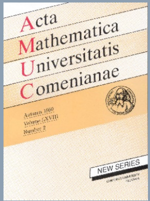

MAXIMAL OPERATORS OF THE FEJÉR MEANS OF THE TWO DIMENSIONAL CHARACTER SYSTEM OF THE p-SERIES FIELD IN THE KACZMARZ REARRANGEMENT

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Abstract. The main aim of this paper is to prove that the maximal operator σ^* of the Fejér means of the two dimensional character system of the p-series field in the Kaczmarz rearrangement is bounded from the Hardy space H_α to the space L_α for $\alpha > 1/2$, provided that the supremum in the maximal operator is taken over a positive cone. We also prove that the maximal operator σ_0^* of Fejér means of the two dimensional character system of the p-series field in the Kaczmarz rearrangement is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

1. Introduction

The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [9] showed that the maximal operator $\sigma^* f$ is of weak type $(1,1)$, from which the a. e. convergence follows on standard argument. Schipp result implies also the boundedness of $\sigma^* : L_\alpha \to L_\alpha$ $(1 < \alpha \leq \infty)$ by interpolation. This fails to hold for $\alpha = 1$ but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 (see also Simon [13]). Fujii's theorem was extened by Weisz [15]. Namely, he proved that the maximal operator of the Fejér

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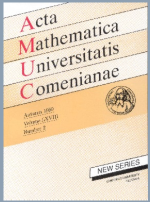
means of the one-dimensional Walsh-Fourier series is bounded from the martingale Hardy space H_α to the space L_α for $\alpha > 1/2$. Simon [11] gave a counterexample, which shows that this boundedness does not hold for $0 < \alpha < 1/2$. In the endpoint case $\alpha = 1/2$ Weisz [17] proved that σ^* is bounded from the Hardy space $H_{1/2}(G_2)$ to the space weak-$L_{1/2}(G_2)$. The author [6] proved that σ^* is not bounded from the Hardy space $H_{1/2}(G_2)$ to the space $L_{1/2}(G_2)$.

If the Walsh system is taken in the Kaczmarz ordening, the analogous to the statement of Schipp [9] is due to Gát [3]. Moreover he proved an (H_1, L_1)-type estimation. Gát result was extended to the Hardy space by Simon [12], who proved that σ^* is of type (H_α, L_α) for $\alpha > 1/2$. Weisz [17] showed that in endpoint case $\alpha = 1/2$ the maximal operator is of weak type $\left(H_{1/2}, L_{1/2}\right)$.

Gát and Nagy [4] proved the a. e. convergence $\sigma_n f \to f$ $(n \to \infty)$ for an integrable function $f \in L_1(G_p)$, where $\sigma_n f$ is the Fejér means of the function f with respect to the character system in the Kaczmarz rearrangement. This result was generalized by the author [7] and it is proved that the maximal operator σ^* of the Fejér means of the one dimensional character system of the p-series field in the Kaczmarz rearrangement is bounded from the Hardy space $H_{1/2}(G_p)$ to the space weak-$L_{1/2}(G_p)$. By interpolation it follows that σ^* is of type (H_α, L_α) for $\alpha > 1/2$. We also prove that the assumption $\alpha > 1/2$ is essentiall, in particular, it is proved that the maximal operator σ^* is not bounded from the Hardy space $H_{1/2}(G_p)$ to the space $L_{1/2}(G_p)$. By interpolation it follows that σ^* is not of type $(H_\alpha, \text{weak-}L_\alpha)$ for $0 < \alpha < 1/2$.

The aim of this paper is to prove that the maximal operator of Fejér means of the two dimensional character system of the p-series field in the Kaczmarz rearrangement is bounded from the Hardy space $H_\alpha(G_p \times G_p)$ to the space $L_\alpha(G_p \times G_p)$ for $\alpha > 1/2$ and is of weak type $(1, 1)$ provided that the supremum in the maximal operator is taken over a positive cone. So we obtain that the Fejer means of a function $f \in L_1(G_p \times G_p)$ converge a. e. to the function in the question, provided again that the limit is taken over a positive cone. We also proved that the maximal operator σ_0^* of Fejér means of the two dimensional character system of the p-series field in the Kaczmarz rearrangement



is not bounded from the Hardy space $H_{1/2}(G_p \times G_p)$ to the space weak-$L_{1/2}(G_p \times G_p)$. Thus, in the question of boundedness of the maximal operator σ_0^* the case of two dimensional character system of the p-series field in the Kaczmarz rearrangement differs from that one-dimensional character system of the p-series field in the Kaczmarz rearrangement. By Theorem 2 and interpolation it follows that σ_0^* is not bounded from $H_\alpha(G_p \times G_p)$ to the space weak-$L_\alpha(G_p \times G_p)$ for $0 < \alpha \leq 1/2$. In particular, from Theorem 2 we have that in Theorem 1 the assumption $\alpha > 1/2$ is essential.

2. Definitions and Notation

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Let $2 \leq p \in \mathbf{N}$ and denote by $\mathbf{Z}_p$ the pth cyclic group, that is, $\mathbf{Z}_p$ can be represented by the set $\{0, 1, \ldots, p-1\}$, where the group operation is the mod p addition and every subset is open. The Haar measure on $\mathbf{Z}_p$ is given in the way that

$$\mu_k(\{j\}) := \frac{1}{j} \qquad (j \in \mathbf{Z}).$$

The group operation on G_p is the coordinate-wise addition, the normalized Haar measure μ is the product measure. The topology on G_p is the product topology, a base for the neighborhoods of G_p can be given in the following way:

$$I_0(x) := G_p,$$
$$I_n(x) := \{y \in G_p : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}, \qquad (x \in G_p, n \in \mathbf{N}).$$

Let $0 = (0 : i \in \mathbf{N}) \in G_p$ denote the null element of G_p, $I_n := I_n(0)$ $(n \in \mathbf{N})$, $\overline{I}_n := G_p \setminus I_n$. Let

$$\Delta := \{I_n(x) : x \in G_p,\ n \in \mathbf{N}\}.$$

The elements of Δ are intervals of G_p. Set $e_i := (0, \ldots, 0, 1, 0, \ldots) \in G_p$ whose i-th coordinate is 1, the rest are zeros.

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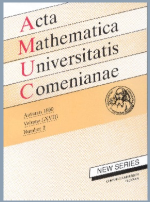

The norm (or quasinorm) of the space $L_\alpha(G_p \times G_p)$ is defined by

$$\|f\|_\alpha := \left(\int_{G_p \times G_p} \left| f\left(x^1, x^2\right) \right|^\alpha \mathrm{d}\mu\left(x^1, x^2\right) \right)^{1/\alpha}, \qquad (0 < \alpha < +\infty).$$

Let $\Gamma(p)$ denote the character group of G_p. We arrange the elements of $\Gamma(p)$ as follows. For $k \in \mathbf{N}$ and $x \in G_p$ denote by r_k the k-th generalized Rademacher function

$$r_k(x) := \exp\left(\frac{2\pi\,\mathrm{i}\,x_k}{p}\right) \qquad \left(\mathrm{i} := \sqrt{-1}, \quad x \in G_p, \quad k \in \mathbf{N}\right).$$

Let $n \in \mathbf{N}$. Then

$$n = \sum_{i=0}^{\infty} n_i p^i, \qquad \text{where} \;\; 0 \le n_i < p \quad (n_i, i \in \mathbf{N}),$$

n is expressed in the number system with base p. Denote by

$$|n| := \max(j \in \mathbf{N} : n_j \neq 0) \qquad \text{i. e.,} \;\; p^{|n|} \le n < p^{|n|+1}.$$

Now, we define the sequence of functions $\psi := (\psi_n : n \in \mathbf{N})$ by

$$\psi_n(x) := \prod_{k=0}^{\infty} \left(r_k(x)\right)^{n_k} \quad (x \in G_p, \; n \in \mathbf{N}).$$

We remark that $\Gamma(p) = \{\psi_n : n \in \mathbf{N}\}$ is a complete orthogonal system with respect to the normalized Haar measure on G_p.

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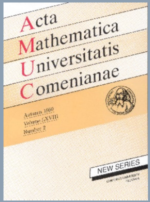


The character group $\Gamma(p)$ can be given in the Kaczmarz rearrangement as follows: $\Gamma(p) = \{\chi_n : n \in \mathbf{N}\}$, where

$$\chi_n(x) := r_{|n|}^{n_{|n|}}(x) \prod_{k=0}^{|n|-1} \left(r_{|n|-1-k}(x)\right)^{n_k} \qquad (x \in G_p, n \in \mathbf{P}),$$

$$\chi_0(x) = 1 \qquad (x \in G_p).$$

Let the transformation $\tau_A : G_p \to G_p$ be defined as follows:

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_0, x_A, x_{A+1}, \ldots).$$

The transformation is measure-preservingand and $\tau_A(\tau_A(x)) = x$. By the definition of τ_A , we have

$$\chi_n(x) = r_{|n|}^{n_{|n|}}(x)\psi_{n-n_{|n|}p^n}(\tau_{|n|}(x)) \qquad (n \in \mathbf{N},\ x \in G_p).$$

The rectangular partial sums of the double Fourier series are defined as follows:

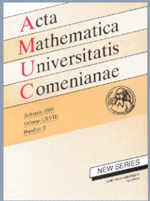
$$S_{M,N}(f; x^1, x^2) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j)\, \chi_i\left(x^1\right) \chi_j\left(x^2\right),$$

where the number

$$\widehat{f}(i,j) = \int_{G_p \times G_p} f\left(x^1, x^2\right) \overline{\chi}_i\left(x^1\right) \overline{\chi}_j\left(x^2\right) \mathrm{d}\mu\left(x^1, x^2\right)$$

is said to be the (i,j)-th Fourier coefficient of the function f. Let

$$I_{n,n}\left(x^1, x^2\right) := I_n\left(x^1\right) \times I_n\left(x^2\right).$$



The σ-algebra generated by the dyadic rectangles

$$\left\{I_{n,n}\left(x^1,x^2\right):\left(x^1,x^2\right)\in G_p\times G_p\right\}$$

will be denoted by $F_{n,n}\,(n\in\mathbf{N})$.

Denote by $f=\left(f^{(n,n)},n\in\mathbf{N}\right)$ martingale with respect to $(F_{n,n},n\in\mathbf{N})$ (for details see, e. g. [14, 16]

The diagonal maximal function of a martingale f is defined by

$$f^*=\sup_{n\in\mathbf{N}}\left|f^{(n,n)}\right|.$$

In case $f\in L_1\left(G_p\times G_p\right)$, diagonal maximal function can also be given by

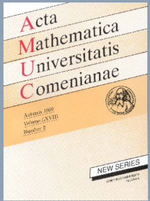
$$f^*\left(x^1,x^2\right)=\sup_{n\in\mathbf{N}}\frac{1}{\mu\left(I_{n,n}(x^1,x^2)\right)}\left|\int\limits_{I_{n,n}(x^1,x^2)}f\left(u^1,u^2\right)\mathrm{d}\mu\left(u^1,u^2\right)\right|,$$
$$\left(x^1,x^2\right)\in G_p\times G_p.$$

For $0<p<\infty$ the Hardy martingale space $H_p(G_p\times G_p)$ consists of all martingales for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

If $f\in L_1\left(G_p\times G_p\right)$ then it is easy to show that the sequence $\left(S_{p^n,p^n}\left(f\right):n\in\mathbf{N}\right)$ is a martingale. If f is a martingale, that is $f=(f^{(n,n)}:n\in\mathbf{N})$, then the Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}\left(i,j\right)=\lim_{k\to\infty}\int\limits_{G\times G}f^{(k,k)}\left(x^1,x^2\right)\overline{\chi}_i\left(x^1\right)\overline{\chi}_j\left(x^2\right)\mathrm{d}\mu\left(x^1,x^2\right).$$



The Fourier coefficients of $f \in L_1(G_p \times G_p)$ are the same as the ones of the martingale $(S_{p^n,p^n}(f) : n \in \mathbf{N})$ obtained from f.

For $n, m \in \mathbf{P}$ and a martingale f the Fejér means of order (n, m) of the two-dimensional character system of the p-series field in the Kaczmarz rearrangement of the martingale f is given by

$$\sigma_{n,m}(f; x^1, x^2) = \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} S_{i,j}(f; x^1, x^2).$$

For the martingale f, the restricted maximal operator of the Fejér means is defined by

$$\sigma_\lambda^* f\left(x^1, x^2\right) = \sup_{p^{-\lambda} \le n/m \le p^\lambda} |\sigma_{n,m}(f; x^1, x^2)|, \qquad \lambda > 0.$$

The Dirichlet kernels and Fejér kernels are defined as follows

$$D_n^\gamma(x) := \sum_{j=0}^{n-1} \gamma_j(x), \qquad K_n^\gamma(x) := \sum_{j=0}^{n-1} D_j^\gamma(x),$$

where γ is either ψ or χ.

The p^nth Dirichlet kernels have a closed form:

$$D_{p^n}^\psi(x) = D_{p^n}^\chi(x) = \begin{cases} p^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n, \end{cases} \qquad \text{where } x \in G_p. \tag{1}$$

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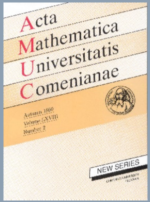

3. Formulation of Main Results

Theorem 1. *Let* $\alpha > 1/2$. *Then the maximal operator* σ_λ^* *is bounded from the Hardy space* $H_\alpha(G_p \times G_p)$ *to the space* $L_\alpha(G_p \times G_p)$. *Especialy, if* $f \in L_1(G_p \times G_p)$ *then*

$$\mu\left(\sigma_\lambda^* > y\right) \leq \frac{c}{y}\left\|f\right\|_1.$$

Corollary 1. *If* $f \in L_1(G_p \times G_p)$, *then*

$$\sigma_{n,m} f\left(x^1, x^2\right) \to f\left(x^1, x^2\right) \qquad a.\ e.$$

as $\min(n,m) \to \infty$ *and* $p^{-\lambda} \leq n/m \leq p^\lambda$ $(\lambda > 0)$.

Theorem 2. *The maximal operator* σ_0^* *is not bounded from the Hardy space* $H_{1/2}(G_p \times G_p)$ *to the space weak-*$L_{1/2}(G_p \times G_p)$.

4. Auxiliary Propositions

We shall need the following lemmas

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Lemma 1 (Gát, Nagy [4]). *Let $A \in \mathbf{N}$ and $n := n_A p^A + n_{A-1} p^{A-1} + \cdots + n_0 p^0$. Then*

$$nK_n^{\chi}(x) = 1 + \sum_{j=0}^{A-1} \sum_{i=1}^{p-1} r_j^i(x)\, p^j K_{p^j}^{\psi}(\tau_j(x)) + \sum_{j=0}^{A-1} p^j D_{p^j}^{\psi}(x) \sum_{l=1}^{p-1} \sum_{i=0}^{l-1} r_j^i(x)$$

$$+ p^A \sum_{l=1}^{n_A-1} r_A^l(x)\, K_{p^A}^{\psi}(\tau_A(x)) + r_A^{n_A}(x)\left(n - n_A p^A\right) K_{n-n_A p^A}^{\psi}(\tau_A(x))$$

$$+ \left(n - n_A p^A\right) \sum_{i=0}^{n_A-1} r_A^i(x)\, D_{p^A}^{\psi}(x) + +p^A \sum_{j=1}^{n_A-1} \sum_{i=0}^{j-1} r_A^i(x)\, D_{p^A}^{\psi}(x).$$

Lemma 2 (Gát, Nagy [4]). *Let $A,\ l \in \mathbf{N}$, $A > l$ and $x \in I_l \setminus I_{l+1}$. Then*

$$K_{p^A}^{\psi}(x) = \begin{cases} 0, & \text{if } x - x_l e_l \notin I_A, \\ \dfrac{p^l}{1 - r_l(x)} & \text{if } x - x_l e_l \in I_A. \end{cases}$$

Lemma 3 ([7]). *Let $n < p^{A+1}$, $A > N$ and $x \in I_N(x_0, \ldots, x_{m-1}, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0)$ $m = -1, 0, \ldots, l-1$, $l = 0, \ldots, N$. Then*

$$\int_{I_N} n\left|K_n^{\psi}(\tau_A(x-t))\right| \mathrm{d}\mu(t) \leq \frac{cp^A}{p^{m+l}},$$

where

$$I_N(x_0, \ldots, x_{m-1}, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0)$$
$$:= I_N(0, \ldots, 0, x_l \neq 0, 0, \ldots, 0) \qquad \textit{for} \quad m = -1,$$

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and

$$I_N\left(x_0,\dots,x_{m-1},x_m\neq 0,0,\dots,0,x_l\neq 0,0,\dots,0\right)$$
$$:=I_N\left(x_0,\dots,x_{m-1},x_m\neq 0,0,\dots,0\right),\qquad \textit{for}\;\; l=N.$$

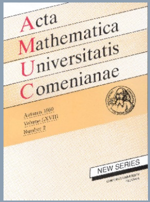
Lemma 4 ([5]). *Let $A\in\mathbf{N}$ and $n_A:=p^{2A}+p^{2A-2}+\ldots+p^2+p^0$. Then*

$$n_{A-1}|K_{n_{A-1}}(x)|\geq cp^{2k+2s}$$

for $x\in I_{2A}(0,\dots,0,x_{2k}\neq 0,0,\dots,0,x_{2s}\neq 0,x_{2s+1},\dots,x_{2A-1})$, $k=0,1,\dots,A-3$, $s=k+2$, $k+3,\dots,A-1$.

Lemma 5. *Let $x\in\overline{I}_N$ and $n\geq p^N$. Then*

$$\int_{I_N}|K_n^{\chi}(x-t)|\mathrm{d}\mu(t)$$
$$\leq c\left\{\sum_{l=0}^{N}\sum_{m=-1}^{l-1}\frac{1}{p^{m+l}}\mathbf{1}_{I_N(x_0,\dots,x_{m-1},x_m\neq 0,0,\dots,0,x_l\neq 0,0,\dots,0)}(x)\right.$$
$$\left.+\frac{1}{p^{2N}}\sum_{j=1}^{N}p^{2j}\sum_{l=0}^{j-1}\frac{1}{p^l}\mathbf{1}_{I_N(0,\dots,0,x_l\neq 0,0,\dots,0,,x_j,\dots,x_{N-1})}(x)\right\}.$$



Proof. From Lemma 1 we write

$$
n\left|K_n^{\chi}(x)\right| \le c\left\{1+\sum_{j=0}^{A}p^j\left|K_{p^j}^{\psi}(\tau_j(x))\right|\right.
$$

$$
\left.+\sum_{j=0}^{A}p^j\left|D_{p^j}^{\psi}(x)\right|+\left(n-n_Ap^A\right)\left|K_{n-n_Ap^A}^{\psi}(\tau_A(x))\right|\right\}. \tag{2}
$$

Using Lemma 3 we obtain

$$
\frac{1}{n}\int\limits_{I_N}\left(n-n_Ap^A\right)\left|K_{n-n_Ap^A}^{\psi}(\tau_A(x-t))\right|\mathrm{d}\mu(t)
$$

$$
\le c\left\{\sum_{l=0}^{N}\sum_{m=-1}^{l-1}\frac{1}{p^{m+l}}\mathbf{1}_{I_N(x_0,\dots,x_{m-1},x_m\neq 0,0,\dots,0,x_l\neq 0,0,\dots,0)}(x)\right\}. \tag{3}
$$

Let $x\in I_N(x_0,\dots,x_{m-1},x_m\neq 0,0,\dots,0,x_l\neq 0,0,\dots,0)$ for some $m=-1,\dots,l-1$, $l=0,\dots,N$. Then using Lemma 2 $K_{p^j}^{\psi}(\tau_j(x-t))\neq 0\ \ (j>N)$ implies

$$
t\in I_j(0,\dots,0,x_N,\dots,x_{j-1}),\qquad m=-1.
$$

Consequently, we can write

$$
\int\limits_{I_N}p^j\left|K_{p^j}^{\psi}(\tau_j(x-t))\right|\mathrm{d}\mu(t)\le\frac{cp^j}{p^j}p^{j-l}\mathbf{1}_{I_N(0,\dots,0,x_l\neq 0,0,\dots,0)}(x)
$$

$$
=\frac{cp^j}{p^l}\mathbf{1}_{I_N(0,\dots,0,x_l\neq 0,0,\dots,0)}(x). \tag{4}
$$

Let $j < N$. Then using Lemma 2 $K^{\psi}_{p^j}\left(\tau_j\left(x-t\right)\right) \neq 0$ implies

$$x \in I_N\left(0,\dots,0,x_l \neq 0,0,\dots,0,x_j,\dots,x_{N-1}\right), l = -1,0,\dots,j-1.$$

Hence we have

$$\begin{aligned}\int\limits_{I_N} p^j \left|K^{\psi}_{p^j}\left(\tau_j\left(x-t\right)\right)\right| \mathrm{d}\mu\left(t\right) &\leq \frac{cp^j}{p^N}\sum_{l=0}^{j-1} p^{j-l}\mathbf{1}_{I_N\left(0,\dots,0,x_l\neq 0,0,\dots,0,x_j,\dots,x_{N-1}\right)}\left(x\right) \\ &= \frac{cp^{2j}}{p^N}\sum_{l=0}^{j-1} p^{-l}\mathbf{1}_{I_N\left(0,\dots,0,x_l\neq 0,0,\dots,0,x_j,\dots,x_{N-1}\right)}\left(x\right). \end{aligned} \tag{5}$$

From (1) we can write

$$\begin{aligned}\sum_{j=0}^{A} p^j \int\limits_{I_N} \left|D^{\psi}_{p^j}\left(x-t\right)\right| \mathrm{d}\mu\left(t\right) &\leq \frac{c}{p^N}\sum_{j=0}^{N-1} p^j \left|D^{\psi}_{p^j}\left(x\right)\right| \\ &\leq \frac{c}{p^N}\sum_{j=0}^{N-1} p^{2j}\mathbf{1}_{I_N\left(0,\dots,0,x_j,\dots,x_{N-1}\right)}\left(x\right). \end{aligned} \tag{6}$$

Combining (2)–(6) we complete the proof of Lemma 5. □

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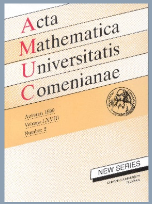

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5. Proofs of Main results

Proof of Theorem 1. In order to prove Theorem 1 it is enough to show that (see Simon [**11**], Theorem 1)

$$\int\limits_{\overline{I}_N}\left(\sup_{n\geq 2^N}\int\limits_{I_N}\left|K_n^{\chi}\left(x-t\right)\right|\mathrm{d}\mu\left(t\right)\right)^{\alpha}\mathrm{d}\mu\left(x\right)\leq c_{\alpha}p^{-N},\qquad \text{for } 1/2<\alpha\leq 1.$$

Applying the inequality

$$\left(\sum_{k=0}^{\infty}a_k\right)^{\alpha}\leq\sum_{k=0}^{\infty}a_k^{\alpha}\qquad\left(a_k\geq 0,\quad 0<\alpha\leq 1\right),$$

from Lemma 5 we can write

$$\begin{aligned}&\int\limits_{\overline{I}_N}\left(\sup_{n\geq 2^N}\int\limits_{I_N}\left|K_n^{\chi}\left(x-t\right)\right|\mathrm{d}\mu\left(t\right)\right)^{\alpha}\mathrm{d}\mu\left(t\right)\\ &\leq c_{\alpha}\left\{\sum_{l=0}^{N}\sum_{m=-1}^{l-1}\frac{1}{p^{\alpha(m+l)}}\int\limits_{G}\mathbf{1}_{I_N(x_0,\ldots,x_{m-1},x_m\neq 0,0,\ldots,0,x_l\neq 0,0,\ldots,0)}\left(x\right)\mathrm{d}\mu\left(x\right)\right.\\ &\quad\left.+\frac{1}{p^{2\alpha N}}\sum_{j=1}^{N}p^{2j\alpha}\sum_{l=0}^{j-1}\frac{1}{p^{l\alpha}}\int\limits_{G}\mathbf{1}_{I_N(0,\ldots,0,x_l\neq 0,0,\ldots,0,,x_j,\ldots,x_{N-1})}\left(x\right)\mathrm{d}\mu\left(x\right)\right\}\end{aligned}$$

$$\leq c_\alpha \left\{ \frac{1}{p^N} \sum_{l=0}^{N} \sum_{m=-1}^{l-1} \frac{p^m}{p^{\alpha(m+l)}} + \frac{1}{p^N p^{2\alpha N}} \sum_{j=1}^{N} p^{2j\alpha} \sum_{l=0}^{j-1} \frac{p^{N-j}}{p^{l\alpha}} \right\} \leq cp^{-N}.$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let $A \in \mathbf{P}$ and

$$f_A\left(x^1, x^2\right) := \left(D_{p^{2A+1}}\left(x^1\right) - D_{p^{2A}}\left(x^1\right)\right)\left(D_{p^{2A+1}}\left(x^2\right) - D_{p^{2A+1}}\left(x^2\right)\right).$$

It is simple to calculate

$$\widehat{f}_A^{\psi}(i,k) = \begin{cases} 1, & \text{if } \; i,k = p^{2A}, \ldots, p^{2A+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$S_{i,j}^{\psi}\left(f_A; x^1, x^2\right) = \begin{cases} \left(D_i^{\psi}\left(x^1\right) - D_{p^{2A}}\left(x^1\right)\right)\left(D_j^{\psi}\left(x^2\right) - D_{p^{2A}}\left(x^2\right)\right), & \\ \qquad \text{if } \; i,j = p^{2A}+1, \ldots, p^{2A+1}-1, & \\ f_A\left(x^1, x^2\right), \qquad \text{if } \; i,j \geq p^{2A+1}, & \\ 0, \qquad \text{otherwise.} & \end{cases} \tag{7}$$

Since

$$f_A^*\left(x^1, x^2\right) = \sup_{n \in \mathbf{N}} \left|S_{p^n,p^n}\left(f_A; x^1, x^2\right)\right| = \left|f_A\left(x^1, x^2\right)\right|,$$

from (1) we get

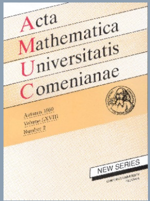
$$\|f_A\|_{H_\alpha} = \|f_A^*\|_\alpha = \left\|D_{p^{2A}}\right\|_\alpha^2 = p^{4A(1-1/\alpha)}. \tag{8}$$

Since

$$D^\chi_{k+p^{2A}}\left(x\right) - D^\chi_{p^{2A}}\left(x\right) = r_{2A}\left(x\right) D_k\left(\tau_{2A}\left(x\right)\right), \qquad k = 1, 2, \ldots, p^{2A},$$

from (7) we obtain

$$\begin{aligned}
\sigma_0^{\chi *} f_A\left(x^1, x^2\right) &= \sup_{n\in\mathbf{N}} \left|\sigma_{n,n} f_A\left(x^1, x^2\right)\right| \geq \left|\sigma_{n_A,n_A} f_A\left(x^1, x^2\right)\right| \\
&= \frac{1}{(n_A)^2}\left|\sum_{i=0}^{n_A-1}\sum_{j=0}^{n_A-1} S^\chi_{i,j} f_A\left(x^1, x^2\right)\right| \\
&= \frac{1}{(n_A)^2}\left|\sum_{i=p^{2A}+1}^{n_A-1}\sum_{j=p^{2A}+1}^{n_A-1}\left(D_i^\chi(x^1) - D_{p^{2A}}\left(x^1\right)\right)\left(D_j^\chi(x^2) - D_{p^{2A}}(x^2)\right)\right| \\
&= \frac{1}{(n_A)^2}\left|\sum_{i=1}^{n_{A-1}-1}\sum_{j=1}^{n_{A-1}-1}\left(D^\chi_{i+p^{2A}}(x^1) - D_{p^{2A}}(x^1)\right)\left(D^\chi_{j+p^{2A}}(x^2) - D_{p^{2A}}(x^2)\right)\right| \qquad (9) \\
&= \frac{1}{(n_A)^2}\left|r_{2A}\left(x^1\right) r_{2A}\left(x^2\right)\sum_{i=1}^{n_{A-1}-1}\sum_{j=1}^{n_{A-1}-1} D_i^\psi\left(\tau_{2A}\left(x^1\right)\right) D_j^\psi\left(\tau_{2A}\left(x^2\right)\right)\right| \\
&= \frac{n_{A-1}^2}{n_A^2}\left|K^\psi_{n_{A-1}}\left(\tau_{2A}\left(x^1\right)\right)\right|\left|K^\psi_{n_{A-1}}\left(\tau_{2A}\left(x^2\right)\right)\right|.
\end{aligned}$$



Denote

$$J_{2A}^{m,s}(x) := I_{2A}(x_0, x_1, \dots, x_{2A-2s-2}, x_{2A-2s-1} = 1, 0, \dots, x_{2A-2m-1} = 1, 0, \dots, 0)$$

and let

$$\left(x^1, x^2\right) \in J_{2A}^{k_l^1, k_l^1+1}\left(x^1\right) \times J_{2A}^{k_l^2, k_l^2+1}\left(x^2\right),$$

where

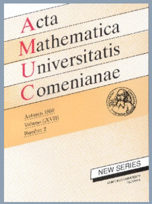
$$k_l^1 := \left[\frac{A}{2}\right] + \left[\frac{1}{8}\log_p A\right] - l, \qquad k_l^2 := \left[\frac{A}{2}\right] + \left[\frac{1}{8}\log_p A\right] + l \qquad l = 0, 1, \dots, \left[\frac{1}{8}\log_p A\right].$$

Then from Lemma 4 and (9) we obtain

$$\sigma_0^* f_A\left(x^1, x^2\right) \geq c\frac{p^{4k_l^1+4k_l^2}}{p^{4A}} \geq \frac{p^{2A+\log_p \sqrt{A}-4l} p^{2A+\log_p \sqrt{A}+4l}}{p^{4A}} \geq cA.$$

On the other hand,

$$\begin{aligned}
&\mu\left\{\left(x^1, x^2\right) \in G_p \times G_p : \left|\sigma_0^{\chi *} f_A\left(x^1, x^2\right)\right| \geq cA\right\} \\
&\geq c \sum_{l=1}^{[\frac{1}{8}\log_q \sqrt{A}]} \sum_{x_0^1=0}^{p-1} \cdots \sum_{x_{2A-2k_l^1-2}^1=0}^{p-1} \sum_{x_0^2=0}^{p-1} \cdots \sum_{x_{2A-2k_l^1-2}^2=0}^{p-1} \mu\left(J_{2A}^{k_l^1, k_l^1+1}\left(x^1\right) \times J_{2A}^{k_l^2, k_l^2+1}\left(x^2\right)\right) \\
&\geq c \sum_{l=1}^{[\frac{1}{8}\log_q \sqrt{A}]} \frac{p^{2A-2k_l^1} p^{2A-2k_l^2}}{p^{4A}} = c \sum_{l=1}^{[\frac{1}{8}\log_q \sqrt{A}]} \frac{1}{p^{2k_l^1} p^{2k_l^2}} \\
&= c \sum_{l=1}^{[\frac{1}{8}\log_q \sqrt{A}]} \frac{1}{p^{A+\log_p \sqrt[4]{A}-2l} p^{A+\log_p \sqrt[4]{A}+2l}} \geq c\frac{\log_p A}{p^{2A+\log_p \sqrt{A}}} = c\frac{\log_p A}{\sqrt{A} p^{2A}}.
\end{aligned}$$



Then from (8) we obtain

$$\frac{cA\left(\mu\left\{\left(x^1,x^2\right)\in G_p\times G_p:\left|\sigma_0^{\chi*}f_A\left(x^1,x^2\right)\right|\geq cA\right\}\right)^2}{\|f_A\|_{H_{1/2}}}$$

$$\geq\frac{cA\log_p^2 A}{p^{-4A}p^{4A}A}\geq c\log_p^2 A\to\infty\qquad\text{as}\qquad A\to\infty.$$

Theorem 2 is proved. □

We remark that in the case $p=2$ Theorem 2 is due to Goginava and Nagy [8].

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