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ANALYSIS OF A FRICTIONAL CONTACT PROBLEM WITH ADHESION

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ABSTRACT. We consider a mathematical model which describes the contact between a deformable body and an obstacle, the so-called foundation. The contact is frictional and is modelled with a version of normal compliance condition and the associated Coulomb's law of dry friction in which the adhesion of contact surfaces is taken into account. The evolution of the bonding field is described by a first order differential equation and the the material's behavior is modelled with a nonlinear elastic constitutive law. We derive a variational formulation of the problem then, under a smallness assumption on the coefficient of friction, we prove the existence of a unique weak solution for the model. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem. Finally, we extend our results in the case when the piezoelectric effect is taken into account, i.e. in the case when the material's behavior is modelled with a nonlinear electro-elastic constitutive law.

1. INTRODUCTION

Processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. For this reason, adhesive contact between bodies, when a glue is added to prevent the surfaces from relative motion, has recently received increased attention in the literature. Basic modelling can be found in [**7, 8, 9, 12**]. Analysis of models for adhesive contact can be found in [**3, 4, 6**] and in the monographs [**15, 17**]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [**13, 14**]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [**7, 8**], the bonding field satisfies the restrictions $0 \le \beta \le 1$; when $\beta = 1$ at a point of the contact surface, the adhesion

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is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [**9, 12, 15, 17**].

The aim of this paper is to continue the study of adhesive problems begun in [**3, 4, 17**]. There, models for dynamic or quasistatic process of frictionless adhesive contact between a deformable body and a foundation have been analyzed and simulated; the contact was described with normal compliance or was assumed to be bilateral, and the behavior of the material was modelled with a nonlinear Kelvin–Voigt viscoelastic constitutive law; the existence of a unique weak solution to the models has been obtained by using arguments of nonlinear evolutionary equations in Banach spaces and a fixed point theorem. With respect to [**3, 4**], the novelty of the present paper is three folds: 1) we model the material's behavior with a nonlinear elastic constitutive law; 2) the contact is frictional; 3) we extend our study to problems in which the piezoelectric effect is taken into account.

The piezoelectric effect is characterized by the coupling between the mechanical and electrical properties of the materials. A deformable material which presents such a behavior is called a piezoelectric material. Piezoelectric materials are used extensively as switches and actuary in many engineering systems, in radioelectronics, electroacoustics and measuring equipments. General models for elastic materials with piezoelectirec effect, called electro-elastic materials, can be found in [**1, 10**]. A static frictional contact problem for electric-elastic materials was considered in [**2, 11**] and a slip-dependent frictional contact problem for electro-elastic materials was studied in [**16**]. In this last reference the variational formulation of the corresponding problem was derived and its unique solvability was proved.

The paper is structured as follows. In Section 2 we present the model of the elastic contact problem with adhesion, normal compliance and friction. In Section 3 we derive a variational formulation of the model; it consists in a system coupling a time-dependent variational inequality for the displacement field with an ordinary differential equations for the bonding field. In Section 4 we state and prove our main existence and uniqueness result, Theorem 4.1. It states that if the coefficient of friction is small enough, then the problem has a unique weak solution. We extend our results in Section 5 to the case when the material's behavior is modelled with a nonlinear electro-elastic constitutive law and we provide our second existence and uniqueness result, Theorem 5.1.

2. Problem statement

We consider an elastic body, which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, with a smooth boundary $\partial\Omega = \Gamma$ divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. Let $[0, T]$ be the time interval of interest, where $T > 0$. The body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there; it is also submitted to the action of volume forces of density $\boldsymbol{f}_0$ in $\Omega \times (0, T)$ and surface tractions of density $\boldsymbol{f}_2$ on $\Gamma_2 \times (0, T)$. On $\Gamma_3 \times (0, T)$, the body is in contact with an obstacle, the so-called foundation. The contact is modelled with a version of normal compliance condition and the associated Coulomb's law of dry friction in which the adhesion of contact surfaces is taken into account. We denote by $\boldsymbol{\nu}$ the outward normal unit vector on Γ and the subscripts ν and τ will represent the normal and tangential components of vectors and tensors, respectively. We also denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and use " $\cdot$ " and $\|\cdot\|$ for the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively. Then, the classical model for the frictional contact process is as follows.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *and a bonding field* $\beta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \qquad \text{in } \Omega \times (0,T), \tag{1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega \times (0,T), \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1 \times (0,T), \tag{3}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{4}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu) \qquad \text{on } \Gamma_3 \times (0,T), \tag{5}$$

$$\left\{\begin{array}{l} \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\| \le \mu p_\nu(u_\nu), \\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\| < \mu p_\nu(u_\nu) \Rightarrow \boldsymbol{u}_\tau = \boldsymbol{0}, \\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\| = \mu p_\nu(u_\nu) \Rightarrow \exists \lambda \ge 0 \\ \qquad \text{such that } \boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau) = -\lambda \boldsymbol{u}_\tau \end{array}\right. \qquad \text{on } \Gamma_3 \times (0,T), \tag{6}$$

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(u_\nu)^2 + \gamma_\tau \|\boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\|^2) - \epsilon_a)_+ \qquad \text{on } \Gamma_3 \times (0,T), \tag{7}$$

$$\beta(0) = \beta_0 \qquad \text{on } \Gamma_3. \tag{8}$$

We now provide some comments on equations and conditions (1)–(8) and send to [**15, 17, 18**] for more details on the conditions (5)–(7) which describe the frictional contact with adhesion.

First, equation (1) represents the elastic constitutive law in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor and $\mathcal{F}$ is the elasticity operator, assumed to be nonlinear. Next, equation (2) is the equilibrium equation in which "Div" denotes the divergence operator; we use it here since we assume that the inertial term in the equation of motion can be neglected. Conditions (3) and (4) are the displacement and traction boundary conditions, respectively, and condition (8) represents an initial condition, in which β_0 is the initial bonding field.

Condition (5) represents the normal compliance condition with adhesion and condition (6) is the associated Coulomb's law of dry friction on the contact surface Γ_3, in its static version. Here p_ν is a given function, μ is the coefficient of friction and γ_ν, γ_τ are material parameters; also, R_ν and $\boldsymbol{R}_\tau$ are truncation operators defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0, \end{cases} \qquad \boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } \|\boldsymbol{v}\| \le L, \\ L\,\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|} & \text{if } \|\boldsymbol{v}\| > L, \end{cases}$$

with $L > 0$ being a characteristic length of the bond, beyond which it stretches without offering any additional resistance, see [**12**] for details. It follows from (5) that the contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \beta^2 R_\nu(u_\nu)$; the adhesive traction is tensile, and is proportional to the square of the adhesion and to the normal displacement, but as long as it does not exceed the bond length L. Also, it follows from (6) that the contribution of the adhesive to the tangential shear on the contact surface is represented by the term $\gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau)$; the adhesive shear is proportional to the square of the adhesion and to the tangential displacement, but again, only up to the bond length L.

Equation (7) describes the evolution of the bonding field in which ϵ_a is a given material parameter and $r_+ = \max\{r, 0\}$. Here and below, for simplicity, we use the notation $R_\nu(u_\nu)^2 = (R_\nu(u_\nu))^2$. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot\beta \le 0$. Also, it is easy to see that if $0 \le \beta_0 \le 1$ a.e. on Γ_3, then $0 \le \beta \le 1$ a.e. on Γ_3 during the process.

Because of the friction condition (6), which is non-smooth, we do not expect the problem to have, in general, any classical solution. For this reason, we derive in the next section a variational formulation of the problem and investigate its solvability.

3. Variational formulation

We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i\,, \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}\,, \qquad \|\tau\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and everywhere in this paper, i, j, k, l run from 1 to d, summation over repeated indices is applied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

Everywhere below we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces:

$$L^2(\Omega)^d = \{\ \boldsymbol{v} = (v_i) \mid v_i \in L^2(\Omega)\ \}, \qquad H^1(\Omega)^d = \{\ \boldsymbol{v} = (v_i) \mid v_i \in H^1(\Omega)\ \},$$
$$\mathcal{H} = \{\ \boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}, \qquad \mathcal{H}_1 = \{\ \boldsymbol{\tau} \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)^d}=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x,\qquad (\boldsymbol{u},\boldsymbol{v})_{H^1(\Omega)^d}=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x+\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}=\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}x,\qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1}=\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}x+\int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\mathrm{Div}\,\boldsymbol{\tau}\,\mathrm{d}x,$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\boldsymbol{\varepsilon}(\boldsymbol{v})=(\varepsilon_{ij}(\boldsymbol{v})),\qquad \varepsilon_{ij}(\boldsymbol{v})\frac{1}{2}\,(v_{i,j}+v_{j,i}),\qquad \mathrm{Div}\,\boldsymbol{\tau}=(\tau_{ij,j})$$

for all $\boldsymbol{v}\in H^1(\Omega)^d$ and $\boldsymbol{\tau}\in\mathcal{H}_1$. For every element $\boldsymbol{v}\in H^1(\Omega)^d$ we also write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$. Similarly, σ_ν and $\boldsymbol{\sigma}_\tau$ denote the normal and the tangential traces of a function $\boldsymbol{\sigma}\in\mathcal{H}_1$. When $\boldsymbol{\sigma}$ is a regular function, then $\sigma_\nu=(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}$, and the following Green's type formula holds:

$$(9)\qquad (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q+(\mathrm{Div}\,\boldsymbol{\sigma},\boldsymbol{v})_{L^2(\Omega)^d}=\int_\Gamma \boldsymbol{\sigma}\nu\cdot\boldsymbol{v}\,\mathrm{d}a\qquad \forall\,\boldsymbol{v}\in H^1(\Omega)^d.$$

Consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V=\{\ \boldsymbol{v}\in H^1(\Omega)^d\mid \boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\ \}.$$

Since $\mathrm{meas}(\Gamma_1)>0$, the following Korn's inequality holds:

$$(10)\qquad \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}\ge c_K\,\|\boldsymbol{v}\|_{H^1(\Omega)^d}\quad \forall\,\boldsymbol{v}\in V,$$

where $c_K>0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(11)\qquad (\boldsymbol{u},\boldsymbol{v})_V=(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (10) that $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega)^d}$ are equivalent norms on V and, therefore, $(V,\|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem combined with (10) and (11), there exists a constant c_0 depending only on the domains Ω, Γ_1 and Γ_3 such that

$$(12)\qquad \|\boldsymbol{v}\|_{L^2(\Gamma_3)^d}\le c_0\|\boldsymbol{v}\|_V\qquad \forall\boldsymbol{v}\in V.$$

For every real Hilbert space X we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1\le p\le\infty$, $k=1,2,\ldots$; we also use the space of continuous functions on $[0,T]$ with values on X, denoted $C([0,T];X)$, equipped with the norm

$$\|x\|_{C([0,T];X)}=\max_{t\in[0,T]}\|x(t)\|_X,$$

and we introduce the set

$$\mathcal{Q}=\{\,\theta:[0,T]\to L^2(\Gamma_3))\mid 0\le\theta(t)\le 1\ \ \forall\,t\in[0,T],\ \text{a.e. on }\Gamma_3\,\}.$$

In the study of the problem $\mathcal{P}$, we assume that the elasticity operator $\mathcal{F}$ and the normal compliance function p_ν satisfy:

$$(13)\quad \begin{cases} \text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \qquad\quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2), \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \qquad\quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable in } \Omega, \\ \qquad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{cases}$$

$$(14)\quad \begin{cases} \text{(a) } p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_\nu > 0 \text{ such that} \\ \qquad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)(r_1 - r_2) \ge 0 \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \text{ for all } r \in \mathbb{R}. \\ \text{(e) } p_\nu(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

Examples of nonlinear operators which satisfy conditions (13) can be find in [**17**]. Also, a simple example of a normal compliance function p_ν which satisfies conditions (14) is $p_\nu(r) = c_\nu r_+$ where $c_\nu \in L^\infty(\Gamma_3)$ is a positive function.

We also suppose that the body forces and surface tractions have the regularity

$$(15)\qquad \boldsymbol{f}_0 \in W^{1,\infty}(0, T; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d),$$

and the adhesion coefficients satisfy the conditions

$$(16)\qquad \gamma_\nu,\ \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \epsilon_a \ge 0 \qquad \text{a.e. on } \Gamma_3.$$

Finally, the friction coefficient and the initial bonding field are such that

$$(17)\qquad \mu \in L^\infty(\Gamma_3), \qquad \mu(x) \ge 0 \qquad \text{a.e. on } \Gamma_3,$$
$$(18)\qquad \beta_0 \in L^2(\Gamma_3), \qquad 0 \le \beta_0 \le 1 \qquad \text{a.e. on } \Gamma_3.$$

We define the function $\boldsymbol{f} : [0, T] \to V$ by

$$(19)\qquad (\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, \mathrm{d}a,$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $t \in [0, T]$, and we note that the condition (15) implies that

$$(20)\qquad \boldsymbol{f} \in W^{1,\infty}(0, T; V).$$

Also, we define the adhesion functional $j_{ad} : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$, the normal compliance functional $j_{nc} : V \times V \to \mathbb{R}$ and the friction functional $j_{fr} : V \times V \to \mathbb{R}$ by equalities

$$j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau) \cdot \boldsymbol{v}_\tau \big)\, \mathrm{d}a, \tag{21}$$

$$j_{nc}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu \, \mathrm{d}a, \tag{22}$$

$$j_{fr}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu p_\nu(u_\nu) \|\boldsymbol{v}_\tau\| \, \mathrm{d}a. \tag{23}$$

By a standard procedure based on Green's formula (9) we derive the following variational formulation of problem $\mathcal{P}$, in terms of displacement and bonding fields.

Problem $\mathcal{P}_V$. *Find a displacement field $\boldsymbol{u} : [0,T] \to V$ and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that*

$$\begin{aligned} &(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} + j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \\ &\qquad + j_{nc}(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + j_{fr}(\boldsymbol{u}(t), \boldsymbol{v}) - j_{fr}(\boldsymbol{u}(t), \boldsymbol{u}(t)) \\ &\geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T], \end{aligned} \tag{24}$$

$$\dot\beta(t) = -\Big(\beta(t) \big(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau \|\boldsymbol{R}_\tau(\boldsymbol{u}_\tau(t))\|^2 \big) - \epsilon_a \Big)_+ \quad \text{a.e. } t \in (0,T), \tag{25}$$

$$\beta(0) = \beta_0. \tag{26}$$

In the rest of this section, we derive some inequalities involving the functionals j_{ad}, j_{nc} and j_{fr} which will be used in the following sections. Below in this section β, β_1, β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \le \beta,\ \beta_1,\ \beta_2 \le 1$ a.e. on Γ_3, $\boldsymbol{u}_1$, $\boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2$, $\boldsymbol{u}$ and $\boldsymbol{v}$ represent elements of V; and c is a generic positive constant which may depend on Ω, Γ_1, Γ_3, p_ν, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, in the following text we suppress the explicit dependence on various functions on $\boldsymbol{x} \in \Omega \cup \Gamma_3$.

First, we remark that the j_{ad} and j_{nc} are linear with respect to the last argument and therefore

$$j_{ad}(\beta, \boldsymbol{u}, -\boldsymbol{v}) = -j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}), \quad j_{nc}(\boldsymbol{u}, -\boldsymbol{v}) = -j_{nc}(\boldsymbol{u}, \boldsymbol{v}). \tag{27}$$

Next, using (21) and inequalities $|R_\nu(u_{1\nu})| \le L$, $\|\boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\| \le L$, $|\beta_1| \le 1$, $|\beta_2| \le 1$, we deduce that

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \le c \int_{\Gamma_3} |\beta_1 - \beta_2|\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|\, da,$$

and, combining this inequality with (12), we obtain

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \le c\, \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V. \tag{28}$$

Next, we choose $\beta_1 = \beta_2 = \beta$ in (28) to find

$$j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq 0. \tag{29}$$

Similar computations, based on the Lipschitz continuity of operators R_ν, $\boldsymbol{R}_\tau$, show that

$$|j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{v}) - j_{ad}(\beta, \boldsymbol{u}_2, \boldsymbol{v})| \leq c\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V. \tag{30}$$

Now, we use (22) to see that

$$|j_{nc}(\boldsymbol{u}_1, \boldsymbol{v}) - j_{nc}(\boldsymbol{u}_2, \boldsymbol{v})| \leq \int_{\Gamma_3} |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|\, |v_\nu| \,\mathrm{d}a,$$

and therefore (14)(b) and (12) imply

$$|j_{nc}(\boldsymbol{u}_1, \boldsymbol{v}) - j_{nc}(\boldsymbol{u}_2, \boldsymbol{v})| \leq c\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V. \tag{31}$$

We use again (22) to obtain

$$j_{nc}(\boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{nc}(\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) = \int_{\Gamma_3} (p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}))(u_{2\nu} - u_{1\nu}) \,\mathrm{d}a,$$

and then, (14)(c) implies

$$j_{nc}(\boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{nc}(\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq 0. \tag{32}$$

Finally, we use (23) to find that

$$\begin{aligned} &j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_2) + j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_2) - j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_1) \\ &\quad \leq \int_{\Gamma_3} \mu\, |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|\, \|\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}\| \,\mathrm{d}a. \end{aligned}$$

Therefore, using (14)(b) and (12) we obtain

$$\begin{aligned} &j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_2) - j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_1) + j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_2) \\ &\quad \leq c_0^2 L_\nu\, \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V. \end{aligned} \tag{33}$$

Inequalities (28)–(33) combined with equalities (27) will be used in various places in the rest of the paper.

4. An existence and uniqueness result

Our main result which states the unique solvability of Problem $\mathcal{P}_V$, is the following.

Theorem 4.1. *Assume that* (13)–(18) *hold. Then, there exists* $\mu_0 > 0$ *which depends on* Ω, Γ_1, Γ_3, $\mathcal{F}$ *and* p_ν *such that Problem* $\mathcal{P}_V$ *has a unique solution* $(\boldsymbol{u}, \beta)$, *if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$. *Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{1,\infty}(0, T; V), \tag{34}$$

$$\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{Q}. \tag{35}$$

Let $(\boldsymbol{u}, \beta)$ be the solution of Problem $\mathcal{P}_V$ obtained in Theorem 4.1 and denote by $\boldsymbol{\sigma}$ the function given by (1). It is easy to check that

$$\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H}_1). \tag{36}$$

A triple of functions $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ which satisfies (1), (24)–(26) is called a *weak solution* of the frictional adhesive contact problem $\mathcal{P}$. We conclude by Theorem 4.1 that, under the assumptions (13)–(18), if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$ there exists a unique weak solution of Problem $\mathcal{P}$ which verifies (34)–(36).

We turn now to the proof of the Theorem 4.1 which will be carried out in several steps. To this end, we assume in the following that (13)–(18) hold; below, c is a generic positive constant which may depend on Ω, Γ_1, Γ_3, p_ν, γ_ν, γ_τ, L, and whose value may change from place to place; and $\mathcal{Z}$ denotes the closed set of the space $C([0,T];L^2(\Gamma_3))$ defined by

$$\mathcal{Z} = \left\{ \beta \in C([0,T];L^2(\Gamma_3)) \cap \mathcal{Q} \mid \beta(0) = \beta_0 \right\}. \tag{37}$$

Let $\beta \in \mathcal{Z}$ be given. In the first step we consider the following variational problem.

Problem $\mathcal{P}^1_\beta$. *Find a displacement field $\boldsymbol{u}_\beta : [0,T] \to V$ such that, for all $t \in [0,T]$,*

$$\begin{aligned}(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\beta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\beta(t)))_{\mathcal{H}} + j_{ad}(\beta(t), \boldsymbol{u}_\beta(t), \boldsymbol{v} - \boldsymbol{u}_\beta(t))\\ + j_{nc}(\boldsymbol{u}_\beta(t), \boldsymbol{v} - \boldsymbol{u}_\beta(t)) + j_{fr}(\boldsymbol{u}_\beta(t), \boldsymbol{v}) - j_{fr}(\boldsymbol{u}_\beta(t), \boldsymbol{u}_\beta(t))\\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\beta(t))_V \qquad \forall \boldsymbol{v} \in V.\end{aligned} \tag{38}$$

We have the following result.

Lemma 4.2. *There exists $\mu_0 > 0$ which depends on $\Omega, \Gamma_1, \Gamma_3, \mathcal{F}$ and p_ν such that Problem $\mathcal{P}^1_\beta$ has a unique solution $\boldsymbol{u}_\beta \in C([0,T];V)$, if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$.*

Proof. Let $t \in [0,T]$ and let $A_{\beta(t)} : V \to V$ be the operator defined by

$$(A_{\beta(t)}\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j_{ad}(\beta(t), \boldsymbol{u}, \boldsymbol{v}) + j_{nc}(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{39}$$

We use (13), (27) and (29)–(32) to prove that

$$\|A_{\beta(t)}\boldsymbol{u}_1 - A_{\beta(t)}\boldsymbol{u}_2\|_V \leq c\,\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in V, \tag{40}$$

$$(A_{\beta(t)}\boldsymbol{u}_1 - A_{\beta(t)}\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_V \geq m_{\mathcal{F}}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^2 \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in V, \tag{41}$$

which shows that $A_{\beta(t)}$ is a strongly monotone Lipschitz continuous operator on V. Next, using (14) we can easily check that, for a given $\boldsymbol{u} \in V$, the functional $j_{fr}(\boldsymbol{u}, \cdot) : V \to \mathbb{R}$ is convex and lower semicontinuous and recall that it satisfies (33). Let

$$\mu_0 = \frac{m_{\mathcal{F}}}{c_0^2 L_\nu} \tag{42}$$

and note that μ_0 depends on $\Omega, \Gamma_1, \Gamma_3, \mathcal{F}$ and p_ν. Assume that $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$. Then

$$c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} < m_{\mathcal{F}} \tag{43}$$

and therefore, using (41), (33) and a standard existence and uniqueness result on elliptic quasivariational inequalities (see, e.g. [**5**]), it follows that there exists a unique element $\boldsymbol{u}_\beta(t) \in V$ which satisfies

$$\begin{aligned}(44)\qquad & (A_{\beta(t)}\boldsymbol{u}_\beta(t), \boldsymbol{v} - \boldsymbol{u}_\beta(t))_V + j_{fr}(\boldsymbol{u}_\beta(t), \boldsymbol{v}) - j_{fr}(\boldsymbol{u}_\beta(t), \boldsymbol{u}_\beta(t)) \\ & \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\beta(t))_V \qquad \forall\, \boldsymbol{v} \in V.\end{aligned}$$

We use now (39) and (44) to see that $\boldsymbol{u}_\beta(t)$ is the unique element which solves (38), at any $t \in [0, T]$.

Consider now $t_1,\ t_2 \in [0, T]$ and, for simplicity, denote $\boldsymbol{u}_\beta(t_i) = \boldsymbol{u}_i$, $\beta(t_i) = \beta_i$, $\boldsymbol{f}(t_i) = \boldsymbol{f}_i$ for $i = 1, 2$. Using (38), (13), the inequalities involving the functionals j_{ad}, j_{nc} and j_{fr} presented at the end of Section 3 and (43), after some algebra we obtain

$$(45)\qquad \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \leq \frac{c}{m_{\mathcal{F}} - c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)}}\, \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} + \frac{1}{m_{\mathcal{F}} - c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)}}\, \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_V,$$

Inequality (45) combined with the regularities of $\boldsymbol{f}$ and β in (20) and (37) implies that the mapping $t \mapsto \boldsymbol{u}_\beta(t) : [0, T] \to V$ is continuous, which concludes the proof. □

We assume in following text that $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$ and therefore (43) is valid. In the next step, we use the displacement field $\boldsymbol{u}_\beta$ obtained in Lemma 4.2, and we consider the following initial value problem.

Problem $\mathcal{P}_\beta^2$. *Find a bonding field $\theta_\beta : [0, T] \to L^2(\Gamma_3)$ such that*

$$(46)\quad \dot{\theta}_\beta(t) = -\Big(\theta_\beta(t)\big(\gamma_\nu R_\nu(u_{\beta\nu}(t))^2 + \gamma_\tau \|\boldsymbol{R}_\tau(\boldsymbol{u}_{\beta\tau}(t))\|^2\big) - \epsilon_a\Big)_+ \ \text{a.e. } t \in (0, T),$$

$$(47)\quad \theta_\beta(0) = \beta_0.$$

We obtain the following result.

Lemma 4.3. *There exists a unique solution to Problem $\mathcal{P}_\beta^2$ and it satisfies $\theta_\beta \in W^{1,\infty}(0, T, L^2(\Gamma_3)) \cap \mathcal{Q}$.*

Proof. Consider the mapping $F_\beta : [0, T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\beta(t, \theta) = -\Big(\theta_\beta(t)\big(\gamma_\nu R_\nu(u_{\beta\nu}(t))^2 + \gamma_\tau \|\boldsymbol{R}_\tau(\boldsymbol{u}_{\beta\tau}(t))\|^2\big) - \epsilon_a\Big)_+,$$

for all $t \in [0, T]$ and $\theta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operators R_ν and $\boldsymbol{R}_\tau$ that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_\beta(t, \theta)$ belongs to $L^\infty(0, T; L^2(\Gamma_3))$. Using now a version of Cauchy-Lipschitz theorem (see for instance [**17**, p. 48]), we obtain the existence of a unique function $\theta_\beta \in W^{1,\infty}(0, T, L^2(\Gamma_3))$ which solves (46), (47). We note that (25) and (26) guarantee that $\beta(t) \leq \beta_0$ and, therefore, assumption (18) shows that $\beta(t) \leq 1$ for

$t \geq 0$, a.e. on Γ_3. On the other hand, if $\beta(t_0) = 0$ at $t = t_0$, then it follows from (25) and (26) that $\dot{\beta}(t) = 0$ for all $t \geq t_0$ and therefore, $\beta(t) = 0$ for all $t \geq t_0$, a.e. on Γ_3. We conclude that $0 \leq \beta(t) \leq 1$ for all $t \in [0, T]$, a.e. on Γ_3. Therefore, according to the definition of the set $\mathcal{Q}$, we find that $\theta_\beta \in \mathcal{Q}$, which concludes the proof of the lemma. □

It follows from Lemma 4.3 that for all $\beta \in \mathcal{Z}$ the solution θ_β of Problem $\mathcal{P}^2_\beta$ belongs to $\mathcal{Z}$, see (37). Therefore, we may consider the operator $\Lambda : \mathcal{Z} \to \mathcal{Z}$ given by

$$\Lambda\beta = \theta_\beta. \tag{48}$$

In the last step we will prove the following result.

Lemma 4.4. *There exists a unique element $\beta^* \in \mathcal{Z}$ such that $\Lambda\beta^* = \beta^*$.*

Proof. Suppose that β_1, β_2, are two functions in $\mathcal{Z}$ and denote by $\boldsymbol{u}_i$, θ_i the functions obtained in Lemmas 4.2 and 4.3, respectively, for $\beta = \beta_i$, $i = 1, 2$. Let $t \in [0, T]$; we use similar arguments to those used in the proof of (45) to deduce that

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leq \frac{c}{m_\mathcal{F} - c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)}} \|\,\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}. \tag{49}$$

On the other hand, it follows from (46) and (47) that

$$\theta_i(t) = \beta_0 - \int_0^t \Big(\theta_i(s)\big(\gamma_\nu R_\nu(u_{i\nu}(s))^2 + \gamma_\tau \|\boldsymbol{R}_\tau(\boldsymbol{u}_{i\tau}(s))\|^2\Big) - \epsilon_a\Big)_+ \,\mathrm{d}s$$

and then

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|\theta_1(s) R_\nu(u_{1\nu}(s))^2 - \theta_2(s) R_\nu(u_{2\nu}(s))^2\|_{L^2(\Gamma_3)} \,\mathrm{d}s$$
$$+ \int_0^t \Big\|\theta_1(s)\|\boldsymbol{R}_\tau(\boldsymbol{u}_{1\tau}(s))\|^2 - \theta_2(s)\|\boldsymbol{R}_\tau(\boldsymbol{u}_{2\tau}(s))\|^2\Big\|_{L^2(\Gamma_3)} \,\mathrm{d}s.$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\theta_1 = \theta_1 - \theta_2 + \theta_2$, we get

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Gamma_3)} \,\mathrm{d}s + c \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)} \,\mathrm{d}s.$$

By Gronwall's inequality, it follows that

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)} \,\mathrm{d}s$$

and using (12) we obtain

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \,\mathrm{d}s. \tag{50}$$

We use (48) in the estimate (50) to find

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \,\mathrm{d}s. \tag{51}$$

We now combine (49) with (51) to deduce

$$\|\Lambda\beta_1(t)-\Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \le \frac{c}{m_{\mathcal{F}}-c_0^2L_\nu\|\mu\|_{L^\infty(\Gamma_3)}}\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}\,\mathrm{d}s$$

and, reiterating this inequality n times, it yields

(52)

$$\|\Lambda^n\beta_1-\Lambda^n\beta_2\|_{C([0,T];L^2(\Gamma_3))} \le \frac{c^nT^n}{(m_{\mathcal{F}}-c_0^2L_\nu\|\mu\|_{L^\infty(\Gamma_3)})^n n!}\,\|\beta_1-\beta_2\|_{C([0,T];L^2(\Gamma_3))}.$$

Recall that $\mathcal{Z}$ is a nonempty closed set in the Banach space $C([0,T];L^2(\Gamma_3))$ and note that inequality (52) shows that for n sufficiently large $\Lambda^n:\mathcal{Z}\to\mathcal{Z}$ is a contraction. We use the Banach fixed point theorem to obtain that Λ has a unique fixed point $\beta^*\in\mathcal{Z}$, which concludes the proof. □

Now, we have all the ingredients to provide the proof of Theorem 4.1.

Proof of Theorem 4.1. *Existence.* Let $\beta^*\in\mathcal{Z}$ be the fixed point of Λ and let $\boldsymbol{u}^*$ be the solution of Problem $\mathcal{P}^1_\beta$ for $\beta=\beta^*$, i.e. $\boldsymbol{u}^*=\boldsymbol{u}_{\beta^*}$. Since $\theta_{\beta^*}=\beta^*$, we conclude by (38), (46), (47) that $(\boldsymbol{u}^*,\beta^*)$ is a solution of Problem $\mathcal{P}_V$ and, moreover, β^* satisfies (35). Also, since $\beta^*=\theta_{\beta^*}\in W^{1,\infty}(0,T,L^2(\Gamma_3))$, inequality (49) implies that the function $\boldsymbol{u}^*$ belongs to $W^{1,\infty}(0,T;V)$, which shows that the functions $\boldsymbol{u}^*$ have the regularity expressed in (34).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ defined by (48). Indeed, let $(\boldsymbol{u},\beta)$ be a solution of Problem $\mathcal{P}_V$ which satisfies (34)–(35). It follows from (38) that $\boldsymbol{u}$ is a solution to Problem $\mathcal{P}^1_\beta$ and, since by Lemma 4.2 this problem has a unique solution denoted $\boldsymbol{u}_\beta$, we obtain

$$\boldsymbol{u}=\boldsymbol{u}_\beta. \tag{53}$$

Then, we replace $\boldsymbol{u}=\boldsymbol{u}_\beta$ in (25) and use the initial condition (26) to see that β is a solution to Problem $\mathcal{P}^2_\beta$. Since by Lemma 4.3, this last problem has a unique solution, denoted θ_β, we find

$$\beta=\theta_\beta. \tag{54}$$

We use now (48) and (54) to obtain that $\Lambda\beta=\beta$, i.e. β is a fixed point of the operator Λ. It follows now from Lemma 4.4 that

$$\beta=\beta^*. \tag{55}$$

The uniqueness part of the theorem is now a consequence of (53) and (55). □

5. A piezoelectric frictional contact problem with adhesion

In this section we extend our results to the case when the piezoelectric effect of the material is taken into account. To this end we consider the physical setting described in Section 2 and we assume that, besides the action of the forces and tractions, the body is submitted to the action of volume charges of density q_0 and to electric constraints on the boundary. To describe them we consider a second partition of Γ into two measurable parts Γ_a and Γ_b such that $meas(\Gamma_a)>0$ and

$\Gamma_3 \subseteq \Gamma_b$. We assume that the electric potential vanishes on Γ_a and surface electric charge of density q_2 is prescribed on Γ_b. Also, we assume that the foundation is insulator and we model the material's behavior with an electro-elastic constitutive law. With these assumptions, the classical model of the process is as follows.

Problem $\widetilde{\mathcal{P}}$. *Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $\boldsymbol{D} : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta : \Omega \times [0,T] \to \mathbb{R}$ such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\boldsymbol{E}(\varphi) \quad \text{in } \Omega \times (0,T), \tag{56}$$

$$\boldsymbol{D} = \mathcal{B}\boldsymbol{E}(\varphi) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega \times (0,T), \tag{57}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0,T), \tag{58}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{59}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{60}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{61}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0,T), \tag{62}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\| \le \mu p_\nu(u_\nu), \\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\| < \mu p_\nu(u_\nu) \Rightarrow \boldsymbol{u}_\tau = \boldsymbol{0}, \\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\| = \mu p_\nu(u_\nu) \Rightarrow \exists \lambda \ge 0 \\ \qquad \text{such that } \boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 \boldsymbol{R}_\tau(\boldsymbol{u}_\tau) = -\lambda \boldsymbol{u}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{63}$$

$$\dot{\beta} = -(\beta\,(\gamma_\nu R_\nu(u_\nu))^2 + \gamma_\tau \|\boldsymbol{R}_\tau(\boldsymbol{u}_\tau)\|^2) - \epsilon_a)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{64}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{65}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{66}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{67}$$

Equations (56) and (57) represent the electro-elastic constitutive law in which $\boldsymbol{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{F}$ is the elasticity operator, $\mathcal{E}$ represents the piezoelectric operator, $\mathcal{E}^*$ is its transposed and $\mathcal{B}$ denotes the electric permittivity operator. Details on the electro-elastic constitutive equations of the form (56), (57) can be find, for instance, in [**1, 2**]. Next, equation (59) is the equilibrium equation for the electric-displacement field in which "div" denote the divergence operator for vector valued functions, whereas (65) and (66) represent the electric boundary conditions. The rest of equations and conditions are identic to the corresponding equations and conditions in Problem $\mathcal{P}$.

To study problem $\widetilde{\mathcal{P}}$ we use the notation in Section 3 and, for the electric unknowns φ and $\boldsymbol{D}$, we introduce the spaces

$$W = \{\ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\ \},$$
$$\mathcal{W}_1 = \{\ \boldsymbol{D} = (D_i) \mid D_i \in L^2(\Omega),\ D_{i,i} \in L^2(\Omega)\ \}.$$

Since meas $(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi\|_{L^2(\Omega)^d} \geq c_F\,\|\psi\|_{H^1(\Omega)} \quad \forall\,\psi \in W, \tag{68}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a and $\nabla\psi = (\psi_{,i})$. Over the space W we consider the inner product given by

$$(\varphi,\psi)_W = \int_\Omega \nabla\varphi\cdot\nabla\psi\,\mathrm{d}x$$

and let $\|\cdot\|_W$ be the associated norm. It follows from (68) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and therefore $(W,\|\cdot\|_W)$ is a real Hilbert space. Moreover, the space $\mathcal{W}_1$ is real Hilbert space with the inner product

$$(\boldsymbol{D},\boldsymbol{E})_{\mathcal{W}_1} = \int_\Omega \boldsymbol{D}\cdot\boldsymbol{E}\,\mathrm{d}x + \int_\Omega \operatorname{div}\boldsymbol{D}\cdot\operatorname{div}\boldsymbol{E}\,\mathrm{d}x,$$

where div $\boldsymbol{D} = (D_{i,i})$, and the associated norm $\|\cdot\|_{\mathcal{W}_1}$.

We assume that the piezoelectric operator $\mathcal{E}$ and the electric permittivity operator $\mathcal{B}$ satisfy the following assumptions.

$$\begin{cases} \text{(a)} & \mathcal{E} : \Omega\times\mathbb{S}^d \to \mathbb{R}^d. \\ \text{(b)} & \mathcal{E}(\boldsymbol{x},\boldsymbol{\tau}) = (e_{ijk}(\boldsymbol{x})\tau_{jk}) \quad \forall\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c)} & e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases} \tag{69}$$

$$\begin{cases} \text{(a)} & \mathcal{B} : \Omega\times\mathbb{R}^d \to \mathbb{R}^d. \\ \text{(b)} & \mathcal{B}(\boldsymbol{x},\boldsymbol{E}) = (b_{ij}(\boldsymbol{x})E_j) \quad \forall\boldsymbol{E} = (E_i) \in \mathbb{R}^d, \quad \text{a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c)} & b_{ij} = b_{ji} \in L^\infty(\Omega). \\ \text{(d)} & \text{There exists } m_{\mathcal{B}} > 0 \text{ such that } b_{ij}(\boldsymbol{x})E_iE_j \geq m_{\mathcal{B}}\|\boldsymbol{E}\|^2 \\ & \forall\boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \end{cases} \tag{70}$$

From the assumptions (69) and (70) we deduce that the operators $\mathcal{E}$ and $\mathcal{B}$ are linear, have measurable bounded components denoted e_{ijk} and b_{ij} and, moreover, $\mathcal{B}$ is symmetric and positive definite. Recall also that the transposed operator $\mathcal{E}^*$ is given by $\mathcal{E}^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$, and the following equality holds :

$$\mathcal{E}\boldsymbol{\sigma}\cdot\boldsymbol{v} = \boldsymbol{\sigma}\cdot\mathcal{E}^*\boldsymbol{v} \qquad \forall\boldsymbol{\sigma}\in\mathbb{S}^d,\ \boldsymbol{v}\in\mathbb{R}^d. \tag{71}$$

We also assume that the densities of electric charges satisfy

$$q_0 \in W^{1,\infty}(0,T;L^2(\Omega)), \qquad q_2 \in W^{1,\infty}(0,T;L^2(\Gamma_b)), \tag{72}$$

$$q_2(t) = 0 \quad \text{on } \Gamma_3 \quad \forall\,t\in[0,T]. \tag{73}$$

Note that assumption (73) represents a compatibility condition; indeed, the foundation is supposed to be insulator, like the gap, which is filled with air; therefore, the normal component of the electric displacement field vanishes both on the contact and the separation zone, which implies that $\boldsymbol{D}\cdot\boldsymbol{\nu} = 0$ on $\Gamma_3 \times (0,T)$. Combining this condition with (66) and using assumption $\Gamma_3 \subseteq \Gamma_b$ we obtain (73).

We define the function $q : [0,T] \to W$ by

$$(q(t),\psi)_W = \int_\Omega q_0(t)\psi\,\mathrm{d}x - \int_{\Gamma_b} q_2(t)\psi\,da, \tag{74}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$, $\psi \in W$ and $t \in [0,T]$, and note that conditions (72) imply that

$$q \in W^{1,\infty}(0,T;W). \tag{75}$$

Using arguments similar to those used to derive Problem $\mathcal{P}_V$, we obtain the following variational formulation of the piezoelectric contact problem $\widetilde{\mathcal{P}}$.

Problem $\widetilde{\mathcal{P}}_V$. *Find a displacement field $\boldsymbol{u} : [0,T] \to V$, an electric potential field $\varphi : [0,T] \to W$ and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that*

$$\begin{aligned}(76)\quad &(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}(t)))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}(t)))_{\mathcal{H}} + j_{ad}(\beta(t),\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))\\ &\quad + j_{nc}(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t)) + j_{fr}(\boldsymbol{u}(t),\boldsymbol{v}) - j_{fr}(\boldsymbol{u}(t),\boldsymbol{u}(t))\\ &\geq (\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V \qquad \forall\,\boldsymbol{v}\in V,\ t\in[0,T],\end{aligned}$$

$$\begin{aligned}(77)\quad &(\mathcal{B}\nabla\varphi(t),\nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\nabla\psi)_{L^2(\Omega)^d}\\ &= (q(t),\psi)_W \qquad \forall\,\psi\in W,\ t\in[0,T],\end{aligned}$$

$$(78)\quad \dot\beta(t) = -\Big(\beta(t)\left(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau\|\boldsymbol{R}_\tau(\boldsymbol{u}_\tau(t))\|^2\right) - \epsilon_a\Big)_+ \quad \text{a.e. } t\in(0,T),$$

$$(79)\quad \beta(0) = \beta_0.$$

In the study of Problem $\widetilde{\mathcal{P}}_V$ we have the following existence and uniqueness result.

Theorem 5.1. *Assume that* (13)–(18) *and* (69), (70), (72) *and* (73) *hold. Then, there exists $\widetilde{\mu}_0 > 0$ which depends on Ω, Γ_1, Γ_3, $\mathcal{F}$, $\mathcal{B}$ and p_ν such that Problem $\widetilde{\mathcal{P}}_V$ has a unique solution $(\boldsymbol{u},\beta)$, if $\|\mu\|_{L^\infty(\Gamma_3)} < \widetilde{\mu}_0$. Moreover, the solution has the regularity expressed in* (34), (35) *and*

$$\varphi \in W^{1,\infty}(0,T;W). \tag{80}$$

Let $(\boldsymbol{u},\beta,\varphi)$ be the solution of Problem $\widetilde{\mathcal{P}}_V$ obtained in Theorem 5.1 and denote by $\boldsymbol{\sigma}$, $\boldsymbol{D}$ the function given by (56), (57), respectively. It is easy to check that $\boldsymbol{\sigma}$ satisfies (36) and

$$\boldsymbol{D} \in W^{1,\infty}(0,T;\mathcal{W}_1). \tag{81}$$

A quintuple of functions ($\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, $\boldsymbol{D}$, β) which satisfy (56), (57), (76)–(79) is called a *weak solution* of the contact problem $\widetilde{\mathcal{P}}_V$. We conclude by Theorem 5.1 that, under the stated assumptions, Problem $\widetilde{\mathcal{P}}_V$ has a unique weak solution which satisfies (34)–(36), (80) and (81).

The proof of Theorem 5.1 is similar to the proof of Theorem 4.1 and it is carried out in several steps. Since the modifications are straightforward, we omit the details. In the first step we fix $\beta \in \mathcal{Z}$ and we consider the following variational problem.

Problem $\widetilde{\mathcal{P}}^1_\beta$. *Find a displacement field $\boldsymbol{u}_\beta : [0,T] \to V$ and an electric potential field $\varphi_\beta : [0,T] \to W$ such that, for all $t \in [0,T]$,*

$$(82)\qquad \begin{aligned} &(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\beta(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{u}_\beta(t))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\beta(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_\beta(t)))_{\mathcal{H}} \\ &\quad + j_{ad}(\beta(t), \boldsymbol{u}_\beta(t), \boldsymbol{v} - \boldsymbol{u}_\beta(t)) + j_{nc}(\boldsymbol{u}_\beta(t), \boldsymbol{v} - \boldsymbol{u}_\beta(t)) \\ &\quad + j_{fr}(\boldsymbol{u}_\beta(t), \boldsymbol{v}) - j_{fr}(\boldsymbol{u}_\beta(t), \boldsymbol{u}_\beta(t)) \\ &\geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\beta(t))_V \qquad \forall\, \boldsymbol{v} \in V, \end{aligned}$$

$$(83)\qquad (\mathcal{B}\nabla\varphi_\beta(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\beta(t)), \nabla\psi)_{L^2(\Omega)^d} = (q(t), \psi)_W \qquad \forall\, \psi \in W.$$

We have the following result.

Lemma 5.2. *There exists $\widetilde{\mu}_0 > 0$ which depends on Ω, Γ_1, Γ_3, $\mathcal{F}$, $\mathcal{B}$ and p_ν such that Problem $\widetilde{\mathcal{P}}^1_\beta$ has a unique solution $(\boldsymbol{u}_\beta, \varphi_\beta) \in C([0,T]; V \times W)$, if $\|\mu\|_{L^\infty(\Gamma_3)} < \widetilde{\mu}_0$.*

Proof. In order to solve (82)–(83) we consider the product space $X = V \times W$ endowed with the inner product

$$(x, y)_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_W \qquad \forall\, x = (\boldsymbol{u}, \psi),\ y = (\boldsymbol{v}, \psi) \in X$$

and the associated norm $\|\cdot\|_X$. We define the operator $\widetilde{A}_{\beta(t)} : X \times X \to \mathbb{R}$, the function $j : X \to \mathbb{R}$ and the element $f(t) \in X$ by equalities:

$$(84)\qquad \begin{aligned} (\widetilde{A}_{\beta(t)}x, y)_X &= (A_\beta(t)\boldsymbol{u}, \boldsymbol{v})_V + (\mathcal{B}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} + (\mathcal{E}^*\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ &\quad - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\psi)_{L^2(\Omega)^d} \qquad \forall\, x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X, \end{aligned}$$

$$(85)\qquad j(x, y) = j_{fr}(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall\, x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X,$$

$$(86)\qquad f(t) = (\boldsymbol{f}(t), -q(t)),$$

for all $t \in [0,T]$, where $A_\beta(t)$ is given by (39). It is easy to see that $x_\beta = (\boldsymbol{u}_\beta, \varphi_\beta)$ is a solution to problem (82)–(83) with regularity $(\boldsymbol{u}_\beta, \varphi_\beta) \in C([0,T]; V \times W)$ if and only if $x_\beta \in C([0,T]; X)$ and

$$(87)\qquad \begin{aligned} &(A_{\beta(t)}x_\beta(t), y - x_\beta(t))_X + j(x_\beta(t), y) - j(x_\beta(t), x_\beta(t))_X \\ &\geq (f(t), y - x_\eta(t))_X \qquad \forall\, y \in X,\ t \in [0,T]. \end{aligned}$$

Next, we use (40), (41), (69)–(71) to see that $A_{\beta(t)}$ is a Lipschitz continuous operator on X and satisfies

$$(88)\qquad (A_{\beta(t)}x_1 - A_{\beta(t)}x_2, x_1 - x_2)_X \geq \min\{m_{\mathcal{F}}, m_{\mathcal{B}}\}\|x_1 - x_2\|^2_X \quad \forall x_1,\, x_2 \in X.$$

Also, using (85) and (33), we can easily check that, for a given $x \in X$, the functional $j(x, \cdot) : X \to \mathbb{R}$ is convex and lower semicontinuous and satisfies

$$(89)\qquad \begin{aligned} &j(x_1, y_2) - j(x_1, x_1) + j(x_2, x_1) - j(x_2, x_2) \\ &\leq c_0^2 L_\nu\, \|\mu\|_{L^\infty(\Gamma_3)}\|x_1 - x_2\|_X\|y_1 - y_2\|_X \qquad \forall x_1\, x_2,\, y_1,\, y_2 \in X. \end{aligned}$$

Finally, (86) and the regularity (20) and (75) show that $f \in W^{1,\infty}(0,T;V)$. Let

$$\widetilde{\mu}_0 = \frac{\min\{m_{\mathcal{F}}, m_{\mathcal{B}}\}}{c_0^2 L_\nu} \tag{90}$$

and assume that $\|\mu\|_{L^\infty(\Gamma_3)} < \widetilde{\mu}_0$; we proceed like in the proof of Lemma 4.2 to see that problem (87) has a unique solution $x_\beta \in C([0,T];X)$ which concludes the proof. □

The rest of the steps in the proof of Theorem 5.1 are as follows.

Proof of Theorem 5.1. We assume in what follows that $\|\mu\|_{L^\infty(\Gamma_3)} < \widetilde{\mu}_0$ and, for a given $\beta \in \mathcal{Z}$ we denote by $(\boldsymbol{u}_\beta, \varphi_\beta)$ the solution of the Problem $\widetilde{\mathcal{P}}_\beta^1$ obtained in Lemma 5.2. We use Lemma 4.3 to prove that, for a given $\beta \in \mathcal{Z}$ there exists a unique element θ_β such that

(91) $\theta_\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Q}$,

(92) $\dot{\theta}_\beta(t) = -\Big(\gamma_\nu \theta_\beta(t)\big(R_\nu(u_{\beta\nu}(t))^2 + \gamma_\tau \|\boldsymbol{R}_\tau(\boldsymbol{u}_{\beta\tau}(t))\|^2\big) - \epsilon_a\Big)_+$ a.e. $t \in (0,T)$,

(93) $\theta_\beta(0) = \beta_0$.

Also, it follows from Lemma 4.4 that the operator $\Lambda : \mathcal{Z} \to \mathcal{Z}$ given by

$$\Lambda\beta = \theta_\beta \tag{94}$$

has unique fixed point $\beta^* \in \mathcal{Z}$. Denote $\boldsymbol{u}^* = \boldsymbol{u}_{\beta^*}$, $\varphi^* = \varphi_{\beta^*}$, where $(\boldsymbol{u}_{\beta^*}, \varphi_{\beta^*})$ is the couple of functions obtained in Lemma 5.2 for $\beta = \beta^*$. Then, we use (82)–(83) and (91)–(94) to see that $(\boldsymbol{u}^*, \varphi^*, \beta^*)$ is a solution of Problem $\widetilde{\mathcal{P}}_V$. The uniqueness of the solution as well as the regularity (34), (35) and (80) follows from arguments similar to those used in the proof of Theorem 4.1. □

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