $V$ be a path of length at least one. We distinguish two cases:

**I.** Suppose that $v_1 \ne A_0(1)$ and $v_{d'-1} \ne A'_0(1)$. We define $d'$ walks $A_1, A_2, \ldots, A_{d'}$ by "shifting forwards". Let

$$A_i(0) = v_i, \quad A_i(1) = A_{i-1}(0), \quad \ldots, \quad A_i(k) = A_{i-1}(k-1),$$

where $i = 1, 2, \ldots, d'$. Since

$$A_1(0) = v_1 \ne A_0(1) = A_1(2) \qquad \text{and} \qquad A_i(0) = v_i \ne v_{i-2} = A_{i-1}(1) = A_i(2),$$

for $i = 2, 3, \ldots, d'$; the walks $A_i$ are admissible and the vertices $a_{i-1}$ and $a_i$, $i = 1, 2, \ldots, d'$, are adjacent in $W_k(G)$. Hence $d_{W_k(G)}(a_0, a_{d'}) \le d'$. Now we have $A_{d'}(0) = v_{d'} = A'_0(0)$ and $A_{d'}(1) = A_{d'-1}(0) = v_{d'-1} \ne A'_0(1)$. By Lemma 8, $d_{W_k(G)}(a_r, a'_0) \le k$, therefore

$$d_{W_k(G)}(a_0, a'_0) \le k + d' \le k + d.$$

**II.** Suppose that at least one of the edges $A_0(0)A_0(1)$, $A'_0(0)A'_0(1)$ belongs to $V$.

a) Let $A_0 \not\subset V$ and $A'_0 \not\subset V$. Let $t$ ($s$) be the smallest positive integer such that $v_{t+1} \neq A_0(t+1)$ ($v_{d'-(s+1)} \neq A'_0(s+1)$). It means that

$$A_0 = (v_0 = A_0(0), \ldots, v_t = A_0(t), A_0(t+1), \ldots, A_0(k)) \qquad \text{and}$$
$$A'_0 = (A'_0(k), \ldots, A'_0(s+1), v_{d'-s} = A'_0(s), \ldots, v_{d'} = A'_0(0)).$$

Since the walks $A_0$ and $A'_0$ are edge disjoint, it is evident that $t \leq d' - s$.

We define $t$ walks $A_1, A_2, \ldots, A_t$ by "shifting backwards" as follows. Let

$$A_i(0) = A_{i-1}(1) = v_i, \quad A_i(1) = A_{i-1}(2), \quad \ldots, \quad A_i(k-1) = A_{i-1}(k)$$

and let $A_i(k)$ be an arbitrary vertex adjacent to $A_i(k-1)$ distinct from $A_i(k-2)$, where $i = 1, 2, \ldots, t$.

Analogously we define $s$ walks $A'_1, A'_2, \ldots, A'_s$. Let

$$A'_i(0) = A'_{i-1}(1) = v_{d'-i}, \quad A'_i(1) = A'_{i-1}(2), \quad \ldots, \quad A'_i(k-1) = A'_{i-1}(k)$$

and let $A'_i(k)$ be an arbitrary vertex adjacent to $A'_i(k-1)$ distinct from $A'_i(k-2)$, where $i = 1, 2, \ldots, s$.

The walks $A_i$ ($A'_i$) are admissible and the vertices $a_{i-1}$ and $a_i$, $i = 1, 2, \ldots, t$ ($a'_{i-1}$ and $a'_i$, $i = 1, 2, \ldots, s$), are adjacent in $W_k(G)$. Therefore

$$d_{W_k(G)}(a_0, a_t) \leq t \qquad \text{and} \qquad d_{W_k(G)}(a'_0, a'_s) \leq s.$$

We have

$$A_t(0) = v_t, \qquad A'_s(0) = v_{d'-s} \qquad \text{and}$$
$$A_t(1) = A_{t-1}(2) = \ldots = A_0(t+1), \qquad A'_s(1) = A'_{s-1}(2) = \ldots = A'_0(s+1).$$

Assume that $t < d'-s$. We have $A_t(1) = A_0(t+1) \neq v_{t+1}$, $A'_s(1) = A'_0(s+1) \neq v_{d'-(s+1)}$ and $d(A_t(0), A'_s(0)) = d' - s - t$. In the same way as in part I of this proof it can be shown that $d_{W_k(G)}(a_t, a'_s) \leq d' - s - t + k$. Then

$$\begin{aligned} d_{W_k(G)}(a_0, a'_0) &\leq d_{W_k(G)}(a_0, a_t) + d_{W_k(G)}(a_t, a'_s) + d_{W_k(G)}(a'_s, a'_0) \\ &\leq t + (d' - s - t + k) + s = d' + k \leq d + k. \end{aligned}$$

Suppose that $t = d' - s$. Then $A_t(0) = A'_s(0)$. Since $A_0$ and $A'_0$ are edge-disjoint, $A_t(1) = A_0(t+1) \neq A'_0(s+1) = A'_s(1)$, and by Lemma 8, $d_{W_k(G)}(a_t, a'_s) \leq k$. Hence

$$d_{W_k(G)}(a_0, a'_0) \leq t + k + s = d' + k \leq d + k.$$

b) Let $A_0 \subset V$. (The case $A'_0 \subset V$ can be solved in a similar manner.) Suppose that $B_0 = A_0$, where $B_0(i) = A_0(k - i)$, $i = 0, 1, \ldots, k$. Then instead of $A_0$ we consider $B_0$, instead of $V$ consider $V' = (v'_0 = v_k = B_0(0), v'_1 = v_{k+1}, \ldots, v'_{d'-k} = v_{d'} = A'_0(0))$ and proceed analogously as above. □

*Proof of Assertion 3.* Led $2 \leq d$ and $k - 1 \leq d$. Assume that $V$, $A$ and $A'$ are three vertex-disjoint paths, where $V = (v_1, v_2, \ldots, v_{d-1})$, $A = (A(0), A(1), \ldots, A(k))$ and $A' = (A'(0), A'(1), \ldots, A'(k))$. Denote by $G$ a graph consisting of $V$, $A$, $A'$ and edges $A(i)v_1$, $A'(i)v_{d-1}$, $i = 0, 1, \ldots, k$. Then the diameter

$$\operatorname{diam}(G) = d(d_G(A(0), A'(0)) = d) \qquad \text{and} \qquad d_{W_k(G)}(a, a') = d + k.$$

By part A of Theorem 1, $\operatorname{diam}(W_k(G)) \leq \operatorname{diam}(G) + k$, therefore

$$\operatorname{diam}(W_k(G)) = d + k.$$

□

*Proof of Theorem 5.* Let $\operatorname{diam}(G) = d_G(v_0, v_d) = d$. Let $V = (v_0, v_1, \ldots, v_d)$ be a path of length $d$ in $G$.

**I.** Suppose $k$ is even. Denote by $A_0$ ($A'$) a walk of length $k$ in $G$, where $A_0\left(\frac{k}{2}\right) = v_0$ ($A'\left(\frac{k}{2}\right) = v_d$). We show that $d_{W_k(G)}(a_0, a') \geq d$. Let us prove it by contradiction and assume that $d_{W_k(G)}(a_0, a') = s \leq d - 1$. Let

$(a_0, a_1, \ldots, a_s = a')$ be a path of length $s$ in $W_k(G)$. The vertices $a_{i-1}, a_i$ are adjacent in $W_k(G)$, therefore $A_i\left(\frac{k}{2}\right) = A_{i-1}\left(\frac{k}{2}+1\right)$ or $A_i\left(\frac{k}{2}\right) = A_{i-1}\left(\frac{k}{2}-1\right)$, $i = 1, 2, \ldots, s$. Hence

$$d_G\left(A_i\left(\frac{k}{2}\right), A_{i-1}\left(\frac{k}{2}\right)\right) = 1, \qquad i = 1, 2, \ldots, s, \qquad \text{and}$$
$$d_G(v_0, v_d) = d_G\left(A_0\left(\frac{k}{2}\right), A'\left(\frac{k}{2}\right)\right) \le s,$$

a contradiction.

**II.** Suppose $k$ is odd. Let $A_0$ be a walk of length $k$ with the central edge $v_0v_1$, where $v_0 = A_0\left(\frac{k-1}{2}\right)$, $v_1 = A_0\left(\frac{k+1}{2}\right)$; and let $A'$ be a walk of length $k$ with the central edge $v_{d-1}v_d = A'\left(\frac{k-1}{2}\right) A'\left(\frac{k+1}{2}\right)$ in $G$. We show that $d_{W_k(G)}(a_0, a') \ge d-1$. Assume the contrary and let $d_{W_k(G)}(a_0, a') = t \le d-2$. Let $(a_0, a_1, \ldots, a_t = a')$ be a path of length $t$ in $W_k(G)$. Since $a_{i-1}$ and $a_i$, $i = 1, 2, \ldots, t$, are adjacent vertices in $W_k(G)$, the central edges of $A_{i-1}$ and $A_i$ have to be adjacent in $G$. Then it is easy to see that

$$\max\left\{d_G\left(A_0\left(\frac{k-1}{2}\right), A_i\left(\frac{k-1}{2}\right)\right), d_G\left(A_0\left(\frac{k-1}{2}\right), A_i\left(\frac{k+1}{2}\right)\right)\right\} \le i+1,$$

$i = 1, 2, \ldots, t$, a contradiction. □

*Proof of Assertion 6.* Let $k$ be odd and $d \ge 2k$. Assume that $C$ and $C'$ are two edge-disjoint cycles;

$$C = (c_0, c_1, \ldots, c_{2k} = c_0) \qquad \text{and} \qquad C' = (c'_0, c'_1, \ldots, c'_{2(d-k)} = c'_0).$$

Let $c_k = c'_{d-k}$ and let $G$ be the graph consisting of $C$ and $C'$. Then

$$V = (c_0, c_1, \ldots, c_k = c'_{d-k}, c'_{d-k-1}, \ldots, c'_0)$$

is the diameter path of $G$ and the diameter is $\mathrm{diam}(G) = k + (d - k) = d$. In the following, we denote by

$$B_1 = (c_0, c_1, \dots, c_k), \qquad B_2 = (c_k, c_{k+1}, \dots, c_{2k} = c_0),$$
$$B_3 = (c'_{d-k}, c'_{d-k-1}, \dots, c'_{d-2k}), \qquad B_4 = (c'_{d-k}, c'_{d-k+1}, \dots, c'_d)$$

the paths of length $k$ in $G$. Note that $d_{W_k(G)}(b_i, b_j) = k$, where $i, j \in \{1, 2, 3, 4\}$, $i < j$.

Suppose that $A$, $A'$ are any two walks of length $k$ in $G$. We prove that

$$d_{W_k(G)}(a, a') \leq d - 1.$$

We partition the walks of length $k$ into three sets $S_1$, $S_2$ and $S_3$, where $S_1$ is the set of those, being part of $C$; $S_2$ those in $C'$; and $S_3$ the remaining ones. There are six cases to distinguish.

**I.** Let $A, A' \in S_1$. Since $A$ and $A'$ are the walks in the cycle $C$ of length $2k$, it is easy to see that

$$d_{W_k(G)}(a, a') \leq k$$

($d_{W_k(G)}(a, a') = k$ if and only if $A$ and $A'$ are edge-disjoint).

**II.** Let $A, A' \in S_2$. By analogy, since $C_2$ is the cycle of length $2(d-k)$ and $A, A' \subset C_2$, we have

$$d_{W_k(G)}(a, a') \leq d - k.$$

**III.** Let $A, A' \in S_3$. Then $A \cap B_i$ and $A' \cap B_j$ are paths of length at least $\frac{k+1}{2}$ for some $i, j \in \{1, 2, 3, 4\}$, in $G$. Consequently, $d_{W_k(G)}(a, b_i) \leq \frac{k-1}{2}$ and $d_{W_k(G)}(a', b_j) \leq \frac{k-1}{2}$. We know that $d_{W_k(G)}(b_i, b_j) = k$ if $i \neq j$, hence

$$d_{W_k(G)}(a, a') \leq 2k - 1 \leq d - 1.$$

**IV.** Let $A \in S_1$ and $A' \in S_2$. Since $A \cap B_1$ or $A \cap B_2$ is a path of length at least $\frac{k+1}{2}$ in $G$, we have $d_{W_k(G)}(a, b_i) \leq \frac{k-1}{2}$, where $i = 1$ or $2$. Let

$$B'_3 = (c'_{d-k}, c'_{d-k-1}, \dots, c'_0), \qquad B'_4 = (c'_{d-k}, c'_{d-k+1}, \dots, c'_{2(d-k)})$$

be the paths of length $d-k$ in $G$. Suppose that $A' \cap B_3'$ is a path of length at least $\frac{k+1}{2}$ in $G$. Then there exists a path $A_s' \subset B_3'$ of length $k$ in $G$, such that $d_{W_k(G)}(a', a_s') \le \frac{k-1}{2}$. Since $B_3'$ is the path of length $d-k$ and $A_s'$, $B_3$ are the subpaths of $B_3'$ of length $k$, $d_{W_k(G)}(a_s', b_3) \le d-2k$. Therefore

$$d_{W_k(G)}(a', b_3) \le d_{W_k(G)}(a', a_s') + d_{W_k(G)}(a_s', b_3) \le \frac{k-1}{2} + d - 2k$$

and hence

$$\begin{aligned} d_{W_k(G)}(a, a') &\le d_{W_k(G)}(a, b_i) + d_{W_k(G)}(b_i, b_3) + d_{W_k(G)}(b_3, a') \\ &\le \frac{k-1}{2} + k + \left(\frac{k-1}{2} + d - 2k\right) = d - 1. \end{aligned}$$

If $A' \cap B_3'$ is a path of length less than $\frac{k+1}{2}$ in $G$, then $A' \cap B_4'$ is a path of length at least $\frac{k+1}{2}$. It can be proved in a similar manner that

$$d_{W_k(G)}(a, a') \le d - 1.$$

**V.** Let $A \in S_1$ and $A' \in S_3$. Since $d_{W_k(G)}(a, b_i) \le \frac{k-1}{2}$, where $i = 1$ or $2$; and $d_{W_k(G)}(a', b_j) \le \frac{k-1}{2}$, where $j \in \{1, 2, 3, 4\}$, we have

$$d_{W_k(G)}(a, a') \le 2k - 1$$

**VI.** Let $A \in S_2$ and $A' \in S_3$. By IV, we know that $d_{W_k(G)}(a, b_i) \le \frac{k-1}{2} + d - 2k$ for $i = 3$ or $i = 4$, and analogously as in IV it can be shown that $d_{W_k(G)}(a, a') \le d - 1$.

By part II of Theorem 5, the diameter of every walk graph is greater than or equal to $d-1$, therefore

$$\mathrm{diam}(W_k(G)) = d - 1.$$

□

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