CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

YONGFU SU, XIAOLONG QIN AND MEIJUAN SHANG

ABSTRACT. Let E be a uniformly convex Banach space, and let K be a nonempty convex closed subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and let $F(T)$ be nonempty, where $F(T)$ denotes the fixed points set of T. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\varepsilon \le \alpha_n, \alpha'_n, \alpha''_n \le 1 - \varepsilon$ for all $n \in N$ and some $\varepsilon > 0$, starting with arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by setting

$$\begin{cases} z_n = P(\alpha''_n T(PT)^{n-1} x_n + \beta''_n x_n + \gamma''_n w_n), \\ y_n = P(\alpha'_n T(PT)^{n-1} z_n + \beta'_n x_n + \gamma'_n v_n), \\ x_{n+1} = P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n u_n), \end{cases}$$

with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma''_n < \infty$, where $\{w_n\}, \{v_n\}$ and $\{u_n\}$ are bounded sequences in K. (i) If E is real uniformly convex Banach space satisfying $Opial's$ condition, then weak convergence of $\{x_n\}$ to some $p \in F(T)$ is obtained; (ii) If T satisfies condition (A), then $\{x_n\}$ convergence strongly to some $p \in F(T)$.

1. Introduction and Preliminaries

Let E be a real Banach space, K be a nonempty subset of X and $F(T)$ denote the set of fixed points of T. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \text{for all } x, y \in K.$$

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972. They proved that, if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of K has a fixed point. Moreover, the fixed point set $F(T)$ of T is closed and convex.

Recently, Chidume et al. have introduced another new concept about asymptotically nonexpansive mappings

Definition 1.1 ([1]). Let E be a real normed linear space, K a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A map $T : K \to E$ is said to be an asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$ such that the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n \|x - y\|, \qquad \forall x, y \in K, \ n \geq 1.$$

T is called uniformly L-lipschitzian if there exists $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L \|x - y\|, \qquad \forall x, y \in K, \ n \geq 1.$$

Many authors have contributed by their efforts to investigate the problem of finding a fixed point of asymptotically nonexpansive mappings and non-self asymptotically nonexpansive mappings. In [5], [6], Schu introduced a modified Mann iteration process to approximate fixed points of

asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space H. More precisely, he proved the following theorems.

Theorem JS1 ([5]). *Let H be a Hilbert space, K a nonempty closed convex and bounded subset of H, and $T : K \to K$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in [0,1] satisfying the condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \geq 1$ and for some $\varepsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1,$$
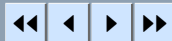
converges strongly to a fixed point of T.

Theorem JS2 ([6]). *Let E be a uniformly convex Banach space satisfying Opial's condition, K a nonempty closed convex and bounded subset of E, and $T : K \to K$ an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0, 1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1$, for all $n \geq 1$ and some $a, b \in (0, 1)$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1,$$

converges weakly to a fixed point of T.

In [4], Rhoades extended Theorem JS1 to a uniformly convex Banach space using a modified Ishikawa iteration method. In [3], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on K, provided that $F(T) = \{x \in K : Tx = x\} \neq \emptyset$.

In [9], Tan and Xu introduced a modified Ishikawa processes to approximate fixed points of nonexpansive mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space E. More precisely, they proved the following theorem.

Theorem TX ([9]). *Let E be a uniformly convex Banach space which satisfies Opial's condition or has a Frechet differentiable norm. Let C be a nonempty closed convex bounded subset of E, $T : C \to C$ a nonexpansive mapping and $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

(1.1) $$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n], \ n \geq 1$$

converges weakly to a fixed point of T.

In the above results, T remains a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space, however if, the domain K of T is a proper subset of E (and this is the case in several applications), and T maps K into E, then iteration processes of Mann and Ishikawa may fail to be well defined.

In 2003, Chidume et al. [1] studied the iteration scheme defined by

$$x_1 \in K, \qquad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \qquad n \geq 1.$$

In the framework of a uniformly convex Banach space, where K is a nonempty closed convex non-expansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. $T : K \to E$ is an asymptotically nonexpansive non-self map with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying the condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \geq 1$ and for some $\varepsilon > 0$. They proved strong and weak convergence theorems for asymptotically nonexpansive nonself-maps.

Recently, Naseer Shahzad [7] studied the sequence $\{x_n\}$ defined by

$$x_1 \in K, \qquad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n TP[(1 - \beta_n)x_n + \beta_n Tx_n]),$$

where K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. He proved weak and strong convergence theorems for non-self nonexpansive mappings in Banach spaces.
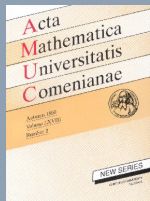
Motivated by the Chidume et al. [1], Nasser Shahzad [7] and some others, the purpose of this paper is to construct an iterative scheme for approximating a fixed point of asymptotically nonexpansive non-self maps (provided that such a fixed point exists) and to prove some strong and weak convergence theorems for such maps.

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. In this paper, the following iteration scheme is studied

$$(1.2) \quad \begin{cases} x_1 \in K \\ z_n = P(\alpha_n'' T(PT)^{n-1} x_n + \beta_n'' x_n + \gamma_n'' w_n) \\ y_n = P(\alpha_n' T(PT)^{n-1} z_n + \beta_n' x_n + \gamma_n' v_n) \\ x_{n+1} = P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n u_n) \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, $\{\gamma_n'\}$, $\{\alpha_n''\}$, $\{\beta_n''\}$ and $\{\gamma_n''\}$ are real sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$.

Our theorems improve and generalize some previous results to some extent.

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to E$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Recall that the mapping $T : K \to E$ with $F(T) \neq \emptyset$ where K is a subset of E, is said to satisfy condition A [8] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in K$

$$\|x - Tx\| \geq f(d(x, F(T)),$$

where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

In order to prove our main results, we shall make use of the following Lemmas.

Lemma 1.1 (Schu [6].). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in N$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n \to \infty} \|x_n\| \leq r, \limsup_{n \to \infty} \|y_n\| \leq r$$

and

$$\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Lemma 1.2 ([1] Demiclosed principle for nonself-map). *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed with respect to zero.*

Lemma 1.3 (Tan and Xu [9]). *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying the following condition*

$$r_{n+1} \leq (1 + s_n) r_n + t_n, \qquad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \to \infty} r_n$ exists.

2. Main results

Lemma 2.1. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined by the recursion (1.2) taking arbitrary $x_1 \in K$, with the restrictions $\sum_{n=1}^{\infty} \gamma_n'' < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then*

$\lim_{n\to\infty} \|x_n - p\|$ *exists, for any* $p \in F(T)$, *where* $F(T)$ *denotes the nonempty fixed point set of* T.

Proof. Since $\{w_n\}$, $\{v_n\}$ and $\{u_n\}$ are bounded sequences in C, for any given $p \in F(T)$, we can set

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\}, \qquad M_2 = \sup\{\|v_n - p\| : n \geq 1\},$$
$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \qquad M = \max\{M_i : i = 1, 2, 3\}.$$

It follows from (1.2) that

$$
\begin{aligned}
\|z_n - p\| &= \|P(\alpha_n'' T(PT)^{n-1} x_n + \beta_n'' x_n + \gamma_n'' w_n) - p\| \\
&\leq \|\alpha_n'' T(PT)^{n-1} x_n + \beta_n'' x_n + \gamma_n'' w_n - p\| \\
&\leq \alpha_n'' \|T(PT)^{n-1} x_n - p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|w_n - p\| \\
&= \alpha_n'' \|T(PT)^{n-1} x_n - T(PT)^{n-1} p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|w_n - p\| \\
&\leq \alpha_n'' k_n \|x_n - p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|w_n - p\| \\
&\leq \alpha_n'' k_n \|x_n - p\| + (1 - \alpha_n'') \|x_n - p\| + \gamma_n'' \|w_n - p\| \\
&\leq k_n \|x_n - p\| + \gamma_n'' M,
\end{aligned}
$$

which implies that

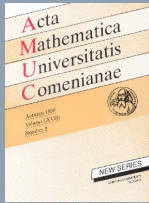$$(2.1) \qquad \|z_n - p\| \leq k_n \|x_n - p\| + \gamma_n'' M.$$

From (1.2) and (2.1) we get

$$\|y_n - p\| = \|P(\alpha'_n T(PT)^{n-1} z_n + \beta'_n x_n + \gamma'_n v_n) - p\|$$
$$\leq \|\alpha'_n T(PT)^{n-1} z_n + \beta'_n x_n + \gamma'_n v_n - p\|$$
$$\leq \alpha'_n \|T(PT)^{n-1} z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\|$$
$$= \alpha'_n \|T(PT)^{n-1} z_n - T(PT)^{n-1} p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\|$$
$$\leq \alpha'_n k_n \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\|$$
$$\leq \alpha'_n k_n \|z_n - p\| + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n \|v_n - p\|$$
$$\leq \alpha'_n k_n (k_n \|x_n - p\| + \gamma''_n M) + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n \|v_n - p\|$$
$$\leq k_n^2 \|x_n - p\| + k_n \gamma''_n M + \gamma'_n M,$$

which inplies that

(2.2) $$\|y_n - p\| \leq k_n^2 \|x_n - p\| + k_n \gamma''_n M + \gamma'_n M.$$

Again, from (1.2) and (2.2) we have

$$\|x_{n+1} - p\| = \|P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n u_n) - p\|$$
$$= \|\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n u_n - p\|$$
$$\leq \alpha_n \|T(PT)^{n-1} y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\|$$
$$\leq \alpha_n \|T(PT)^{n-1} y_n - T(PT)^{n-1} p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\|$$
$$\leq \alpha_n k_n \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\|$$
$$\leq \alpha_n k_n \|y_n - p\| + (1 - \alpha_n) \|x_n - p\| + \gamma_n \|u_n - p\|$$
$$\leq \alpha_n k_n (k_n^2 \|x_n - p\| + k_n \gamma''_n M + \gamma'_n M) + (1 - \alpha'_n) \|x_n - p\| + \gamma_n M$$
$$\leq k_n^3 \|x_n - p\| + k_n^2 \gamma''_n M + k_n \gamma'_n M + \gamma_n M.$$

Therefore

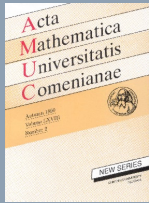$$(2.3) \qquad \|x_{n+1} - p\| \leq (1 + (k_n^3 - 1))\|x_n - p\| + (k_n^2 \gamma_n'' + k_n \gamma_n' + \gamma_n')M.$$

Note that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$, therefore by Lemma 1.3, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$. This completes the proof. $\qquad\square$

Lemma 2.2. *Let E be a normed linear space, K a nonempty closed convex subset which is also a nonexpansive retract of E, $T : K \to E$ a uniformly L-Lipschitzian mapping. Let $\{x_n\}$ be the sequence defined by the recursion (1.2) taking arbitrary $x_1 \in K$, with the restrictions $\sum_{n=1}^{\infty} \gamma_n'' < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ and set $C_n = \|x_n - T(PT)^{n-1}x_n\|$, $\forall n \geq 1$. If $\lim_{n\to\infty} C_n = 0$, then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof. Since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded, it follows from Lemma 2.1 that $\{u_n - x_n\}$, $\{v_n - x_n\}$, $\{w_n - x_n\}$ are all bounded, now, we set

$$r_1 = \sup\{\|u_n - x_n\| : n \geq 1\}, \qquad\qquad r_2 = \sup\{\|v_n - x_n\| : n \geq 1\},$$
$$r_3 = \sup\{\|w_n - x_n\| : n \geq 1\}, \qquad\qquad r_4 = \sup\{\|v_{n-1} - x_n\| : n \geq 1\},$$
$$r_5 = \sup\{\|u_{n-1} - T(PT)^{n-2}x_n\| : n \geq 1\}, \qquad r = \max\{r_i : i = 1,2,3,4,5\}.$$

It follows from (1.2) that

$$\|x_{n+1} - x_n\| \leq \|\alpha_n T(PT)^{n-1}y_n + \beta_n x_n + \gamma_n u_n - x_n\|$$
$$\leq \|T(PT)^{n-1}y_n - x_n\| + \gamma_n r$$
$$\leq \|T(PT)^{n-1}x_n - x_n\| + \|T(PT)^{n-1}y_n - T(PT)^{n-1}x_n\| + \gamma_n r$$
$$\leq C_n + L\|y_n - x_n\| + \gamma_n r$$
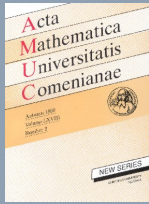$$\leq C_n + L\|\alpha_n' T(PT)^{n-1}z_n + \beta_n' x_n + \gamma_n' v_n - x_n\| + \gamma_n r$$

$$\leq C_n + L\|T(PT)^{n-1}z_n - x_n\| + \gamma'_n Lr + \gamma_n r$$

$$\leq C_n + L\|T(PT)^{n-1}x_n - x_n\| + L\|T(PT)^{n-1}z_n - T(PT)^{n-1}x_n\|$$
$$+ \gamma'_n Lr + \gamma_n r$$

$$\leq C_n + LC_n + L^2\|z_n - x_n\| + \gamma'_n Lr + \gamma_n r$$

$$\leq C_n + LC_n + L^2\|\alpha''_n T(PT)^{n-1}x_n + \beta''_n x_n + \gamma''_n w_n - x_n\|$$
$$+ \gamma'_n Lr + \gamma_n r$$

$$(2.4) \qquad = C_n(1 + L + L^2) + \gamma''_n L^2 r + \gamma'_n Lr + \gamma_n r$$

and

$$\|y_{n-1} - x_n\| \leq \|\alpha'_{n-1}T(PT)^{n-2}z_{n-1} + \beta'_{n-1}x_{n-1} + \gamma'_{n-1}v_{n-1} - x_n\|$$

$$\leq \|T(PT)^{n-2}z_{n-1} - x_n\| + \|x_{n-1} - x_n\| + \gamma'_{n-1}r$$

$$\leq \|T(PT)^{n-2}x_{n-1} - x_{n-1}\| + \|T(PT)^{n-2}z_{n-1} - T(PT)^{n-2}x_{n-1}\|$$
$$+ 2\|x_{n-1} - x_n\| + \gamma'_{n-1}r$$

$$(2.5) \qquad \leq C_{n-1} + LC_{n-1} + L\gamma''_{n-1}r + 2\|x_{n-1} - x_n\| + \gamma'_{n-1}r.$$

Substituting (2.4) into (2.5) we obtain

$$(2.6) \qquad \|y_{n-1} - x_n\| \leq C_{n-1}(3 + 3L + 2L^2) + (1 + 2L)r(L\gamma''_{n-1} + \gamma'_{n-1})$$
$$+ 2\gamma'_{n-1}r.$$

On the other hand, from (2.4) and (2.6) we have

$$\|x_n - (PT)^{n-1}x_n\|$$
$$\leq \|\alpha_{n-1}T(PT)^{n-2}y_{n-1} + \beta_{n-1}x_{n-1} + \gamma_{n-1}u_{n-1} - T(PT)^{n-2}x_n\|$$

$$\leq \|T(PT)^{n-2}y_{n-1} - T(PT)^{n-2}x_n\| + \|x_{n-1} - T(PT)^{n-2}x_n\|$$
$$+ \gamma_{n-1}r$$
$$\leq L\|y_{n-1} - x_n\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\|$$
$$+ \|T(PT)^{n-2}x_{n-1} - T(PT)^{n-2}x_n\| + \gamma_{n-1}r$$
$$\leq L\|y_{n-1} - x_n\| + C_{n-1} + L\|x_{n-1} - x_n\| + \gamma_{n-1}r$$
$$\leq LC_{n-1}(4 + 4L + 3L^2) + C_{n-1} + L^2 r\gamma''_{n-1}(1 + 3L)$$
$$(2.7) \qquad + 3Lr\gamma'_{n-1}(1 + L) + (1 + L)r\gamma_{n-1}.$$

It follows from (2.7) that

$$\|x_n - Tx_n\| \leq \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{n-1}x_n - Tx_n\|$$
$$\leq C_n + L\|(PT)^{n-1}x_n - x_n\|$$
$$\leq C_n + L^2 C_{n-1}(4 + 4L + 3L^2) + LC_{n-1} + L^3 r\gamma''_{n-1}(1 + 3L)$$
$$+ 3L^2 r\gamma'_{n-1}(1 + L) + L(1 + L)r\gamma_{n-1}.$$

It follows from $\lim_{n\to\infty} C_n = 0$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

Theorem 2.1. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}, \{\alpha''_n\}, \{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\varepsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \varepsilon$ for all $n \in N$*
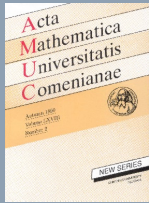
and some $\varepsilon > 0$. Let $\{x_n\}$ be the sequence defined by the recursion (1.2) *taking arbitrary $x_1 \in K$. Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. Take $p \in F(T)$, by Lemma 2.1 we know, $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = c$. If $c = 0$, then by the continuity of T the conclusion follows. Now suppose $c > 0$. We claim $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Taking limsup on both the sides in the inequality (2.1), we have

$$(2.8) \qquad \limsup_{n\to\infty} \|z_n - p\| \le c.$$

Similarly, taking limsup on both sides of the inequality (2.2), we have

$$(2.9) \qquad \limsup_{n\to\infty} \|y_n - p\| \le c.$$

Next, we consider

$$\|T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)\| \le \|T(PT)^{n-1}y_n - p\| + \gamma_n\|u_n - x_n\|$$
$$\le k_n\|y_n - p\| + \gamma_n r.$$

Taking limsup on both the sides in the above inequality and using (2.9) we get

$$\limsup_{n\to\infty} \|T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)\| \le c.$$

and

$$\|x_n - p + \gamma_n(u_n - x_n)\| \le \|x_n - p\| + \gamma_n\|u_n - x_n\|$$
$$\le \|x_n - p\| + \gamma_n r,$$

which imply that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(u_n - x_n)\| \le c.$$

Again, $\lim_{n\to\infty} \|x_{n+1} - p\| = c$ means that

$$(2.10) \qquad \liminf_{n\to\infty} \|\alpha_n(T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)) \\ + (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \geq c.$$

On the other hand, using (2.1) yields

$$\|\alpha_n(T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)) + (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \\ \leq \alpha_n\|T(PT)^{n-1}y_n - p\| + (1 - \alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ \leq \alpha_n k_n\|y_n - p\| + (1 - \alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ \leq \alpha_n k_n(k_n^2\|x_n - p\| + k_n\gamma_n''r + \gamma_n'r) + (1 - \alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ \leq k_n^3\|x_n - p\| + k_n^2\gamma_n''r + k_n\gamma_n'r + \gamma_n r.$$

Therefore,

$$(2.11) \qquad \limsup_{n\to\infty} \|\alpha_n(T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)) \\ + (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \leq c.$$

Combining (2.10) with (2.11) we obtain

$$\lim_{n\to\infty} \|\alpha_n(T(PT)^{n-1}y_n - p + \gamma_n(u_n - x_n)) \\ + (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| = c.$$

By applying Lemma 1.1, we have

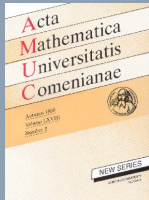$$(2.12) \qquad \lim_{n\to\infty} \|T(PT)^{n-1}y_n - x_n\| = 0.$$

Notice that

$$\|x_n - p\| \le \|T(PT)^{n-1}y_n - x_n\| + \|T(PT)^{n-1}y_n - p\|$$
$$\le \|T(PT)^{n-1}y_n - x_n\| + k_n\|y_n - p\|,$$

which yields

$$c \le \liminf_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|y_n - p\| \le c.$$

This implies that

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

Again, $\lim_{n\to\infty} \|y_n - p\| = c$ gives

(2.13)
$$\liminf_{n\to\infty} \|\alpha'_n(Tz_n - p + \gamma'_n(v_n - x_n))$$
$$+ (1 - \alpha'_n)(x_n - p + \gamma'_n(v_n - x_n))\| \ge c.$$

Similarly, we have

$$\|\alpha'_n(T(PT)^{n-1}z_n - p + \gamma'_n(v_n - x_n)) + (1 - \alpha'_n)(x_n - p + \gamma'_n(v_n - x_n))\|$$
$$\le \alpha'_n\|T(PT)^{n-1}z_n - p\| + (1 - \alpha'_n)\|x_n - p\| + \gamma'_n\|v_n - x_n\|$$
$$\le \alpha'_n k_n\|z_n - p\| + (1 - \alpha'_n)\|x_n - p\| + \gamma'_n\|v_n - x_n\|$$
$$\le \alpha'_n k_n(k_n\|x_n - p\| + \gamma''_n r) + (1 - \alpha'_n)\|x_n - p\| + \gamma'_n\|v_n - x_n\|$$
$$\le k_n^2\|x_n - p\| + k_n\gamma''_n r + \gamma'_n r.$$

Therefore,

(2.14)
$$\limsup_{n\to\infty} \|\alpha'_n(T(PT)^{n-1}z_n - p + \gamma'_n(v_n - x_n))$$
$$+ (1 - \alpha'_n)(x_n - p + \gamma'_n(v_n - x_n))\| \le c.$$

Combining (2.13) with (2.14) yields that

$$(2.15) \quad \lim_{n \to \infty} \| \alpha'_n (T(PT)^{n-1} z_n - p + \gamma'_n (v_n - x_n)) + (1 - \alpha'_n)(x_n - p + \gamma'_n (v_n - x_n)) \| = c.$$

On the other hand, we have
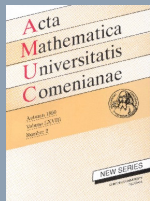
$$\| T(PT)^{n-1} z_n - p + \gamma'_n (v_n - x_n) \| \leq \| T(PT)^{n-1} z_n - p \| + \gamma'_n \| v_n - x_n \|$$
$$\leq k_n \| z_n - p \| + \gamma'_n r$$

Taking limsup on both sides of the above inequality and using (2.1), we have

$$(2.16) \quad \limsup_{n \to \infty} \| T(PT)^{n-1} z_n - p + \gamma'_n (v_n - x_n) \| \leq c$$

and

$$\| x_n - p + \gamma'_n (v_n - x_n) \| \leq \| x_n - p \| + \gamma'_n \| v_n - x_n \|$$
$$\leq \| x_n - p \| + \gamma'_n r,$$

which yields

$$(2.17) \quad \limsup_{n \to \infty} \| x_n - p + \gamma'_n (v_n - x_n) \| \leq c.$$

Applying Lemma 1.1, it follows from (2.15), (2.16) and (2.17) that

$$(2.18) \quad \lim_{n \to \infty} \| T(PT)^{n-1} z_n - x_n \| = 0.$$

Notice that

$$\| x_n - p \| \leq \| T(PT)^{n-1} z_n - x_n \| + \| T(PT)^{n-1} z_n - p \|$$
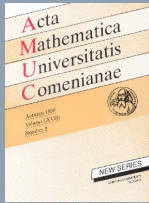$$\leq \| T(PT)^{n-1} z_n - x_n \| + k_n \| z_n - p \|.$$

We have
$$c \leq \liminf_{n \to \infty} \|z_n - p\| \leq \limsup_{n \to \infty} \|z_n - p\| \leq c.$$
That implies that
(2.19)
$$\lim_{n \to \infty} \|z_n - p\| = c.$$
By the same method, we have
(2.20)
$$\lim_{n \to \infty} \|\alpha_n''(T(PT)^{n-1}x_n - p + \gamma_n''(w_n - x_n)) + (1 - \alpha_n'')(x_n - p + \gamma_n''(w_n - x_n)) - p\| = c.$$
Moreover,
$$\|T(PT)^{n-1}x_n - p + \gamma_n''(w_n - x_n)\| \leq \|T(PT)^{n-1}x_n - p\| + \gamma_n''\|w_n - x_n\|$$
$$\leq k_n\|x_n - p\| + \gamma_n''r$$
which implies that
(2.21)
$$\limsup_{n \to \infty} \|T(PT)^{n-1}x_n - p + \gamma_n''(w_n - x_n)\| \leq c.$$
It follows from
$$\|x_n - p + \gamma_n''(w_n - x_n)\| \leq \|x_n - p\| + \gamma_n''\|w_n - x_n\|$$
$$\leq \|x_n - p\| + \gamma_n''r.$$
we obtain
(2.22)
$$\limsup_{n \to \infty} \|x_n - p + \gamma_n''(w_n - x_n)\| \leq c.$$
Combining (2.20), (2.21) with (2.22) yields
(2.23)
$$\lim_{n \to \infty} \|T(PT)^{n-1}x_n - x_n\| = 0.$$

Since T is uniformly L-Lipschitzian for some $L > 0$, it follows form Lemma 2.2 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. $\qquad\square$

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E satisfying Opial's condition. Suppose that $T : K \to E$ is an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $k_n \to 1$ as $n \to \infty$. Let $\{x_n\}$ be defined by (1.2), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\varepsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1 - \varepsilon$ for all $n \in N$ and some $\varepsilon > 0$. Then $\{x_n\}$ converges weakly to a fixed point of $F(T)$.*

Proof. For any $p \in F(T)$, it follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - p\|$ exists. We now prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Firstly, let p_1 and p_2 be weak limits of subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemmas 2.1 and 2.2, we know that $p \in F(T)$. Secondly, let us assume $p_1 \neq p_2$, then by Opial's condition, we obtain

$$\lim_{n\to\infty}\|x_n - p_1\| = \lim_{k\to\infty}\|x_{n_k} - p_1\| < \lim_{k\to\infty}\|x_{n_k} - p_2\| = \lim_{j\to\infty}\|x_{n_j} - p_2\|$$
$$< \lim_{k\to\infty}\|x_{n_k} - p_1\| = \lim_{n\to\infty}\|x_n - p_1\|$$

which is a contradiction. Hence $p_1 = p_2$. Then $\{x_n\}$ converges weakly to a fixed point of T. The proof is complete. $\qquad\square$

Next, we shall prove a strong convergence theorem.

Theorem 2.3. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be a nonexpansive mapping with $p \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$*
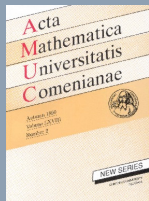
be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and and $\varepsilon \le \alpha_n, \alpha'_n, \alpha''_n \le 1 - \varepsilon$ for all $n \in N$ and some $\varepsilon > 0$. Let $\{x_n\}$ be the sequence defined by the recursion (1.2) taking arbitrary $x_1 \in K$. Suppose T satisfies condition (A). Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 2.1, $\lim_{n \to \infty} \|x_n - p\|$ exists for all $p \in F = F(T)$. Let $\lim_{n \to \infty} \|x_n - p\| = c$ for some $c \ge 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Theorem 2.1, $\lim_{n \to \infty} \|Tx_n - x_n\| = 0$, and (2.5) give

$$\inf_{p \in F} \|x_{n+1} - p\| \le \inf_{p \in F} (1 + (k_n^3 - 1))\|x_n - p\| + (k_n^2 \gamma''_n + k_n \gamma'_n + \gamma_n)M.$$

This means that

$$d(x_{n+1}, F) \le (1 + (k_n^3 - 1))d(x_n, F) + (k_n^2 \gamma''_n + k_n \gamma'_n + \gamma_n)M.$$

Thus $\lim_{n \to \infty} d(x_n, F)$ exists by virtue of Lemma 1.3. Now by condition (A), $\lim_{n \to \infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n \to \infty} d(x_n, F) = 0$. Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and sequence $\{y_j\} \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. Then following the method in the proof of Tan and Xu [9], we get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let $y_j \to y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \to y$. As $\lim_{n \to \infty} \|x_n - p\|$ exists, $x_n \to y \in F = F(T)$ thereby completing the proof. $\square$

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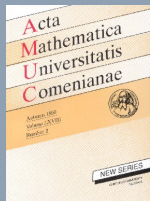
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Yongfu Su, Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China,
e-mail: `qxlxajh@163.com`

Xiaolong Qin, Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China

Meijuan Shang, Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China and Department of Mathematics, Shijiazhuang University, Shijiazhuang 300160, China