Acta Math. Univ. Comenianae
Vol. LXXVII, 2 (2008), pp. 167–174

DISTRIBUTIVE, STANDARD AND NEUTRAL ELEMENTS IN TRELLISES

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Abstract. In this paper, the concepts of distributive, standard and neutral elements introduced in lattices by O. Ore, G. Grätzer and G. Birkhoff, respectively, have been extended to trellises (also called weakly associative lattices) and some of their analogous characterizations are obtained. Also, the concept of a normal trellis is introduced as a generalization of a lattice and it is proved that an element d of a normal trellis L is neutral if and only if for any $x, y \in L$, the elements d, x, y generate a distributive subtrellis of L.

1. Introduction

Any reflexive and antisymmetric binary relation $\trianglelefteq$ on a set A is called a *pseudo-order* on A and $\langle A\ ;\ \trianglelefteq\rangle$ is called a *pseudo-ordered set* or a *psoset*. For $a, b \in A$ if $a \trianglelefteq b$ and $a \neq b$, then we write $a \triangleleft b$. For a subset B of A, the notions of a *lower bound*, an *upper bound, the greatest lower bound* (g.l.b or meet), *the least upper bound* (l.u.b or join) are defined analogous to the corresponding notions in a poset.

Any psoset can be regarded as a digraph (possibly infinite) in which for any pair of distinct points u and v either there is no directed line between u and v, or if there is a directed line from u to v, there is no directed line from v to u. The digraph in Figure 1 represents the psoset $A = \{0, a, b, c, 1\}$ with $0 \triangleleft a \triangleleft b \triangleleft c \triangleleft 1, 0 \triangleleft x \triangleleft 1$ for every $x \in \{a, b, c\}$ and $0 \triangleleft 1$ while a and c are noncomparable.

Define a relation $\sqsubseteq_B$ on a subset B of a psoset $\langle A; \trianglelefteq\rangle$ by setting $b \sqsubseteq_B b'$ for two elements b and b' of B if and only if there is a directed path in B from b to b' say $b = b_0 \trianglelefteq b_1 \trianglelefteq \cdots \trianglelefteq b_n = b'$ for some $n \geq 0$. The relation $\sqsupseteq_B$ is defined dually. If for each pair of elements b and b' of B at least one of the relations $b \sqsubseteq_B b'$ or $b' \sqsubseteq_B b$ holds, then B will be called a pseudo-chain or a p-chain. If for each pair of elements b and b' of B both the relations $b \sqsubseteq_B b'$ and $b' \sqsubseteq_B b$ hold, then B will be called a cycle. The empty set and a single element set in a psoset are cycles. A nontrivial cycle contains at least three elements. A psoset is said to be acyclic if it does not contain any nontrivial cycle.

Received May 24, 2006; revised April 17, 2008.
2000 *Mathematics Subject Classification.* Primary 06B05.
Key words and phrases. Pseudo-ordered set; trellis; distributive element; standard element; neutral element; normal trellis.

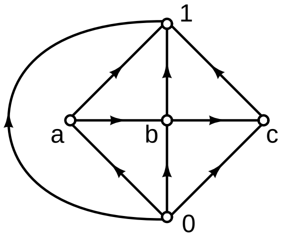


Figure 1

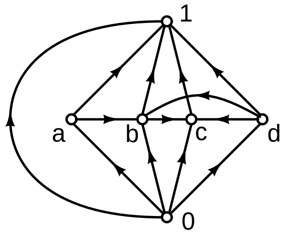


Figure 2

A psoset $\langle A\ ;\ \trianglelefteq\rangle$ in which every pair of elements has a l.u.b and a g.l.b is called a trellis. In other words, a trellis is an algebra $\langle L\ ;\ \wedge;\vee\rangle$ where the binary operations $\wedge$ and $\vee$ satisfy the following properties:

(i) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for all $a, b \in A$

(ii) $a \vee (b \wedge a) = a \wedge (b \vee a)$ for all $a, b \in A$

(iii) $a \vee ((a \wedge b) \vee (a \wedge c)) = a = a \wedge ((a \vee b) \wedge (a \vee c))$ for all $a, b, c \in A$.

The notions of a psoset and a trellis are due to [**3**] and [**8**]. By a join (meet)-semitrellis we mean a psoset, any two of whose elements have a l.u.b (g.l.b). A subtrellis S of a trellis $\langle L;\ \wedge;\vee\rangle$ is a nonempty subset of L such that $a, b \in S$ implies $a \wedge b, a \vee b$ belong to S. An ideal I of a trellis L is a subtrellis of L such that $i \in I$ and $a \in L$ imply that $a \wedge i \in I$ or equivalently $i \in I, a \in L$ and $a \trianglelefteq i$ imply that $a \in I$. A dual ideal or a filter of a trellis is defined dually. In [**8**], the empty set is also regarded as an ideal of a trellis. If B is a nonempty subset of a trellis L, then the ideal generated by B is the intersection of all ideals of L containing B and it is denoted by $(B]$. An ideal generated by a single element a is called the principal ideal generated by a and is denoted by $(a]$. The dual notions $[B)$ and $[a)$ are defined dually. As in the case of a lattice, the set of all ideals of a trellis L forms a lattice with respect to set inclusion and it is denoted by $I(L)$.

2. On distributive elements in trellises

The following definitions are due to [**8**]. An element d of a trellis L is said to be $\vee -$ *associative* if $d \vee (x \vee y) = (d \vee x) \vee y$ for all $x, y \in L$. $\wedge$-associativity of an element will be defined dually. An element d of a trellis L is said to be

(i) *left transitive* if $x \trianglelefteq y \trianglelefteq d$ imply $x \trianglelefteq d$

(ii) *right transitive* if $d \trianglelefteq x \trianglelefteq y$ imply $d \trianglelefteq y$

(iii) *middle transitive* if $x \trianglelefteq d \trianglelefteq y$ imply $x \trianglelefteq y$ for $x, y \in L$.

d is said to be *transitive* if it is left, right and middle transitive.

In the following, we introduce the notion of a weakly $\vee$-associative (weakly $\wedge$-associative) element in a trellis.

Definition 2.1. An element d of a trellis L is called *weakly* $\vee$*-associative* if $d \vee (x \vee y) = (d \vee x) \vee (d \vee y)$ for all $x, y \in L$.

Weak $\wedge$-associativity of d is defined dually.

Remark 2.2. In a trellis L,

1. a $\vee$-associative ($\wedge$-associative) element is weakly $\vee$-associative (weakly $\wedge$-associative) whereas the converse is not true. For, the element a in the trellis L of Figure 1 satisfies $a \vee (x \vee y) = (a \vee x) \vee (a \vee y)$ for all $x, y \in L$ but it is not $\vee$-associative as $a \vee (b \vee c) = 1 \neq c = (a \vee b) \vee c$.
2. a $\vee$-associative element is transitive whereas a transitive element need not be associative [**8**].
3. transitivity and weak $\vee$-associativity (weak $\wedge$-associativity) of an element are independent. For, in the trellis of Figure 1, the element a is weakly $\vee$-associative but not transitive whereas in the trellis of Figure 2, d is transitive but not weakly $\vee$-associative as $d \vee (a \vee c) = 1 \neq c = (d \vee a) \vee (d \vee c)$.
4. an element d is left transitive if and only if $(d] = \{x \in L | x \trianglelefteq d\}$.

In the following we introduce the notion of a distributive element in a trellis.

Definition 2.3. An element d of a trellis L is said to be *distributive* if

(i) d is $\vee$-associative and
(ii) $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$ for all $x, y \in L$.

A dually distributive element is defined dually.

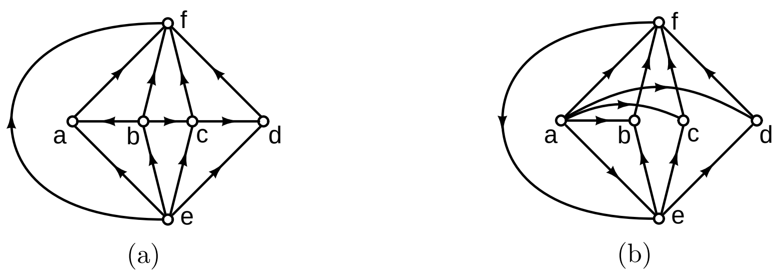
Remark 2.4. (i) and (ii) in Definition 2.3 are independent. For, every element d of a lattice L is associative whereas it is not necessary that an element d of L should satisfy (ii) of Definition 2.3. On the other hand, in the trellis L of Figure 1, a satisfies (ii) of the definition but not (i).

Examples.

1. The least element and the greatest element of a trellis are distributive.
2. Element a in the trellis of Figure 3(a) and Figure 3(b) is distributive.

The following three definitions are as in the case of lattices.

Definition 2.5. Let L, K be two trellises. For a homomorphism $F : L \to K$ (not necessarily onto), the relation Θ on L defined by $x \equiv y(\Theta)$ if and only if $f(x) = f(y)$ is called the *congruence kernel of the homomorphism* f and is denoted by $\ker(f)$.



(a) (b)

Figure 3

Definition 2.6. Let L be a trellis and Θ be a congruence relation on L. If the quotient trellis L/Θ has a zero, $[a]\Theta$, then $[a]\Theta$ as a subset of L is an ideal, called the *ideal kernel* of the congruence relation Θ.

Definition 2.7. Let H be a nonempty subset of a trellis L and $\Theta[H]$ denote the smallest congruence relation of L under which any two elements a, b of H are congruent. Then $\Theta[H]$ is called the *congruence relation generated by* H.

The following theorem is a generalization of the corresponding result in lattices (see [**4**]).

Theorem 2.8. *For an element d of a trellis L, the following statements are equivalent.*

(i) *d is distributive.*
(ii) *d is $\vee$-associative and the map $f : x \to d \vee x$ is a homomorphism of L onto $[d)$.*
(iii) *d is $\vee$-associative and the binary relation Θ_d on L defined by, for $x, y \in L$, $x \equiv y(\Theta_d)$ if and only if $d \vee x = d \vee y$, is a congruence relation.*

Proof. (i) $\Rightarrow$ (ii): By (i), d is $\vee$-associative. Now for any $x \in L, f(x) = d \vee x \in [d)$. Let $x, y \in L$. Then $f(x \wedge y) = d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y) = f(x) \wedge f(y)$. Also, $f(x \vee y) = d \vee (x \vee y) = (d \vee x) \vee (d \vee y) = f(x) \vee f(y)$.

f is onto: For, if $y \in [d)$, then $y \in L$ and $y \trianglerighteq d$ by the dual of (4) of Remark 2.2. Then $f(y) = d \vee y = y$. Hence (ii) holds.

(ii) $\Rightarrow$ (iii): By (ii), d is $\vee$-associative. We have for $x, y \in L, x \equiv y(\Theta_d)$ if and only if $d \vee x = d \vee y$. In other words, $x \equiv y(\Theta_d)$ if and only if $f(x) = f(y)$. Thus $\Theta_d = \ker(f)$, the congruence kernel of the homomorphism f. Hence (iii) holds.

(iii) $\Rightarrow$ (i): Let $x, y \in L$. We have $x \equiv d \vee x(\Theta_d)$ since $d \vee x = d \vee (d \vee x)$. Similarly, $y \equiv d \vee y(\Theta_d)$. But then $x \wedge y \equiv (d \vee x) \wedge (d \vee y)(\Theta_d)$, which implies, $d \vee (x \wedge y) = d \vee ((d \vee x) \wedge (d \vee y)) = (d \vee x) \wedge (d \vee y)$ since $d \trianglelefteq d \vee x, d \vee y$ so that $d \trianglelefteq (d \vee x) \wedge (d \vee y)$. Also d is $\vee$-associative by (iii). Hence (i) holds. □

Corollary 2.9. *Let d be a distributive element of a trellis L. Then*

(i) *$L/\Theta_d \cong [d)$ where Θ_d is as in Theorem 2.8.*
(ii) *$(d]$ is the ideal kernel of Θ_d.*
(iii) *$\Theta_d = \Theta[(d]]$.*

Proof. (i) $f : x \to d \vee x$ is a homomorphism of L onto $[d)$ and $\Theta_d = \ker(f)$. Hence, by the Homomorphism Theorem (Every homomorphic image of a trellis L is isomorphic to a suitable quotient trellis of L), $L/\Theta_d \cong [d)$.

(ii) Let $x, y \in (d]$. Then $x, y \trianglelefteq d$. Therefore $d \vee x = d = d \vee y$ which implies $x \equiv y(\Theta_d)$. Further, if $y \in L$ with $y \equiv x(\Theta_d), x \in (d]$, then $d = d \vee x = d \vee y$ and hence $y \trianglelefteq d$ so that $y \in (d]$. Hence $(d]$ is the ideal kernel of Θ_d.

(iii) Let d be a distributive element of L and $x \equiv y(\Theta_d)$. Then $d \vee x = d \vee y$. For any element $u \in L$ with $u \trianglelefteq d$, we have $u \equiv d(\Theta[(d]])$. Now $x = x \vee (d \wedge x) \equiv x \vee d = y \vee d \equiv y \vee (d \wedge y) = y(\Theta[(d]])$. Thus $\Theta_d \leq \Theta[(d]]$. Since $\Theta[(d]]$ is the congruence relation of L generated by $(d]$, $\Theta_d = \Theta[(d]]$ by Definition 2.7. □

3. On standard elements in tellises

In the following we introduce the notion of a standard element in a trellis.

Definition 3.1. An element d of a trellis L is said to be standard if

(i) d is $\vee$-associative and
(ii) $x \wedge (d \vee y) = (x \wedge d) \vee (x \wedge y)$ for all elements $x, y \in L$.

A dually standard element is defined dually.

Remark 3.2.

1. As in Remark 2.4, one can show the independence of (i) and (ii) of Definition 3.1.
2. A standard element of a trellis need not be $\wedge$-associative. For, in the trellis of Figure 2, the element d is standard but not $\wedge$-associative as $d \wedge (a \wedge c) = 0 \neq c = (d \wedge a) \wedge c$.

Definition 3.3. An element d of a trellis L is said to be modular if d satisfies any of the following equivalent conditions:

(i) For $x, y \in L$, $x \trianglerighteq y$ implies $x \wedge (d \vee y) = (x \wedge d) \vee y$
(ii) $x \wedge (d \vee (x \wedge y)) = (x \wedge d) \vee (x \wedge y)$ for $x, y \in L$.
(iii) $(x \vee y) \wedge (d \vee y) = ((x \vee y) \wedge d) \vee y$ for $x, y \in L$.

The equivalence of the three conditions in the above definition can easily be proved. A modular trellis is one in which every element is modular.

Remark 3.4.

1. Any standard element d of a trellis L is modular.
2. If d is a modular element in a trellis L, then for $x \trianglerighteq y$,

$$d \wedge x = d \wedge y \quad \text{and} \quad d \vee x = d \vee y \quad \text{imply} \quad x = y \tag{1}$$

   whereas the converse is not true. For, in the trellis of Figure 3(a), the element a satisfies (1) but it is not modular.
3. If d is a standard element in a trellis L, then for $x, y \in L$, $d \wedge x = d \wedge y$ and $d \vee x = d \vee y$ imply $x = y$.

Theorem 3.5. *For an element d of a trellis L, the following statements are equivalent.*

(i) *d is standard.*
(ii) *d is distributive and modular.*

Proof. (i) $\Rightarrow$ (ii): d is $\vee$-associative by the definition of a standard element. Now for all $x, y \in L$,

$$\begin{aligned}
(d \vee x) \wedge (d \vee y) &= ((d \vee x) \wedge d) \vee ((d \vee x) \wedge y) \quad \text{since } d \text{ is standard;} \\
&= d \vee ((d \vee x) \wedge y); \\
&= d \vee ((y \wedge d) \vee (y \wedge x)) \quad \text{since } d \text{ is standard;} \\
&= (d \vee (y \wedge d)) \vee (y \wedge x) \quad \text{since } d \text{ is } \vee-\text{associative;} \\
&= d \vee (x \wedge y).
\end{aligned}$$

Thus d is a distributive element of L. By (1) of Remark 3.4, d is modular. Hence (ii) holds.

(ii) $\Rightarrow$ (i): d is $\vee$-associative by the definition of a distributive element. Further, for all $x, y \in L$,

$$\begin{aligned}(x \wedge d) \vee (x \wedge y) &= x \wedge (d \vee (x \wedge y)) \quad \text{since } x \trianglerighteq x \wedge y \text{ and } d \text{ is modular;}\\ &= x \wedge ((d \vee x) \wedge (d \vee y)) \text{ since } d \text{ is distributive;}\\ &= x \wedge (d \vee (x \wedge (d \vee y))) \text{ since } d \text{ is distributive;}\\ &= (x \wedge d) \vee (x \wedge (d \vee y)) \text{ since } x \trianglerighteq x \wedge (d \vee y) \text{ and } d \text{ is modular;}\\ &= x \wedge (d \vee y) \text{ since } x \wedge d \trianglelefteq x \wedge (d \vee y) \text{ as } x \wedge d \trianglelefteq d \vee y \text{ by the}\\ &\quad \text{middle transitivity of } d \text{ and } x \wedge d \trianglelefteq x.\end{aligned}$$

Thus d is standard. □

Corollary 3.9. *In a modular trellis, every distributive element is standard.*

4. On neutral elemnts in trellises

Definition 4.1. An element d of a trellis L is said to be *neutral* if there exists an embedding f of L into the direct product $A \times B$ of trellises A and B, where A has the largest element 1 and B has the smallest element 0 with $f(d) = (1, 0)$.

As in the case of lattices, one can easily prove the equivalence of the statements in the following remark.

Remark 4.2. For an element d of a trellis L, the following statements are equivalent.

(i) d is a neutral element.
(ii) d is standard and dually distributive.
(iii) d is distributive, dually distributive and modular.
(iv) d is dually standard and distributive.

It is known that an element d of a lattice L is neutral if and only if the sublattice generated by $\{d, x, y\}$ is distributive [**4**]. We shall try to generalize this result for arbitrary trellises. Since a distributive trellis is a lattice [**8**], for elements d, x, y of a trellis L, $\{d, x, y\}$ generate a distributive subtrellis if and only if it generates a distributive lattice. A free distributive lattice generated by $\{d, x, y\}$ has eighteen elements (see [**4**, Fig. I.5.6]). Therefore, if $\{d, x, y\}$ generates a distributive subtrellis, then $x \wedge y \trianglelefteq x \vee y$ in L. However, in a trellis L for $x, y \in L$, we need not have $x \wedge y \trianglelefteq x \vee y$. To overcome this difficulty, we introduce the following definition.

Definition 4.3. A trellis L is said to be *normal* if $x \wedge y \trianglelefteq x \vee y$ for every $x, y \in L$.

Remark 4.4.

1. Every lattice is a normal trellis.
2. Every tournament is a normal trellis.

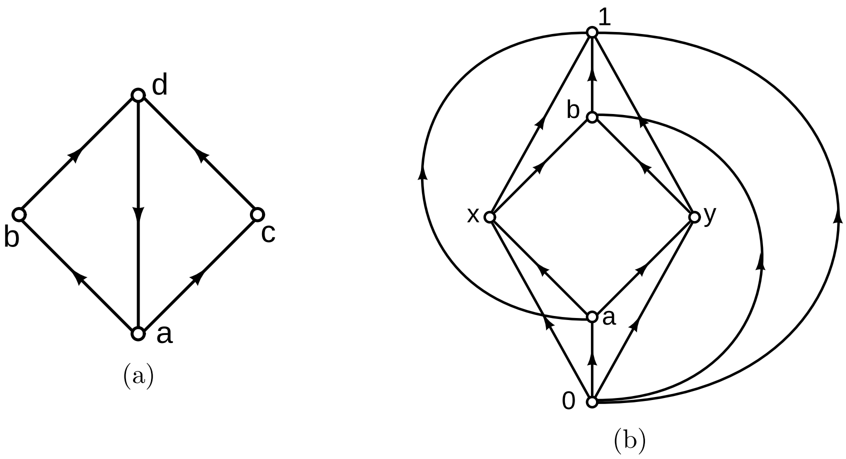


Figure 4

It is not true that all trellises are normal. In fact, the trellis L of Figure 4(a) is not normal as $b \wedge c \ntrianglelefteq b \vee c$ for $b, c \in L$. Also, the trellis of Figure 4(b), which is acyclic and modular, is not normal as $x \wedge y \ntrianglelefteq x \vee y$.

The following theorem generalizes a corresponding result of lattices [**4**] to normal trellises.

Theorem 4.5. *The following statements are equivalent for an element d of a normal trellis L.*

(i) *d is neutral.*
(ii) *d is standard and dually distributive.*
(iii) *For any $x, y \in L$, the subtrellis generated by $\{d, x, y\}$ is distributive.*

Proof. (i) is equivalent to (ii) follows by Remark 4.2.

(i) $\Rightarrow$ (iii) : If d is a neutral element in a normal trellis L, it can be observed that for any $x, y \in L, d, x, y$ generate a distributive lattice.

(iii) $\Rightarrow$ (i): Obvious. $\square$

Acknowledgement. The author thanks Prof. S. Parameshwara Bhatta for his valuable suggestions in the preparation of this paper.

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