Acta Math. Univ. Comenianae
Vol. LXXVII, 2(2008), pp. 245–253

ITERATIVE SOLUTIONS OF NONLINEAR EQUATIONS WITH ϕ-STRONGLY ACCRETIVE OPERATORS

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ABSTRACT. Suppose that X is an arbitrary real Banach space and $T : X \to X$ is a Lipschitz continuous ϕ-strongly accretive operator or uniformly continuous ϕ-strongly accretive operator. We prove that under different conditions the three--step iteration methods with errors converge strongly to the solution of the equation $Tx = f$ for a given $f \in X$.

1. INTRODUCTION

Let X be a real Banach space with norm $\|\cdot\|$ and dual X^*, and J denote the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \{f \in X^* : \|f\|^2 = \|x\|^2 = \langle x, f\rangle\}, \qquad x \in X,$$

where $\langle\cdot,\cdot\rangle$ is the generalized duality pairing. In this paper, I denotes the identity operator on X, R^+ and $\delta(K)$ denote the set of nonnegative real numbers and the diameter of K for any $K \subseteq X$, respectively. An operator T with domain $D(T)$ and range $R(T)$ in X is called *ϕ-strongly accretive* if there exists a strictly increasing function $\phi : R^+ \to R^+$ with $\phi(0) = 0$ such that for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|. \tag{1.1}$$

If there exists a positive constant $k > 0$ such that (1.1) holds with $\phi(\|x-y\|)$ replaced by $k\|x-y\|$, then T is called *strongly accretive*. The accretive operators were introduced independently in 1967 by Browder [**1**] and Kato [**8**]. An early fundamental result in the theory of accretive operator, due to Browder, states the initial value problem

$$\frac{du}{dt} + Tu = 0, \quad u(0) = u_0 \tag{1.2}$$

is solvable if T is locally Lipschitz and accretive on X. Martin [**11**] proved that if $T : X \to X$ is strongly accretive and continuous, then T is subjective so that the

Received April 9, 2007.
2000 *Mathematics Subject Classification.* Primary 47H05, 47H10, 47H15.
Key words and phrases. ϕ-strongly accretive operators; three-step iteration method with errors; Banach spaces.

equation

$$Tx = f \tag{1.3}$$

has a solution for any given $f \in X$. Using the Mann and Ishikawa iteration methods with errors, Chang [**3**], Chidume [**4**], [**5**], Ding [**7**], Liu and Kang [**10**] and Osilike [**12**], [**13**] obtained a few convergence theorems for Lipschitz ϕ-strongly accretive operators. Chang [**2**] and Yin, Liu and Lee [**16**] also got some convergence theorems for uniformly continuous ϕ-strongly accretive operators.

The purpose of this paper is to study the three-step iterative approximation of solution to equation (1.3) in the case when T is a Lipschitz ϕ-strongly accretive operator and X is a real Banach space. We also show that if $T : X \to X$ is a uniformly continuous ϕ-strongly accretive operator, then the three-step iteration method with errors converges strongly to the solution of equation (1.3). Our results generalize, improve the known results in [**2**]–[**7**], [**10**], [**12**], [**13**] and [**15**].

2. Preliminaries

The following Lemmas play a crucial role in the proofs of our main results.

Lemma 2.1 ([**7**]). *Suppose that $\phi : R^+ \to R^+$ is a strictly increasing function with $\phi(0) = 0$. Assume that $\{r_n\}_{n=0}^\infty$, $\{s_n\}_{n=0}^\infty$, $\{k_n\}_{n=0}^\infty$ and $\{t_n\}_{n=0}^\infty$ are sequences of nonnegative numbers satisfying the following conditions:*

$$\sum_{n=0}^{\infty} k_n < \infty, \quad \sum_{n=0}^{\infty} t_n < \infty, \quad \sum_{n=0}^{\infty} s_n = \infty \tag{2.1}$$

and

$$r_{n+1} \le (1 + k_n) r_n - s_n r_n \frac{\phi(r_{n+1})}{1 + r_{n+1} + \phi(r_{n+1})} + t_n \qquad \text{for } n \ge 0. \tag{2.2}$$

Then $\lim_{n\to\infty} r_n = 0$.

Lemma 2.2 ([**10**]). *Suppose that X is an arbitrary Banach space and $T : X \to X$ is a continuous ϕ-strongly accretive operator. Then the equation $Tx = f$ has a unique solution for any $f \in X$.*

Lemma 2.3 ([**9**]). *Let $\{\alpha_n\}_{n=0}^\infty, \{\beta_n\}_{n=0}^\infty$ and $\{\gamma_n\}_{n=0}^\infty$ be three nonnegative real sequences satisfying the inequality*

$$\alpha_{n+1} \le (1 - \omega_n)\alpha_n + \omega_n \beta_n + \gamma_n \qquad \text{for } n \ge 0,$$

where $\{\omega_n\}_{n=0}^\infty \subset [0, 1]$, $\sum_{n=0}^\infty \omega_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^\infty \gamma_n < \infty$. Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main Results

Theorem 3.1. *Suppose that X is an arbitrary real Banach space and $T : X \to X$ is a Lipschitz ϕ-strongly accretive operator. Assume that $\{u_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$,*

$\{w_n\}_{n=0}^{\infty}$ are sequences in X and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ such that $\{\|w_n\|\}_{n=0}^{\infty}$ is bounded and

$$\sum_{n=0}^{\infty} a_n^2 < \infty, \quad \sum_{n=0}^{\infty} a_n b_n < \infty, \quad \sum_{n=0}^{\infty} \|u_n\| < \infty, \quad \sum_{n=0}^{\infty} \|v_n\| < \infty, \tag{3.1}$$

$$\sum_{n=0}^{\infty} a_n = \infty. \tag{3.2}$$

For any given $f \in X$, define $S : X \to X$ by $Sx = f + x - Tx$ for all $x \in X$. Then the three-step iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ defined for arbitrary $x_0 \in X$ by

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n S x_n + w_n, \\ y_n &= (1-b_n)x_n + b_n S z_n + v_n, \\ x_{n+1} &= (1-a_n)x_n + a_n S y_n + u_n, \qquad n \geq 0 \end{aligned} \tag{3.3}$$

converges strongly to the unique solution q of the equation $Tx = f$. Moreover

$$\begin{aligned} \|x_{n+1} - q\| \leq {} & [1 + (3 + 3L^3 + L^4)a_n^2 + L(1+L^2)a_n b_n]\|x_n - q\| \\ & - A(x_{n+1}, q)a_n\|x_n - q\| + a_n b_n L^2(3+L)\|w_n\| \\ & + a_n L(3+L)\|v_n\| + (3+L)\|u_n\| \end{aligned} \tag{3.4}$$

for $n \geq 0$, where $A(x,y) = \frac{\phi(\|x-y\|)}{1+\|x-y\|+\phi(\|x-y\|)} \in [0,1)$ for $x, y \in X$.

Proof. It follows from Lemma 2.2 that the equation $Tx = f$ has a unique solution $q \in X$. Let L' denote the Lipschitz constant of T. From the definition of S we know that q is a fixed point of S and S is also Lipschitz with constant $L = 1 + L'$. Thus for any $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle (I-S)x - (I-S)y, j(x-y) \rangle \geq A(x,y)\|x-y\|^2.$$

This implies that

$$\langle (I - S - A(x,y))x - (I - S - A(x,y))y, j(x-y) \rangle \geq 0$$

and it follows from Lemma 1.1 of Kato [**8**] that

$$\|x - y\| \leq \|x - y + r[(I - S - A(x,y))x - (I - S - A(x,y))y]\| \tag{3.5}$$

for $x, y \in X$ and $r > 0$. From (3.3) we conclude that for each $n \geq 0$

$$\begin{aligned} x_n &= x_{n+1} + a_n x_n - a_n S y_n - u_n \\ &= (1+a_n)x_{n+1} + a_n(I - S - A(x_{n+1},q))x_{n+1} - (I - A(x_{n+1},q))a_n x_n \\ &\quad + a_n(Sx_{n+1} - Sy_n) + (2 - A(x_{n+1},q))a_n^2(x_n - Sy_n) \\ &\quad - [1 + (2 - A(x_{n+1},q))a_n]u_n \end{aligned} \tag{3.6}$$

and

$$q = (1+a_n)q + a_n(I - S - A(x_{n+1},q))q - (I - A(x_{n+1},q))a_n q. \tag{3.7}$$

It follows from (3.5)–(3.7) that

$$\begin{aligned}
&\|x_n - q\| \\
&= \|(1+a_n)x_{n+1} + a_n(I - S - A(x_{n+1}, q))x_{n+1} - (I - A(x_{n+1}, q))a_n x_n \\
&\quad + a_n(Sx_{n+1} - Sy_n) + (2 - A(x_{n+1}, q))a_n^2(x_n - Sy_n) \\
&\quad - [1 + (2 - A(x_{n+1}, q))a_n]u_n - (1+a_n)q - a_n(I - S - A(x_{n+1}, q))q \\
&\quad + (I - A(x_{n+1}, q))a_n q\| \\
&\geq (1+a_n)\left\| x_{n+1} - q + \frac{a_n}{1+a_n}[(I - S - A(x_{n+1}, q))x_{n+1} \right. \\
&\quad \left. - (I - S - A(x_{n+1}, q))q \right\| - a_n(1 - A(x_{n+1}, q))\|x_n - q\| \\
&\quad - (2 - A(x_{n+1}, q))a_n^2\|x_n - Sy_n\| - a_n\|Sx_{n+1} - Sy_n\| \\
&\quad - [1 + (2 - A(x_{n+1}, q))a_n]\|u_n\| \\
&\geq (1+a_n)\|x_{n+1} - q\| - a_n(1 - A(x_{n+1}, q))\|x_n - q\| \\
&\quad - (2 - A(x_{n+1}, q))a_n^2\|x_n - Sy_n\| - a_n\|Sx_{n+1} - Sy_n\| \\
&\quad - [1 + (2 - A(x_{n+1}, q))a_n]\|u_n\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_{n+1} - q\| \\
&\leq \frac{1 + (1 - A(x_{n+1}, q))a_n}{1+a_n}\|x_n - q\| + (2 - A(x_{n+1}, q))a_n^2\|x_n - Sy_n\| \\
&\quad + a_n\|Sx_{n+1} - Sy_n\| + [1 + (2 - A(x_{n+1}, q))a_n]\|u_n\| \\
&\leq (1 - A(x_{n+1}, q)a_n + a_n^2)\|x_n - q\| + 2a_n^2\|x_n - Sy_n\| \\
&\quad + a_n\|Sx_{n+1} - Sy_n\| + (1 + 2a_n)\|u_n\|
\end{aligned} \tag{3.8}$$

for $n \geq 0$. By (3.3) we get that

$$\begin{aligned}
\|z_n - q\| &\leq (1 - c_n)\|x_n - q\| + c_n\|Sx_n - q\| + \|w_n\| \\
&\leq (1 - c_n)\|x_n - q\| + Lc_n\|x_n - q\| + \|w_n\| \\
&\leq L\|x_n - q\| + \|w_n\|,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\|y_n - q\| &\leq (1 - b_n)\|x_n - q\| + b_n\|Sz_n - q\| + \|v_n\| \\
&\leq (1 - b_n)\|x_n - q\| + Lb_n\|z_n - q\| + \|v_n\|,
\end{aligned} \tag{3.10}$$

$$\|x_n - Sz_n\| \leq \|x_n - q\| + \|Sz_n - q\| \leq \|x_n - q\| + L\|z_n - q\|, \tag{3.11}$$

$$\|x_n - y_n\| \leq b_n\|x_n - Sz_n\| + \|v_n\| \tag{3.12}$$

and

$$\|Sy_n - y_n\| \leq \|Sy_n - q\| + \|y_n - q\| \leq (1+L)\|y_n - q\| \tag{3.13}$$

for $n \geq 0$. From (3.9)–(3.13) we obtain that

$$\|x_n - Sy_n\| \leq (1 + L^3)\|x_n - q\| + L^2 b_n\|w_n\| + L\|v_n\| \tag{3.14}$$

and

$$\begin{aligned}\|Sx_{n+1} - Sy_n\| \le (Lb_n + L^3b_n - La_nb_n - L^3a_nb_n + L^3a_n + L^4a_n)\|x_n - q\| \\ + (L^2b_n + L^3a_nb_n)\|w_n\| + (L + L^2a_n)\|v_n\| + L\|u_n\|\end{aligned} \tag{3.15}$$

for $n \ge 0$. It follows from (3.8), (3.14) and (3.15) that

$$\begin{aligned}\|x_{n+1} - q\| \le [1 + (3 + 3L^3 + L^4)a_n^2 + L(1 + L^2)a_nb_n]\|x_n - q\| \\ - A(x_{n+1}, q)a_n\|x_n - q\| + a_nb_nL^2(3 + L)\|w_n\| \\ + (3 + L)a_n\|v_n\| + (3 + L)\|u_n\|\end{aligned} \tag{3.16}$$

for $n \ge 0$. Set

$$r_n = \|x_n - q\|, \quad k_n = (3 + 3L^3 + L^4)a_n^2 + L(1 + L^2)a_nb_n, \quad s_n = a_n,$$
$$t_n = a_nb_nL^2(3 + L)\|w_n\| + a_nL(3 + L)\|v_n\| + (3 + L)\|u_n\| \quad \text{for } n \ge 0.$$

Then (3.16) yields that

$$r_{n+1} \le (1 + k_n)r_n - s_nr_n\frac{\phi(r_{n+1})}{1 + r_{n+1} + \phi(r_{n+1})} + t_n \quad \text{for } n \ge 0. \tag{3.17}$$

It follows from (3.1), (3.2), (3.17) and Lemma 2.1 that $r_n \to 0$ as $n \to \infty$. That is $x_n \to q$ as $n \to \infty$. This completes the proof. □

Remark 3.2. Theorem 3.1 extends Theorem 5.2 of [**3**], Theorem 1 of [**4**], Theorem 2 of [**5**], Theorem 1 of [**6**], Theorem 3.1 of [**10**], Theorem 1 of [**12**], Theorem 1 of [**13**] and Theorem 4.1 of [**15**].

Theorem 3.3. *Let* X, $\{u_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$, $\{w_n\}_{n=0}^\infty$, $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$ *and* $\{c_n\}_{n=0}^\infty$ *be as in Theorem 3.1 and* $T : D(T) \subset X \to X$ *be a Lipschitz* ϕ*-strongly accretive operator. Suppose that the equation* $Tx = f$ *has a solution* $q \in D(T)$ *for some* $f \in X$. *Assume that the sequences* $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$ *and* $\{z_n\}_{n=0}^\infty$ *generated from an arbitrary* $x_0 \in D(T)$ *by* (3.3) *are contained in* $D(T)$. *Then* $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$ *and* $\{z_n\}_{n=0}^\infty$ *converge strongly to* q *and satisfied* (3.4).

The proof of Theorem 3.3 uses the same idea as that of Theorem 3.1. So we omit it.

Remark 3.4. Theorem 3.1 in [**7**] and Theorem 3.2 in [**10**] are special cases of our Theorem 3.3.

Theorem 3.5. *Suppose that* X *is an arbitrary real Banach space and* $T : X \to X$ *is a uniformly continuous* ϕ*-strongly accretive operator, and the range of either* $(I - T)$ *or* T *is bounded. For any* $f \in X$, *define* $S : X \to X$ *by* $Sx = f + x - Tx$ *for all* $x \in X$ *and the three-step iteration sequence with errors* $\{x_n\}_{n=0}^\infty$ *by*

$$\begin{aligned}&x_0, u_0, v_0, w_0 \in X, \\ &z_n = a''_nx_n + b''_nSx_n + c''_nw_n, \\ &y_n = a'_nx_n + b'_nSz_n + c'_nv_n, \\ &x_{n+1} = a_nx_n + b_nSy_n + c_nu_n, \quad n \ge 0,\end{aligned} \tag{3.18}$$

where $\{u_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$ *and* $\{w_n\}_{n=0}^\infty$ *are arbitrary bounded sequences in* X *and* $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$, $\{c_n\}_{n=0}^\infty$, $\{a'_n\}_{n=0}^\infty$, $\{b'_n\}_{n=0}^\infty$, $\{c'_n\}_{n=0}^\infty$, $\{a''_n\}_{n=0}^\infty$, $\{b''_n\}_{n=0}^\infty$ *and* $\{c''_n\}_{n=0}^\infty$ *are real sequences in* $[0,1]$ *satisfying the following conditions*

$$\begin{aligned} &a_n + b_n + c_n = 1, \quad a'_n + b'_n + c'_n = 1, \\ &a''_n + b''_n + c''_n = 1, \quad b_n + c_n \in (0,1), \quad n \geq 0, \end{aligned} \tag{3.19}$$

$$\sum_{n=0}^{\infty} b_n = +\infty, \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = \lim_{n\to\infty} \frac{c_n}{b_n + c_n} = 0. \tag{3.20}$$

Then the sequence $\{x_n\}_{n=0}^\infty$ *converges strongly to the unique solution of the equation* $Tx = f$.

Proof. It follows from Lemma 2.2 that the equation $Tx = f$ has a unique solution $q \in X$. By (1.2) we have

$$\langle Tx - Ty, j(x-y)\rangle = \langle (I-S)x - (I-S)y, j(x-y)\rangle \geq A(x,y)\|x-y\|^2,$$

where $A(x,y) = \frac{\phi(\|x-y\|)}{1+\|x-y\|+\phi(\|x-y\|)} \in [0,1)$ for $x, y \in X$. This implies that

$$\langle (I - S - A(x,y))x - (I - S - A(x,y))y, j(x-y)\rangle \geq 0$$

for $x, y \in X$. It follows from Lemma 1.1 of Kato [**8**] that

$$\|x - y\| \leq \|x - y + r[(I - S - A(x,y))x - (I - S - A(x,y))y]\| \tag{3.21}$$

for $x, y \in X$ and $r > 0$. Now we show that $R(S)$ is bounded. If $R(I-T)$ is bounded, then

$$\|Sx - Sy\| = \|(I-T)x - (I-T)y\| \leq \delta(R(I-T))$$

for $x, y \in X$. If $R(T)$ is bounded, we get that

$$\begin{aligned} \|Sx - Sy\| &= \|(x-y) - (Tx - Ty)\| \\ &\leq \phi^{-1}(\|Tx - Ty\|) + \|Tx - Ty\| \\ &\leq \phi^{-1}(\delta(R(T))) + \delta(R(T)) \end{aligned}$$

for $x, y \in X$. Hence $R(S)$ is bounded. Put

$$d_n = b_n + c_n, \qquad d'_n = b'_n + c'_n, \qquad d''_n = b''_n + c''_n \qquad \text{for } n \geq 0$$

and

$$\begin{aligned} D = \max\{&\|x_0 - q\|, \\ &\sup\{\|x - q\| : x \in \{u_n, v_n, w_n, Sx_n, Sy_n, Sz_n : n \geq 0\}\}\}. \end{aligned} \tag{3.22}$$

By (3.18) and (3.22) we conclude that

$$\max\{\|x_n - q\|, \|y_n - q\|, \|z_n - q\|\} \leq D \qquad \text{for } n \geq 0. \tag{3.23}$$

Using (3.18) we obtain that

$$\begin{aligned} (1 - d_n)x_n &= x_{n+1} - d_n S y_n - c_n(u_n - Sy_n) \\ &= [1 - (1 - A(x_{n+1}, q))d_n]x_{n+1} + d_n(I - S - A(x_{n+1}, q))x_{n+1} \\ &\quad + d_n(Sx_{n+1} - Sy_n) - c_n(u_n - Sy_n). \end{aligned} \tag{3.24}$$

Note that

$$(1 - d_n)q = [1 - (1 - A(x_{n+1}, q))d_n]q + d_n(I - S - A(x_{n+1}, q))q. \tag{3.25}$$

It follows from (3.21) and (3.23)–(3.25) that

$$\begin{aligned}
&(1 - d_n)\|x_n - q\| \\
&\geq [1 - (1 - A(x_{n+1}, q))d_n]\|x_{n+1} - q \\
&\quad + \frac{d_n}{1 - (1 - A(x_{n+1}, q))d_n}[(I - S - A(x_{n+1}, q))x_{n+1} \\
&\quad - (I - S - A(x_{n+1}, q))q]\| - d_n\|Sx_{n+1} - Sy_n\| - c_n\|u_n - Sy_n\| \\
&\geq [1 - (1 - A(x_{n+1}, q))d_n]\|x_{n+1} - q\| - d_n\|Sx_{n+1} - Sy_n\| - 2Dc_n.
\end{aligned}$$

That is

$$\begin{aligned}
&\|x_{n+1} - q\| \\
&\leq \frac{1 - d_n}{1 - (1 - A(x_{n+1}, q))d_n}\|x_n - q\| \\
&\quad + \frac{d_n}{1 - (1 - A(x_{n+1}, q))d_n}\|Sx_{n+1} - Sy_n\| + \frac{2Dc_n}{1 - (1 - A(x_{n+1}, q))d_n} \\
&\leq [1 - (1 - A(x_{n+1}, q))d_n]\|x_n - q\| + Md_n\|Sx_{n+1} - Sy_n\| + Mc_n
\end{aligned} \tag{3.26}$$

for $n \geq 0$, where M is some constant. In view of (3.18)–(3.20) we infer that

$$\begin{aligned}
\|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \\
&\leq b_n\|Sy_n - x_n\| + c_n\|u_n - x_n\| + b'_n\|Sz_n - x_n\| + c'_n\|v_n - x_n\| \\
&\leq b_n\|Sy_n - x_n\| + c_n\|u_n - x_n\| + b'_n\|Sz_n - z_n\| + c'_n\|v_n - x_n\| \\
&\quad + b'_n(b''_n\|Sx_n - x_n\| + c''_n\|w_n - x_n\|) \\
&\leq 2D(d_n + d'_n + b'_n d''_n) \to 0
\end{aligned}$$

as $n \to \infty$. Since S is uniformly continuous, we have

$$\|Sx_{n+1} - Sy_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.27}$$

Set $\inf\{A(x_{n+1}, q) : n \geq 0\} = r$. We claim that $r = 0$. If not, then $r > 0$. It is easy to check that

$$\|x_{n+1} - q\| \leq (1 - rd_n)\|x_n - q\| + Md_n\|Sx_{n+1} - Sy_n\| + Mc_n \quad \text{for } n \geq 0.$$

Put

$$\begin{aligned}
&c_n = t_n d_n, \quad \alpha_n = \|x_n - q\|, \quad \omega_n = rd_n, \\
&\beta_n = Mr^{-1}(\|Sx_{n+1} - Sy_n\| + t_n), \quad \gamma_n = 0 \quad \text{for } n \geq 0.
\end{aligned}$$

(3.2) ensures that $t_n \to 0$ as $n \to \infty$. It follows from (3.20), (3.27) and Lemma 2.3 that $\omega_n \in (0, 1]$ with $\sum_{n=0}^{\infty} \omega_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \gamma_n < \infty$. So $\|x_n - q\| \to 0$ as $n \to \infty$, which means that $r = 0$. This is a contradiction. Thus $r = 0$ and there exists a subsequence $\{\|x_{n_i+1} - q\|\}_{i=0}^{\infty}$ of $\{\|x_{n+1} - q\|\}_{n=0}^{\infty}$ satisfying

$$\|x_{n_i+1} - q\| \to 0 \quad \text{as } i \to \infty. \tag{3.28}$$

From (3.28) and (3.29) we conclude that for given $\varepsilon > 0$ there exists a positive integer m such that for $n \geq m$,

$$\|x_{n_m+1} - q\| < \varepsilon \tag{3.29}$$

and

$$M\|Sx_{n+1} - Sy_n\| + M\frac{c_n}{d_n} < \min\left\{\frac{1}{2}\varepsilon, \frac{\phi(\varepsilon)\varepsilon}{1+\phi(\frac{3}{2}\varepsilon)+\frac{3}{2}\varepsilon}\right\}. \tag{3.30}$$

Now we claim that

$$\|x_{n_m+j} - q\| < \varepsilon \quad \text{for } j \geq 1. \tag{3.31}$$

In fact (3.29) means that (3.31) holds for $j = 1$. Assume that (3.31) holds for $j = k$. If $\|x_{n_m+k+1} - q\| > \varepsilon$, we get that

$$\begin{aligned} &\|x_{n_m+k+1} - q\| \\ &\leq \|x_{n_m+k} - q\| + Md_{n_m+k}\|Sx_{n_m+k+1} - Sy_{n_m+k}\| + Mc_{n_m+k} \\ &\leq \varepsilon + \min\left\{\frac{1}{2}\varepsilon, \frac{\phi(\varepsilon)\varepsilon}{1+\phi(\frac{3}{2}\varepsilon)+\frac{3}{2}\varepsilon}\right\}d_{n_m+k} \\ &\leq \frac{3}{2}\varepsilon. \end{aligned} \tag{3.32}$$

Note that $\phi(\|x_{n_m+k+1} - q\|) > \phi(\varepsilon)$. From (3.32) we get that

$$A(x_{n_m+k+1}, q) \geq \frac{\phi(\varepsilon)}{1+\phi(\frac{3}{2}\varepsilon)+\frac{3}{2}\varepsilon}. \tag{3.33}$$

By virtue of (3.26) (3.30) and (3.33) we obtain that

$$\begin{aligned} &\|x_{n_m+k+1} - q\| \\ &\leq \left(1 - \frac{\phi(\varepsilon)\varepsilon}{1+\phi(\frac{3}{2}\varepsilon)+\frac{3}{2}\varepsilon}d_{n_m+k}\right)\|x_{n_m+k} - q\| \\ &\quad + Md_{n_m+k}\|Sx_{n_m+k+1} - Sy_{n_m+k}\| + Mc_{n_m+k} \\ &\leq \left(1 - \frac{\phi(\varepsilon)\varepsilon}{1+\phi(\frac{3}{2}\varepsilon)+\frac{3}{2}\varepsilon}d_{n_m+k}\right)\varepsilon + \min\left\{\frac{1}{2}\varepsilon, \frac{\phi(\varepsilon)\varepsilon}{1+\phi(\frac{3}{2}\varepsilon)+\frac{3}{2}\varepsilon}\right\}d_{n_m+k} \\ &\leq \varepsilon. \end{aligned}$$

That is

$$\varepsilon < \|x_{n_m+k+1} - q\| \leq \varepsilon,$$

which is a contradiction. Hence $\|x_{n_m+k+1} - q\| \leq \varepsilon$. By induction (3.29) holds for $j \geq 1$. Thus (3.31) yields that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Remark 3.6. Theorem 3.5 extends and improves Theorem 3.4 in [**2**] and Theorem 3.1 in [**16**].

Acknowledgement. This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (20060467).

References

1. Browder F. E., *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875–882.

2. Chang S. S., *Some problems and results in the study of nonlinear analysis*, Nonlinear Anal. **30** (1997), 4197–4208.

3. Chang S. S., ChoY. J., Lee B. S. and S. M. Kang, *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces*, J. Math. Anal. Appl. **224** (1998) 149–165.

4. Chidume C. E., *An iterative process for nonlinear Lipschitzian strongly accretive mapping in L_p spaces*, J. Math. Anal. Appl. **151** (1990), 453–461.

5. ______, *Iterative solution of nonlinear equations with strongly accretive operators*, J. Math. Anal. Appl. **192** (1995), 502–518.

6. Deng L., *On Chidume's open questions*, J. Math. Anal. Appl. **174** (1993), 441–449.

7. Ding X. P., *Iterative process with errors to nonlinear ϕ-strongly accretive operator equations in arbitrary Banach spaces*, Computers Math. Applic. **33** (1997), 75–82.

8. Kato T., *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508–520.

9. Liu L. S., *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mapping in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.

10. Liu Z. and Kang S. M. *Convergence theorems for ϕ-strongly accretive and ϕ-hemicontractive operators*, J. Math. Anal. Appl. **253** (2001), 35–49.

11. Martin R. H., Jr. *A global existence theorem for autonomous differential equations in Banach spaces*, Proc. Amer. Math. Soc. **26** (1970), 307–314.

12. Osilike M. O., *Iterative solution of nonlinear equations of the ϕ-strongly accretive type*, J. Math. Anal. Appl. **200** (1996), 259–271.

13. ______, *Ishikawa and Mann iteration methods with errors for nonlinear equations of the accretive type*, J. Math. Anal. Appl. **213** (1997), 91–105.

14. ______, *Iterative solution of nonlinear ϕ-strongly accretive operator equations in arbitrary Banach spaces*, Nonlinear Anal. TMA **36** (1999), 1–9.

15. Tan K. K. and Xu H. K., *Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces*, J. Math. Anal. Appl. **178** (1993), 9–21.

16. Yin Q., Liu Z. and Lee B. S., *Iterative solutions of nonlinear equations with ϕ-strongly accretive operators*, Nonlinear Anal. Forum **5** (2000), 87–89.

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