

PREDICTABLE REPRESENTATION PROPERTY OF SOME HILBERTIAN MARTINGALES

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ABSTRACT. We prove as for the real case that a martingale with values in a separable real Hilbert space is extremal if and only if it satisfies the predictable representation property.

1. INTRODUCTION.

In this article we shall use the stochastic integral with respect to a local vectorial martingale as it is defined in [2].

Let (Ω, \mathcal{F}, P) be a probability space equipped with a filtration (\mathcal{F}_t) satisfying the usual conditions, that is, (\mathcal{F}_t) is complete and right continuous. Let \mathbf{H} be a real separable Hilbert space whose inner product and norm are respectively denoted by $\langle \cdot, \cdot \rangle_{\mathbf{H}}$ and $\| \cdot \|_{\mathbf{H}}$. The dual of \mathbf{H} will be denoted by \mathbf{H}' . For every process X with values in \mathbf{H} , we denote by (\mathcal{F}_t^X) the complete and right continuous filtration generated by X . We also denote by E_P the expectation with respect to the probability law P .

Let M be a continuous local (\mathcal{F}_t) -martingale with values in \mathbf{H} defined on (Ω, \mathcal{F}, P) . We say that M is (\mathcal{F}_t) -extremal if P is an extreme point of the convex set of probabilities law on $\mathcal{F}_{\infty} = \sigma(\mathcal{F}_s, s \geq 0)$ for which M is a local (\mathcal{F}_t) -martingale.

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We say that M has the predictable representation property with respect to the filtration (\mathcal{F}_t) if for every real local (\mathcal{F}_t) -martingale N on $(\Omega, \mathcal{F}_\infty, P)$ there exists an (\mathcal{F}_t) -predictable process (H_t) with values in \mathbf{H}' such that

$$N_t = N_0 + \int_0^t H_s dM_s$$

for every $t \geq 0$. As in the real case, this is equivalent to the existence, for every $Y \in L^2(\Omega, \mathcal{F}_\infty)$, of an (\mathcal{F}_t) -predictable process (H_t) with values in \mathbf{H}' such that

$$Y = E_P(Y) + \int_0^{+\infty} H_s dM_s$$

and $\int_0^{+\infty} \|H_s\|^2 d\langle M \rangle_s < +\infty$ (this can be proved in the same way as in [3, Proposition 3.2]).

We say that M is extremal or has the predictable representation property if it has this property with respect to the filtration (\mathcal{F}_t^M) .

When $\mathbf{H} = \mathbf{R}$ the notions of extremal martingales and predictable representation property coincide with the same usual ones. We recall here that this case was studied by Strook and Yor in a remarkable paper [4].

In this paper we will prove that a local (\mathcal{F}_t) -martingale on (Ω, \mathcal{F}, P) with values in \mathbf{H} is extremal if and only if it has the predictable representation property.

We also give some examples of extremal martingales with values in \mathbf{H} , that are defined by stochastic integrals of real predictable processes with respect to a cylindrical Brownian motion in \mathbf{H} .

In the whole paper we will fix a hilbertian basis $\{e_n : n \in \mathbf{N}\}$ of \mathbf{H} .

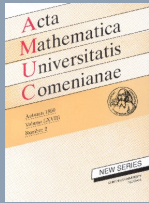


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2. PRELIMINARY RESULTS

We denote by H_0^2 the space of bounded continuous real (\mathcal{F}_t) -martingales vanishing at 0. Equipped with the inner product

$$\langle M, N \rangle_{H_0^2} = E_P(\langle M, N \rangle_\infty), \quad \forall M, N \in H_0^2,$$

H_0^2 is a Hilbert space.

If M is a continuous local (\mathcal{F}_t) -martingale of integrable square with values in \mathbf{H} , we denote by $\langle M \rangle$ the increasing predictable process such that $\|M\|_{\mathbf{H}}^2 - \langle M \rangle$ is a local (\mathcal{F}_t) -martingale. If M and N are two local (\mathcal{F}_t) -martingales of integrable squares, we define the process $\langle M, N \rangle$ by standard polarisation.

If M is a bounded continuous local (\mathcal{F}_t) -martingale with values in \mathbf{H} , we denote by $\mathcal{L}_p^2(\mathbf{H}', M)$ the space of (\mathcal{F}_t) -predictable processes $h = (H_t)$ with values in \mathbf{H}' such that $E_P(\int_0^{+\infty} \|H_s\|^2 d\langle M \rangle_s) < \infty$. We denote by $H \cdot M$ the martingale $(\int_0^t H_s dM_s)$.

For any stopping time T of the filtration (\mathcal{F}_t) and any process $X = (X_t)$, X^T denotes the process $(X_{t \wedge T})$. We say that a stopping time T reduces a local martingale Z if Z^T is a bounded martingale.

Proposition 2.1. *Let M be a continuous local (\mathcal{F}_t) -martingale with values in \mathbf{H} , then every real continuous local (\mathcal{F}_t^M) -martingale $X = (X_t)$, vanishing at 0, can be uniquely written as*

$$X = H \cdot M + L$$

where H is a (\mathcal{F}_t^M) -predictable process with values in \mathbf{H}' and L is a real local (\mathcal{F}_t^M) -martingale such that, for any stopping time T such that the martingales M^T and X^T are bounded, L^T is orthogonal in H_0^2 to the subspace

$$G = \{H \cdot M^T : H \in \mathcal{L}_p^2(\mathbf{H}', M^T)\}.$$



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Proof. The unicity of the decomposition is easy, let us prove the existence. Since M and X are local martingales, there exists a sequence (T_n) of stopping times reducing X and M ; let T be one of them. Put

$$G = \{H \cdot M^T : H \in \mathcal{L}_p^2(\mathbf{H}', M^T)\}.$$

It is then clear that G is a closed subspace of H_0^2 and that we can write $X^T = \bar{H} \cdot M^T + \bar{L}$, where $\bar{L} \in G^\perp$. For any bounded stopping time S , we have

$$\begin{aligned} E_P(M_S^T L_S) &= E_P(M_S^T E_P(L_\infty | \mathcal{F}_s)) \\ &= E_P(M_S^T L_\infty) = 0 \end{aligned}$$

since $M^{T \wedge S} \in G$. Because of the unicity, H and L extend to processes satisfying the desired conditions. \square

We will also need a vectorial version of a theorem in the measure theory due to Douglas. Let us consider the set \mathcal{P} of sequences $\pi = (P_n)$, $n \geq 1$, of probability measures on (Ω, \mathcal{F}) . For any probability measure P on (Ω, \mathcal{F}) , denote by $\pi(P)$ the element of \mathcal{P} defined on E by $P_n = P$ for every $n \geq 1$.

For any function f with values in \mathbf{H} defined on a set E , let f_n be the functions defined by $f_n(x) = \langle f(x), e_n \rangle_{\mathbf{H}}$ for every $x \in E$.

Let \mathcal{L} be a set of \mathcal{F} -measurable functions with values in \mathbf{H} , we denote by $\mathcal{K}_{\mathcal{L}}$ the set of sequences $\pi = (P_n) \in \mathcal{P}$ such that $f_n \in L_{P_n}^1(\Omega, \mathcal{F})$ and $\int f_n dP_n = 0$ for any $f \in \mathcal{L}$ and any $n \geq 1$. It is easy to see that the set $\mathcal{K}_{\mathcal{L}}$ is convex.

The following theorem is a vectorial version of a classical theorem in the measure theory due to Douglas ([3, Chap. V, Theorem 4.4])



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Theorem 2.2. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{L} a set of \mathcal{F} -measurable functions with values in \mathbf{H} and \mathcal{L}^* the vector space generated by \mathcal{L} and the constants in \mathbf{H} . Then \mathcal{L}^* is dense in $L^1_P(\Omega, \mathbf{H})$ if and only if, $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$.

Proof. The idea of the proof is the same as for the classical theorem of Douglas. Assume that \mathcal{L}^* is dense in $L^1(\Omega, \mathbf{H})$, let $\pi_1 = (P_n^1)$ and let $\pi_2 = (P_n^2)$ in $\mathcal{K}_{\mathcal{L}}$ such that $\pi(P) = \alpha\pi_1 + (1 - \alpha)\pi_2$, with $0 \leq \alpha \leq 1$. If $\alpha \neq 0, 1$, then π_1, π_2 and $\pi(P)$ would be identical on \mathcal{L}^* , and therefore on $L^1_P(\Omega, \mathbf{H})$ by density, hence $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$.

Conversely, assume that $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$. Then if \mathcal{L}^* is not dense in $L^1_P(\Omega, \mathbf{H})$, there exists, following Hahn-Banach theorem, a continuous linear form φ not identically 0 on $L^1_P(\Omega, \mathbf{H})$ which vanishes on \mathcal{L}^* . But such a form can be identified with an element of $L^\infty(\Omega, \mathbf{H})$, hence we can find functions $g_n \in L^\infty_P(\Omega), n \in \mathbf{N}$, such that

$$\phi(f) = \sum_{n \geq 1} \int g_n f_n dP$$

for any $f = \sum_n f_n e_n \in L^1(\Omega, \mathbf{H})$. We can assume that $\|g_n\|_\infty \leq 1$ for any n . Put, for any $n \geq 1$, $P_n^1 = (1 - g_n)P$ and $P_n^2 = (1 + g_n)P$, then $\pi_1 = (P_n^1) \in \mathcal{K}_{\mathcal{L}}$, $\pi_2 = (P_n^2) \in \mathcal{K}_{\mathcal{L}}$ and $\pi(P) = \alpha\pi_1 + (1 - \alpha)\pi_2$, but $\pi_1 \neq \pi_2$, which is a contradiction with the fact that $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$. \square

Remark. If $\mathbf{H} = \mathbf{R}$, then $\mathcal{K}_{\mathcal{L}}$ is simply the convex set of probability laws Q on (Ω, \mathcal{F}) such that $\mathcal{L} \subset L^1_Q(\Omega)$ and $\int f dQ = 0$ for any $f \in \mathcal{L}$. In this case, Theorem 2.2 is reduced to the classical Douglas Theorem.

Let $X = (X_t)$ be an integrable process with values in \mathbf{H} (i.e. X_t is integrable for every t). Put

$$\mathcal{L} = \{1_A(X_t - X_s) : A \in \mathcal{F}_s^X, s \leq t\}.$$

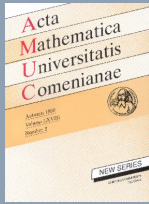


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Then X is a martingale under the law P , with values in \mathbf{H} , if and only if $\pi(P) \in \mathcal{L}^*$. If $\mathcal{F} = \mathcal{F}_\infty^X$, this is equivalent to $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$ or to P is an extreme point of the set of probability laws Q on $(\Omega, \mathcal{F}_\infty^X)$, X being a local martingale of (\mathcal{F}^X) under the law Q .

Proposition 2.3. *Assume that $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$. Then every \mathbf{H} -valued local (\mathcal{F}_t^X) -martingale has a continuous version.*

Proof. As in the real case it suffices to prove that for any $Y \in L^1(\Omega, \mathbf{H})$, the martingale N defined by

$$N_t = E_P(Y | \mathcal{F}_t^X)$$

is continuous. It is not hard to see that this result is true if $Y \in \mathcal{L}^*$. Now, by Theorem 2.2, if $Y \in L^1(\Omega, \mathbf{H})$, one can find a sequence (Y_n) in \mathcal{L}^* which converges to Y in $L^1(\Omega, \mathbf{H})$ - norm. For any $\varepsilon > 0$, one has

$$P[\sup_{s \leq t} \|E_P(Y_n | \mathcal{F}_s^X) - E_P(Y | \mathcal{F}_s^X)\|_{\mathbf{H}} \geq \varepsilon] \leq \varepsilon^{-1} E_P(\|Y_n - Y\|_{\mathbf{H}}).$$

Hence, by reasoning as for the real martingales, we obtain the desired result. □

It follows easily from the above proposition that if $\pi(P)$ is an extreme point of $\mathcal{K}_{\mathcal{L}}$, then every local (\mathcal{F}_t^X) -martingale with values in a real separable Hilbert space \mathbf{K} has a continuous version.

Theorem 2.4. *Let M be a continuous local (\mathcal{F}_t) -martingale defined on (Ω, \mathcal{F}, P) with values in \mathbf{H} . The following statements are equivalent:*

- i) M is extremal.
- ii) M has the predictable representation property with respect to (\mathcal{F}_t^M) , and the σ -algebra \mathcal{F}_0^M is P -a.s. trivial.

Proof. Put

$$\mathcal{L} = \{1_A(M_t - M_s) : A \in \mathcal{F}_s^M, s \leq t\}.$$

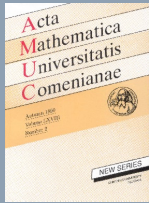


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Assume first that M is extremal, that is $\pi(P)$ is an extremal point of $\mathcal{K}_{\mathcal{L}}$, and let $Y \in L_P^\infty(\Omega, \mathcal{F}_\infty^M)$. Then by Proposition 2.1, there exist a predictable process $H = (H_t)$ with values in \mathbf{H}' and a real continuous martingale $L = (L_t)$ such that

$$E_P(Y|\mathcal{F}_t^M) = E_P(Y) + \int_0^t H_s dM_s + L_t, \quad \forall t \geq 0,$$

with $\langle H \cdot M^T, L \rangle = 0$ for any (\mathcal{F}_t^M) -stopping time T and any $K \in \mathcal{L}_P^2(\mathbf{H}, M^T)$. By stopping and by virtue of the relation $\langle M, L^T \rangle = \langle M, Y \rangle^T$, we can assume that $|L|$ is bounded by a constant $k > 0$. Put $P_n^1 = (1 + \frac{L_\infty}{2k})P$ and $P_n^2 = (1 - \frac{L_\infty}{2k})P$. Let $\pi_1 = (P_n^1)$ and $\pi_n^2 = (P_n^2)$; then we have $\pi(P) = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. Hence $\pi_1 = \pi_2 = \pi(P)$ because of the extremality of $\pi(P)$, then $L_\infty = 0$. We then deduce that $\bar{L} = 0$, furthermore

$$E_P(Y|\mathcal{F}_t^M) = E_P(Y) + \int_0^t H_s dM_s,$$

or

$$Y = E_P(Y) + \int_0^{+\infty} H_s dM_s.$$

Conversely, if $P = \alpha P_1 + (1 - \alpha)P_2$, where $\pi(P_1), \pi(P_2) \in \mathcal{K}_{\mathcal{L}}$, then the real martingale $(\frac{dP_1|_{\mathcal{F}_t^M}}{dP})$ admits a continuous version L because P has the predictable representation property, and $L \cdot M$ is a martingale under the probability P , hence $\langle M, L \rangle = 0$. But we have $L_t = L_0 + \int_0^t H_s dM_s$, henceforth $\langle X, L \rangle_t = \int_0^t H_s d\langle X \rangle_s$. It then follows that P -a.s. we have $H_s = d\langle M \rangle_s$ -p.s., hence $\int_0^t H_s dM_s = 0$, for every $t \geq 0$. Then L is constant, i.e. $L_t = L_0$ for every $t \geq 0$. Since \mathcal{F}_0^M is trivial, we have $L_0 = 1$ and then $P = P_1 = P_2$. This proves that P is extremal. \square



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3. EXAMPLES

Proposition 3.1. *The cylindrical Brownian motion in \mathbf{H} (defined on a probability space (Ω, \mathcal{F}, P)) is an extremal martingale.*

Proof. Let us remark that since \mathbf{H} is separable, then the σ -algebra generated by the continuous linear forms on \mathbf{H} is identical to the Borel σ -algebra of \mathbf{H} .

Let Q be a probability measure on \mathcal{F}_∞^B for which B is a local martingale. Then for any non identically 0 continuous linear form ϕ on \mathbf{H} , the process $\phi(B) = (\phi(B_t))$ is a real Brownian motion under the probabilities measures P and Q ; then $Q = P$ on $\mathcal{F}_\infty^{\phi(B)}$. Since ϕ is arbitrary, it follows that $Q = P$ on $\sigma(\{\phi(B_t) : t \geq 0, \phi \in \mathbf{H}'\}) = \mathcal{F}_\infty^B$. Hence P is the unique probability measure on \mathcal{F}_∞^B for which B is a local martingale. Then the martingale B is extremal. \square

Proposition 3.2. *Let (H_t) be a real process (\mathcal{F}_t^B) -predictable such that P -almost every $\omega \in \Omega$, the set $\{s \geq 0 : H_s(\omega) \neq 0\}$ is of null Lebesgue measure, and such that $E^P(\int_0^{+\infty} H_s^2 ds) < +\infty$. Then the martingale M defined by $M_t = \int_0^t H_s dB_s$ is extremal.*

Proof. By replacing if necessary the Brownian motion (B_t) by the Brownian motion $(\int_0^t \text{sgn}(H_s) dB_t)$, we may assume that the process (H_t) is non-negative. We have, up to a multiplicative constant,

$$\langle M \rangle_t = \int_0^t H_s^2 ds \cdot I, \quad \forall t \geq 0,$$

where I is the identity operator on \mathbf{H} (see [2]), hence the process (H_t) is (\mathcal{F}_t^B) -adapted. On the other hand, we have

$$B_t = \int_0^t \frac{1}{H_s} dM_s,$$



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hence B is (\mathcal{F}_t^M) -adapted. Then $(\mathcal{F}_t^M) = (\mathcal{F}_t^B)$. If $N = (N_t)$ is a real \mathcal{F}_t^M -martingale, we have, following the above proposition,

$$N_t = c + \int_0^t X_s dB_s, \quad \forall t \geq 0,$$

where $X = (X_t)$ is a (\mathcal{F}_t^B) -predictable process with values in \mathbf{H} and c is a constant, hence

$$N_t = c + \int_0^t \frac{X_s}{H_s} dM_s, \quad \forall t \geq 0.$$

Then M is extremal by Theorem 2.4. □

Let (H_t) be as in the above proposition. Then for P -almost every $\omega \in \Omega$, the mapping from \mathbf{H} in \mathbf{H} defined by $x \mapsto H_t(\omega)x$ is for almost every $t \geq 0$ (in the Lebesgue measure sense) an isomorphism from \mathbf{H} into itself. This suggests the following problem:

Problem: Let \mathbf{H} and \mathbf{K} be two real separable Hilbert spaces, (B_t) a cylindrical Brownian motion in \mathbf{H} , and let (H_t) be a predictable process with values in $\mathcal{L}(\mathbf{H}, \mathbf{K})$ such that the stochastic integral $\int_0^t H_s dB_s$ is well defined and that for any $t \geq 0$, and P -almost any $\omega \in \Omega$, H_t is for almost every $t \geq 0$ an isomorphism from \mathbf{H} in \mathbf{K} . Is $H \cdot B$ extremal?

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