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# BEHAVIOR AT INFINITY OF CONVOLUTION TYPE INTEGRALS

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Abstract. Behavior at infinity of convolution type integrals on abstract spaces is studied.

## 1. Introduction

Let $0 < \alpha < n$. The operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, \mathrm{d}y$$

is known as the classical Riesz potential. We refer to the monographs [**1**], [**5**], [**6**] for various properties of the Riesz potentials. Their behavior at infinity was investigated in [**3**], [**4**], [**7**].

It is easy to see that if $f$ is non-negative and compactly supported, then $I_\alpha f(x)$ has the order $|x|^{\alpha-n}$ at infinity. D. Siegel and E. Talvila [**7**] found necessary and sufficient conditions on $f$ for the validity of $I_\alpha f(x) = O(|x|^{\alpha-n})$ as $|x| \to \infty$ even when $f$ is not compactly supported.

**Theorem A.** ([**7**]) *If $f \geq 0$, then a necessary and sufficient condition for $I_\alpha f(x)$ to exist on $\mathbb{R}^n$ and be $O(|x|^{\alpha-n})$ as $|x| \to \infty$ is such that*

$$\int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, (1+|y|)^{n-\alpha}\, \mathrm{d}y$$

*is bounded on $\mathbb{R}^n$.*

We generalize this fact for convolution type integrals on abstract spaces with a monotone decreasing kernel satisfying the so-called “doubling” condition. The limit at infinity of convolution type integrals, on normal homogeneous spaces, which are generalizations of classic Riesz potentials is also studied.

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## 2. The Necessary and Sufficient Condition

**Definition 1.** Let $X$ be a set. A function $\rho : X \times X \to [0, \infty)$ is called quasi-metric if

1. $\rho(x, y) = 0 \Leftrightarrow x = y$;
2. $\rho(x, y) = \rho(y, x)$;
3. there exists a constant $c \geq 1$ such that for every $x, y, z \in X$

$$\rho(x, y) \leq c(\rho(x, z) + \rho(z, y)).$$

If $(X, \rho)$ is a set endowed with a quasi-metric, then the balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$, where $x \in X$ and $r > 0$, satisfy the axioms of a complete system of neighborhoods in $X$, and therefore induce a (separated) topology. With respect to this topology, the balls $B(x, r)$ need not be open.

We denote $\operatorname{diam} X = \sup\{\rho(x, y) : x \in X, y \in X\}$.

**Lemma 1.** *Let $(X, \rho)$ be a set with a quasi-metric, $\operatorname{diam} X = \infty$ and $m > c$. Then $B(x, m\rho(0, x)) \to X$ as $\rho(0, x) \to \infty$.*

*Proof.* Assume the contrary. Suppose that there is an $y \in X$ such that for all $\delta > 0$ there exists an $x \in X$ such that the inequality $\rho(0, x) > \delta$ implies $\rho(x, y) \geq m\rho(0, x)$. Then by Definition 1 we have

$$m\rho(0, x) \leq \rho(x, y) \leq c(\rho(0, x) + \rho(0, y)).$$

Hence $\rho(0, x) \leq \dfrac{c}{m - c}\rho(0, y)$. Choosing $\delta > \dfrac{c}{m - c}\rho(0, y)$, we arrive at the contradiction. Lemma 1 is proved. □

Let $X$ be a set with a quasi-metric $\rho$ and a nonnegative measure $\mu$ and $\operatorname{diam} X = \infty$. Consider the integral

$$K_\mu(x) = \int_X K(\rho(x, y)) \mathrm{d}\mu(y) \tag{1}$$

where $K : (0, \infty) \to [0, \infty)$ is a monotone decreasing function and there exists a constant $C \geq 1$ such that $K(r) \leq CK(2r)$ for $r > 0$.

**Lemma 2.** *Let $K_\mu(x) = O(K(\rho(0, x)))$ as $\rho(0, x) \to \infty$. Then $\int_X \mathrm{d}\mu(y) < \infty$.*

*Proof.* Let $m > c$. Then

$$K_\mu(x) \geq \int_{B(x, m\rho(0,x))} K(\rho(x, y)) \mathrm{d}\mu(y) \geq K(m\rho(0, x)) \int_{B(x, m\rho(0,x))} \mathrm{d}\mu(y)$$
$$\geq C_1 K(\rho(0, x)) \int_{B(x, m\rho(0,x))} \mathrm{d}\mu(y).$$

Hence $\int_{B(x, m\rho(0,x))} \mathrm{d}\mu(y) < \infty$. By Lemma 1, $B(x, m\rho(0, x)) \to X$ as $\rho(0, x) \to \infty$. Then $\int_X \mathrm{d}\mu(y) < \infty$. Lemma 2 is proved. □

**Theorem 1.** *A necessary and sufficient condition for integral* (1) *to exist on* $X$ *and be* $O(K(\rho(0,x)))$*, as* $\rho(0,x)\to\infty$*, is that*

$$\int\limits_X \frac{K(\rho(x,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) \tag{2}$$

*is bounded on* $X$*.*

*Proof.* Let integral (1) exist on $X$ and $K_\mu(x)=O(K(\rho(0,x)))$ as $\rho(0,x)\to\infty$. Fix any $z\in X$. To prove that $\int_X \frac{K(\rho(z,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y)<\infty$, take $m>c$ such that $m\rho(0,z)>1$. Then

$$\begin{aligned}\int\limits_X \frac{K(\rho(z,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) &= \int\limits_{B(0,1)} \frac{K(\rho(z,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y)\\ &\quad+\int\limits_{B(0,m\rho(0,z))\setminus B(0,1)} \frac{K(\rho(z,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y)\\ &\quad+\int\limits_{X\setminus B(0,m\rho(0,z))} \frac{K(\rho(z,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y)\\ &= I_1(z)+I_2(z)+I_3(z).\end{aligned}$$

It is clear that

$$I_1(z)\le \frac{1}{K(1+\rho(0,1))}\int\limits_{B(0,1)} K(\rho(z,y))\mathrm{d}\mu(y)<\infty.$$

If $1\le\rho(z,y)<m\rho(0,z)$, then

$$1+\rho(0,y)\le 1+c(\rho(0,z)+\rho(z,y))<1+c(1+m)\rho(0,z)<d\rho(0,z),$$

where $d=m+c(1+m)$. Hence

$$I_2(z)\le\frac{1}{K(d\rho(0,z))}\int\limits_{B(0,m\rho(0,z))\setminus B(0,1)} K(\rho(z,y))\mathrm{d}\mu(y)<\infty.$$

Consider $I_3(z)$. If $1<m\rho(0,z)\le\rho(z,y)$, then there exists $C_1\ge 1$ such that

$$\begin{aligned}\frac{K(\rho(z,y))}{K(1+\rho(0,y))}&\le\frac{K(\rho(z,y))}{K(1+c(\rho(0,z)+\rho(z,y)))}\le\frac{K(\rho(z,y))}{K(1+c\left(1+\frac{1}{m}\right)\rho(z,y))}\\ &\le\frac{K(\rho(z,y))}{K((1+c\left(1+\frac{1}{m}\right))\rho(z,y))}\le C_1\end{aligned}$$

Then $I_3(z)\le C_1\int_X \mathrm{d}\mu(y)$. By Lemma 2, we have $I_3(z)<\infty$. Therefore

$$\int\limits_X \frac{K(\rho(z,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y)<\infty.$$

The necessary part of the theorem has been proved.

Now let $\int_X \frac{K(\rho(x,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) < \infty$ for any $x \in X$. To prove that integral (1) exists on $X$ and is $O(K(\rho(0,x)))$ as $\rho(0,x) \to \infty$, take $a \in (0, c^{-1})$. Then

$$K_\mu(x) = \int\limits_{X\setminus B(x,a\rho(0,x))} K(\rho(x,y))\mathrm{d}\mu(y) + \int\limits_{B(x,a\rho(0,x))} K(\rho(x,y))\mathrm{d}\mu(y)$$
$$= J_1(x) + J_2(x).$$

It is clear that

$$\int\limits_X \mathrm{d}\mu(y) \le \int\limits_X \frac{K(\rho(0,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) < \infty.$$

Then

$$J_1(x) \le K(a\rho(0,x)) \int\limits_{X\setminus B(x,a\rho(0,x))} \mathrm{d}\mu(y) \le C_2 K(\rho(0,x)).$$

Consider $J_2(x)$. If $\rho(x,y) < a\rho(0,x)$, then

$$1 + \rho(0,y) > c^{-1}\rho(0,x) - \rho(x,y) > (c^{-1} - a)\rho(0,x).$$

Hence

$$J_2(x) \le K((c^{-1}-a)\rho(0,x)) \int\limits_{B(x,a\rho(0,x))} \frac{K(\rho(x,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) = C_3 K(\rho(0,x)).$$

From the estimates of $J_1(x)$ and $J_2(x)$ the proof of the sufficiency of the condition follows. Theorem 1 is proved. □

## 3. Limit at Infinity

For Riesz potentials, Lemmas 3 and 4 were formulated in [**2**] and [**4**].

**Lemma 3.** *Let $X$ be a set with a quasi-metric $\rho$ and a nonnegative Borel measure $\mu$ on $X$ with* $\operatorname{supp}\mu = X$, $\operatorname{diam} X = \infty$ *and $f$ be a nonnegative $\mu$-locally integrable function on $X$. Suppose that a function $K : (0,\infty) \to [0,\infty)$ satisfies the following conditions:*

($K_1$) *$K(t)$ is an almost decreasing function, i.e., there exists a constant $D > 1$ such that*

$$K(s_2) \le DK(s_1) \quad \text{for} \quad 0 < s_1 < s_2 < \infty;$$

($K_2$) *there exists a constant $M \ge 1$ such that $K(r) \le MK(2r)$ for $r > 0$;*

($K_3$)

$$\int\limits_{B(x,1)} K(\rho(x,y))\mathrm{d}\mu(y) < \infty.$$

*Then for the existence of*

$$U_K f(x) = \int_X K(\rho(x,y)) f(y) \mathrm{d}\mu(y) \tag{3}$$

*$\mu$-almost everywhere on $X$, it is necessary and sufficient that one of the following equivalent conditions is fulfilled:*

1. *there exists $x_0 \in X$ such that*

$$\int_{X \setminus B(x_0,1)} K(\rho(x_0,y)) f(y) \mathrm{d}\mu(y) < \infty;$$

2. *for arbitrary $x \in X$*

$$\int_{X \setminus B(x,1)} K(\rho(x,y)) f(y) \mathrm{d}\mu(y) < \infty;$$

3.

$$\int_X K(1 + \rho(0,y)) f(y) \mathrm{d}\mu(y) < \infty. \tag{4}$$

*Proof.* First we show that from condition 1. it follows that integral (3) is finite $\mu$-a.e. on $X$. For this purpose we write

$$\begin{aligned}\int_{B(x_0,1)} U_K f(x) d\mu(x) &= \int_{B(x_0,1)} d\mu(x) \int_{B(x_0,1+c)} K(\rho(x,y)) f(y) \mathrm{d}\mu(y) \\ &\quad + \int_{B(x_0,1)} d\mu(x) \int_{X \setminus B(x_0,1+c)} K(\rho(x,y)) f(y) \mathrm{d}\mu(y) \\ &= J_1 + J_2.\end{aligned}$$

Consider $J_1$. If $y \in B(x_0, 1+c)$ and $x \in B(x_0, 1)$, then

$$\begin{aligned}\{y : \rho(x_0,y) < 1+c\} &\subset \{y : \rho(0,y) < c(1+c+\rho(0,x_0))\}\,; \\ \{x : \rho(x_0,x) < 1\} &\subset \{x : \rho(x,y) < c(2+c)\}\,.\end{aligned}$$

By Fubini's theorem, we have

$$\begin{aligned}J_1 &= \int_{B(x_0,1+c)} f(y) \mathrm{d}\mu(y) \int_{B(x_0,1)} K(\rho(x,y)) d\mu(x) \\ &\le \int_{B(0,c(1+c+\rho(0,x_0)))} f(y) \mathrm{d}\mu(y) \int_{B(y,c(2+c))} K(\rho(x,y)) d\mu(x) < \infty.\end{aligned}$$

Consider $J_2$. If $x \in B(x_0, 1)$ and $y \in X \setminus B(x_0, 1+c)$, then

$$\rho(x,y) > c^{-1}\rho(x_0,y) - 1 \ge \frac{c^{-1}}{1+c}\rho(x_0,y).$$

It is clear that there exists a positive integer $n$ such that $\frac{c^{-1}}{1+c} \geq 2^{-n}$. Then from $(K_1)$ and $(K_2)$ we have

$$J_2 \leq DM^n \int\limits_{B(x_0,1)} d\mu(x) \int\limits_{X \setminus B(x_0,1+c)} K(\rho(x_0,y)) f(y) \mathrm{d}\mu(y)$$
$$= DM^n \mu(B(x_0,1)) \int\limits_{X \setminus B(x_0,1+c)} K(\rho(x_0,y)) f(y) \mathrm{d}\mu(y).$$

From condition 1. it follows that $J_2 < \infty$. Therefore integral (3) is finite a.e. on $G$.

Now we show that condition 1. implies condition 2. If $\rho(x,y) \geq 1$, then

$$\rho(x_0,y) \leq c(\rho(x,y) + \rho(x,x_0)) \leq c(1+\rho(x,x_0))\rho(x,y).$$

Let $n_x$ be a positive integer such that $c(1+\rho(x,x_0)) \leq 2^{n_x}$. Then

$$K(\rho(x,y)) \leq DK(2^{-n_x}\rho(x_0,y)) \leq DM^{n_x} K(\rho(x_0,y))$$

and

$$\int\limits_{X \setminus B(x,1)} K(\rho(x,y)) f(y) \mathrm{d}\mu(y) \leq DK(1) \int\limits_{B(x_0,1)} f(y) \mathrm{d}\mu(y)$$
$$+ \int\limits_{(X \setminus B(x_0,1)) \cap (X \setminus B(x,1))} K(\rho(x,y)) f(y) \mathrm{d}\mu(y)$$
$$\leq DK(1) \int\limits_{B(x_0,1)} f(y) \mathrm{d}\mu(y)$$
$$+ DM^{n_x} \int\limits_{X \setminus B(x_0,1)} K(\rho(x_0,y)) f(y) \mathrm{d}\mu(y).$$

Hence condition 1. implies condition 2. Let us show that conditions 1. and 3. are equivalent. Since $\rho(x_0,y) < c(1+\rho(0,x_0))(1+\rho(0,y))$, we have

$$K(1+\rho(0,y)) \leq M_1 K(\rho(x_0,y)).$$

Then

$$\int\limits_X K(1+\rho(0,y)) f(y) \mathrm{d}\mu(y) \leq DK(1) \int\limits_{B(x_0,1)} f(y) \mathrm{d}\mu(y)$$
$$+ \int\limits_{X \setminus B(x_0,1)} K(1+\rho(0,y)) f(y) \mathrm{d}\mu(y)$$
$$\leq DK(1) \int\limits_{B(x_0,1)} f(y) \mathrm{d}\mu(y)$$
$$+ M_1 \int\limits_{X \setminus B(x_0,1)} K(\rho(x_0,y)) f(y) \mathrm{d}\mu(y)$$

so that condition 1. involves condition 3.

If $\rho(x_0, y) \geq 1$, then

$$1 + \rho(0, y) \leq \rho(x_0, y)(1 + c(\rho(0, x_0) + 1)).$$

Hence

$$\int_{X \setminus B(x_0,1)} K(\rho(x_0, y))f(y)\mathrm{d}\mu(y) \leq M_2 \int_X K(1 + \rho(0, y))f(y)\mathrm{d}\mu(y).$$

Therefore condition 1. follows from 3. The proof is completed. □

**Definition 2.** Let $\beta > 0$. A space $(X, \rho, \mu)_\beta$ is a set $X$ with a quasi-metric $\rho$ and a nonnegative Borel measure $\mu$ on $X$ with $\operatorname{supp}\mu = X$, $\operatorname{diam} X = \infty$ such that

$$C^{-1}r^\beta \leq \mu(B(x, r)) \leq Cr^\beta$$

for all $r > 0$ and all $x \in X$, where the constant $C \geq 1$ does not depend on $x$ and $r$.

**Lemma 4.** *Let $K : (0, \infty) \to [0, \infty)$ be a continuous function satisfying conditions $(K_1)$, $(K_2)$ and*

$(K_4)$ *there exist a constant $F > 0$ and $0 < \sigma < \beta$ such that*

$$\int_{B(x,r)} K(\rho(x, y))\mathrm{d}\mu(y) < Fr^\sigma \text{ for any } r > 0.$$

*Let $f$ be a nonnegative $\mu$-locally integrable function on $X$ satisfying the condition*

$$\int_X f(y)^p w(f(y))\mathrm{d}\mu(y) < \infty,$$

*where $p = \frac{\beta}{\sigma}$ and the following conditions are fulfilled*

$(w_1)$ *$w$ is a positive, monotone increasing function on the interval $(0, \infty)$;*

$(w_2)$
$$\int_1^\infty w(r)^{-\frac{1}{p-1}} r^{-1}\mathrm{d}r < \infty;$$

$(w_3)$ *there exists a constant $A > 0$ such that*

$$w(2r) < Aw(r) \text{ for any } r > 0.$$

*Then there exists a positive constant $L$ such that*

$$\int_{\{y \in X : f(y) \geq a\}} K(\rho(x, y))f(y)\mathrm{d}\mu(y)$$

$$< L \left( \int_{\{y \in X : |f(y)| \geq a\}} f(y)^p w(f(y))\mathrm{d}\mu(y) \right)^{\frac{1}{p}} \left( \int_a^\infty w(t)^{-\frac{1}{p-1}} t^{-1}\mathrm{d}t \right)^{\frac{1}{p'}},$$

*for any $a > 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

*Proof.* For $j = 1, 2, \dots$ define

$$X_j = \left\{ y \in X : 2^{j-1}a \le f(y) < 2^j a \right\}.$$

Let $r_j = \mu(X_j)^{\frac{1}{\beta}}$. Then

$$C^{-1}\mu(X_j) \le \mu(B(0, r_j)) \le C\mu(X_j).$$

Hence

$$\begin{aligned}
\int\limits_{X_j} K(\rho(x,y))\mathrm{d}\mu(y) &\le \int\limits_{B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) + \int\limits_{X_j \setminus B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) \\
&\le \int\limits_{B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) + DK(r_j) \int\limits_{X_j \setminus B(x,r_j)} \mathrm{d}\mu(y) \\
&\le \int\limits_{B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) + DCK(r_j)\mu(B(x,r_j)) \\
&\le (1 + D^2 C) \int\limits_{B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) \le M_1 r_j^{\sigma},
\end{aligned}$$

where $M_1 = (1 + D^2 C)F$. Therefore

$$\begin{aligned}
&\int\limits_{\{y\in X:|f(y)|\ge a\}} K(\rho(x,y))f(y)\mathrm{d}\mu(y) \\
&= \sum_{j=1}^{\infty} \int\limits_{X_j} K(\rho(x,y))f(y)\mathrm{d}\mu(y) \le \sum_{j=1}^{\infty} 2^j a \int\limits_{X_j} K(\rho(x,y))\mathrm{d}\mu(y) \\
&\le M_1 \sum_{j=1}^{\infty} 2^j a r_j^{\sigma} = 2M_1 \sum_{j=1}^{\infty} 2^{j-1} a w(2^j a)^{\frac{1}{p}} (\mu(X_j))^{\frac{1}{p}} w(2^j a)^{-\frac{1}{p}} \\
&\le 2M_1 A^{\frac{1}{p}} \sum_{j=1}^{\infty} 2^{j-1} a w(2^{j-1} a)^{\frac{1}{p}} (\mu(X_j))^{\frac{1}{p}} w(2^j a)^{-\frac{1}{p}} \\
&\le 2M_1 A^{\frac{1}{p}} \left[ \sum_{j=1}^{\infty} (2^{j-1}a)^p w(2^{j-1}a)\mu(X_j) \right]^{\frac{1}{p}} \times \left[ \sum_{j=1}^{\infty} w(2^j a)^{-\frac{1}{p-1}} \right]^{\frac{1}{p'}} \\
&\le 2M_1 A^{\frac{1}{p}} \left( \int\limits_{\{y\in X: f(y)\ge a\}} f(y)^p w(f(y))\mathrm{d}\mu(y) \right)^{\frac{1}{p}} \\
&\quad \times \left( \int\limits_a^{\infty} w^{-\frac{1}{p-1}}(t)^{-\frac{1}{p-1}} t^{-1}\mathrm{d}t \right)^{\frac{1}{p'}}.
\end{aligned}$$

Lemma 4 is proved. □

**Lemma 5.** *Let* $(X, \rho)$ *be a set with a quasi-metric,* $\operatorname{diam} X = \infty$ *and* $m < c^{-1}$. *Then*

$$X \setminus B(x, m\rho(0,x)) \to X, \quad \textit{as} \quad \rho(0,x) \to \infty.$$

*Proof.* Assume the contrary. Suppose that there is a $y \in X$ such that for all $\delta > 0$ there exists an $x \in X$ such that $\rho(0,x) > \delta$ yields $\rho(x,y) < m\rho(0,x)$. Then by Definition 1 we have

$$\rho(0,x) \leq c(\rho(x,y) + \rho(0,y)) \leq c(m\rho(0,x) + \rho(0,y)).$$

Hence

$$\rho(0,x) \leq \frac{c}{1-mc}\rho(0,y),$$

which is impossible under the choice $\delta > \dfrac{c}{1-mc}\rho(0,y)$. Lemma 5 is proved. □

The following theorem generalizes the corresponding theorem in [**4**].

**Theorem 2.** *Let the assumptions of Lemma 4 and condition* (4) *be fulfilled and let also* $K$ *and* $w$ *satisfy the conditions*

($K_5$) $\lim\limits_{r\to\infty} K(r) = 0$

($w_4$) $w(r^2) \leq A_1 w(r)$, *for* $r \in (1, \infty)$. *Then*

$$w^*(\rho(0,x)^{-1})^{\frac{1}{p}} U_K f(x) \to 0 \quad \textit{as} \quad \rho(0,x) \to \infty,$$

*where* $w^*(r) = \left(\displaystyle\int_r^\infty w(t)^{-\frac{1}{p-1}} t^{-1} dt\right)^{1-p}$.

*Proof.* Let $m < c^{-1}$. For $x \in X \setminus \{0\}$, we write

$$U_K f(x) = \int\limits_{X\setminus X(x,m\rho(0,x))} K(\rho(x,y)) f(y) \mathrm{d}y + \int\limits_{B(x,m\rho(0,x))} K(\rho(x,y)) f(y) \mathrm{d}y$$
$$= J_1(x) + J_2(x).$$

If $y \in X \setminus B(x, m\rho(0,x))$, then

$$\rho(0,x) + \rho(0,y) \leq \rho(0,x) + c(\rho(0,x) + \rho(x,y))$$
$$\leq ((c+1)m^{-1} + 1)\rho(x,y).$$

Then one has by ($K_2$),

$$J_1(x) \leq \int\limits_{X\setminus B(x,m\rho(0,x))} K\Big(\frac{1}{(c+1)m^{-1}+1}(\rho(0,x) + \rho(0,y))\Big) f(y) \mathrm{d}y$$
$$\leq C_1 \int\limits_X K(\rho(0,x) + \rho(0,y)) f(y) \mathrm{d}y.$$

By conditions (4), ($K_5$) and Lebesgue's dominated convergence theorem,

$$J_1(x) \to 0, \text{ as } \rho(0,x) \to \infty.$$

Consider $J_2(x)$. Let $l > \sigma$. It is clear that

$$
\begin{aligned}
J_2(x) &= \int\limits_{\{y;\rho(x,y)<m\rho(0,x),f(y)<\rho(0,x)^{-l}\}} K(\rho(x,y))f(y)\mathrm{d}y \\
&\quad + \int\limits_{\{y;\rho(x,y)<m\rho(0,x),f(y)\geq\rho(0,x)^{-l}\}} K(\rho(x,y))f(y)\mathrm{d}y \\
&= J_{21}(x) + J_{22}(x).
\end{aligned}
$$

By $(K_4)$, we have

$$
\begin{aligned}
J_{21}(x) &\leq \rho(0,x)^{-l} \int\limits_{B(x,m\rho(0,x))} K(\rho(x,y))\mathrm{d}y \\
&\leq Fm^{\sigma}\rho(0,x)^{\sigma-l} \to 0, \qquad \text{as } \rho(0,x)\to\infty.
\end{aligned}
$$

By Lemma 4 and the assumptions of the theorem,

$$
\begin{aligned}
J_{22}(x) &< L\left(\int\limits_{B(x,\rho(0,x))} f(y)^p w(f(y))\mathrm{d}\mu(y)\right)^{\frac{1}{p}} \left(\int\limits_{\rho(0,x)^{-l}}^{\infty} w(t)^{-\frac{1}{p-1}} t^{-1} dt\right)^{\frac{1}{p'}} \\
&\leq L\left(\int\limits_{B(x,\rho(0,x))} f(y)^p w(f(y))\mathrm{d}\mu(y)\right)^{\frac{1}{p}} w^*(\rho(0,x)^{-1}).
\end{aligned}
$$

Using Lemma 5, we have

$$w^*(\rho(0,x)^{-1})J_{22}(x) \to 0, \text{ as } \rho(0,x)\to\infty.$$

So that

$$w^*(\rho(0,x)^{-1})^{\frac{1}{p}} U_K f(x) \to 0, \text{ as } \rho(0,x)\to\infty.$$

Theorem 2 is proved. □

**Remark.** Typical examples of functions $w$ satisfying conditions $(w_1)$-$(w_4)$, one may take

$$w(r) = [\log(2+r)]^{\delta}, \ \ [\log(2+r)]^{p-1}[\log(2+\log(2+r))]^{\delta}, ...,$$

where $\delta > p-1 > 0$.

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