ON WEAKLY REVERSIBLE RINGS

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ABSTRACT. Let R be a ring. We introduce weakly reversible rings, which are a generalization of reversible rings, and investigate their properties. Moreover, we show that a ring R is weakly reversible if and only if for any n, the n-by-n upper triangular matrix ring $T_n(R)$ is weakly reversible. Also some kinds of examples needed in the process are given.

1. INTRODUCTION

Throughout this paper, all rings are associative with identity. According to Cohn [1], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson-Camillo [2], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa-Niewieczerzal [3] took the term C_0 for it. A generalization of reversible rings is investigated in this paper. We call a ring R weakly reversible if $ab = 0$ implies that $Rbra$ is a nil left ideal of R for all $a, b, r \in R$. Clearly semicommutative rings are weakly reversible. It is well known that semicommutative does not imply reversible (e.g. see [9, Examples 3.9 and 3.11]), examples are given to show that weakly reversible rings are not necessarily semicommutative. It is shown that a ring R is weakly reversible if and only if for any n, the n-by-n upper triangular matrix $T_n(R)$ is a weakly reversible ring.

2. WEAKLY REVERSIBLE RINGS

Definition. A ring R is called weakly reversible if for all $a, b, r \in R$ such that $ab = 0$, $Rbra$ is a nil left ideal of R (equivalently, $braR$ is nil right ideal of R).

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A ring R is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left(right) annihilator over R is an ideal of R [6, Lemma 1.1].

Since every reversible ring is semicommutative, clearly every reversible ring is weakly reversible. In the following we will see the converse is not true. Note the class of weakly reversible rings is closed under subrings and finite direct products.

Proposition 2.1. *Let R be a ring and I an ideal of R such that R/I is weakly reversible. If $I \subseteq \mathrm{nil}(R)$, then R is weakly reversible.*

Proof. Let $a, b \in R$ and $ab = 0$, then $\bar{a}\bar{b} = \bar{0}$. Thus$(\bar{r_1}\bar{b}\bar{r_2}\bar{a}\bar{r_3})^n = \bar{0}$ for some positive integer n, where $\bar{r_1}, \bar{r_2}, \bar{r_3} \in R/I$. Hence $(r_1 b r_2 a r_3)^n \in I \subseteq \mathrm{nil}(R)$. This means that R is weakly reversible. □

Theorem 2.2. *Suppose S and T are rings, and M is an (S,T)-bimodule. Let*

$$R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}.$$

Then R is weakly reversible if and only if S and T are weakly reversible.

Proof. Note that any subring of a weakly reversible ring is weakly reversible, so if R is weakly reversible then S and T are weakly reversible.

Conversely, suppose S and T are weakly reversible. Put

$$I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \triangleleft R.$$

The ring $R/I \cong S \times T$ is weakly reversible, and now Proposition 2.1 implies that R is weakly reversible. □

The following proposition follows immediately by induction on n.

Proposition 2.3. *A ring R is a weakly reversible ring if and only if, for any n, the n-by-n upper triangular matrix ring $T_n(R)$ is a weakly reversible ring.*

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrix $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$, $m \in M$ and the usual matrix operations are used.

Corollary 2.4. *A ring R is a weakly reversible ring if and only if its trivial extension is a weakly reversible ring.*

Corollary 2.5. *Let R be a ring, then R is a weakly reversible ring if and only if for any $n \in N$, $R[x]/(x^n)$ is a weakly reversible ring, where (x^n) is the ideal of $R[x]$ generated by x^n.*

Now we can give examples of weakly reversible rings which are not reversible. As we know, reversible rings are both semicommutative [6, Lemma 1.4] and weakly reversible by definition. So we may conjecture that weakly reversible rings may be semicommutative. But the following example eliminates the possibility.

Example 2.6. Let S be a weakly reversible ring. Then

$$T = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \middle| a_{ij} \in S \right\}$$

is a weakly reversible ring by Proposition 2.3. Note that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

But we have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

So T is not reversible. In fact, T is not semiccomutative by the argument from the last sentence of [9, Example 3.17] (with n=3).

Also let S be a weakly reversible ring. Then the ring

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \middle| a, a_{ij} \in S, n \geq 3 \right\}$$

is not reversible by [6, Example 1.5]. But R_n is weakly reversible by Proposition 2.3 since any subring of weakly reversible rings is weakly reversible. It is obvious that R_4 is not semicommutative and it can be proved similarly that R_n is not semicommutative for $n \geq 5$.

The next example demonstrates that the condition *n-by-n upper triangular matrix ring $T_n(R)$* in Proposition 2.3 cannot be weakened to *n-by-n full matrix ring $M_n(R)$, where n is any integer greater than 1.*

Example 2.7. If R is a weakly reversible ring and n is any integer greater than 1, then $M_n(R)$ is not weakly reversible.

For n=2, observe that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

but

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bar{\in} \ \mathrm{nil}(M_2(R)).$$

So $M_2(R)$ is not weakly reversible. One can augment these matrices in a similar way if $n > 2$.

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