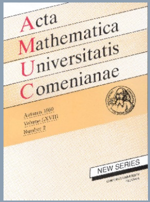

BEHAVIOR AT INFINITY OF CONVOLUTION TYPE INTEGRALS

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Abstract. Behavior at infinity of convolution type integrals on abstract spaces is studied.

1. Introduction

Let $0 < \alpha < n$. The operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, \mathrm{d}y$$

is known as the classical Riesz potential. We refer to the monographs [1], [5], [6] for various properties of the Riesz potentials. Their behavior at infinity was investigated in [3], [4], [7].

It is easy to see that if f is non-negative and compactly supported, then $I_\alpha f(x)$ has the order $|x|^{\alpha-n}$ at infinity. D. Siegel and E. Talvila [7] found necessary and sufficient conditions on f for the validity of $I_\alpha f(x) = O(|x|^{\alpha-n})$ as $|x| \to \infty$ even when f is not compactly supported.

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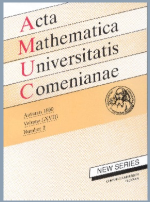


Theorem A. ([7]) *If $f \geq 0$, then a necessary and sufficient condition for $I_\alpha f(x)$ to exist on $\mathbb{R}^n$ and be $O(|x|^{\alpha-n})$ as $|x| \to \infty$ is such that*

$$\int\limits_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\,(1+|y|)^{n-\alpha}\,\mathrm{d}y$$

is bounded on $\mathbb{R}^n$.

We generalize this fact for convolution type integrals on abstract spaces with a monotone decreasing kernel satisfying the so-called “doubling” condition. The limit at infinity of convolution type integrals, on normal homogeneous spaces, which are generalizations of classic Riesz potentials is also studied.

2. The Necessary and Sufficient Condition

Definition 1. Let X be a set. A function $\rho : X \times X \to [0, \infty)$ is called quasi-metric if

1. $\rho(x, y) = 0 \Leftrightarrow x = y$;
2. $\rho(x, y) = \rho(y, x)$;
3. there exists a constant $c \geq 1$ such that for every $x, y, z \in X$

$$\rho(x, y) \leq c\,(\rho(x, z) + \rho(z, y)).$$

If (X, ρ) is a set endowed with a quasi-metric, then the balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$, where $x \in X$ and $r > 0$, satisfy the axioms of a complete system of neighborhoods in X, and therefore induce a (separated) topology. With respect to this topology, the balls $B(x, r)$ need not be open.

We denote $\operatorname{diam} X = \sup\{\rho(x, y) : x \in X, y \in X\}$.

Lemma 1. *Let* (X,ρ) *be a set with a quasi-metric,* $\operatorname{diam} X=\infty$ *and* $m>c$*. Then* $B(x,m\rho\,(0,x))\to X$ *as* $\rho\,(0,x)\to\infty$.

Proof. Assume the contrary. Suppose that there is an $y\in X$ such that for all $\delta>0$ there exists an $x\in X$ such that the inequality $\rho\,(0,x)>\delta$ implies $\rho\,(x,y)\geq m\rho\,(0,x)$. Then by Definition 1 we have

$$m\rho\,(0,x)\leq\rho\,(x,y)\leq c\,(\rho\,(0,x)+\rho\,(0,y))\,.$$

Hence $\rho\,(0,x)\leq\dfrac{c}{m-c}\rho\,(0,y)$. Choosing $\delta>\dfrac{c}{m-c}\rho\,(0,y)$, we arrive at the contradiction. Lemma 1 is proved. $\square$

Let X be a set with a quasi-metric ρ and a nonnegative measure μ and $\operatorname{diam} X=\infty$. Consider the integral

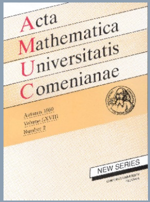
$$K_\mu(x)=\int\limits_X K(\rho(x,y))\mathrm{d}\mu(y) \tag{1}$$

where $K:(0,\infty)\to[0,\infty)$ is a monotone decreasing function and there exists a constant $C\geq 1$ such that $K(r)\leq CK(2r)$ for $r>0$.

Lemma 2. *Let* $K_\mu(x)=O(K(\rho(0,x)))$ *as* $\rho(0,x)\to\infty$*. Then* $\int_X \mathrm{d}\mu(y)<\infty$.

Proof. Let $m>c$. Then

$$\begin{aligned}K_\mu(x)\geq\int\limits_{B(x,m\rho(0,x))}K(\rho(x,y))\mathrm{d}\mu(y)&\geq K(m\rho(0,x))\int\limits_{B(x,m\rho(0,x))}\mathrm{d}\mu(y)\\&\geq C_1K(\rho(0,x))\int\limits_{B(x,m\rho(0,x))}\mathrm{d}\mu(y).\end{aligned}$$



Hence $\int_{B(x,m\rho(0,x))} \mathrm{d}\mu(y) < \infty$. By Lemma 1, $B(x, m\rho(0,x)) \to X$ as $\rho(0,x) \to \infty$. Then $\int_X \mathrm{d}\mu(y) < \infty$. Lemma 2 is proved. $\square$

Theorem 1. *A necessary and sufficient condition for integral (1) to exist on X and be $O(K(\rho(0,x)))$, as $\rho(0,x) \to \infty$, is that*

$$\int_X \frac{K(\rho(x,y))}{K(1+\rho(0,y))} \mathrm{d}\mu(y) \tag{2}$$

is bounded on X.

Proof. Let integral (1) exist on X and $K_\mu(x) = O(K(\rho(0,x)))$ as $\rho(0,x) \to \infty$. Fix any $z \in X$. To prove that $\int_X \frac{K(\rho(z,y))}{K(1+\rho(0,y))} \mathrm{d}\mu(y) < \infty$, take $m > c$ such that $m\rho(0,z) > 1$. Then

$$\begin{aligned}\int_X \frac{K(\rho(z,y))}{K(1+\rho(0,y))} \mathrm{d}\mu(y) &= \int_{B(0,1)} \frac{K(\rho(z,y))}{K(1+\rho(0,y))} \mathrm{d}\mu(y) \\ &\quad + \int_{B(0,m\rho(0,z))\setminus B(0,1)} \frac{K(\rho(z,y))}{K(1+\rho(0,y))} \mathrm{d}\mu(y) \\ &\quad + \int_{X\setminus B(0,m\rho(0,z))} \frac{K(\rho(z,y))}{K(1+\rho(0,y))} \mathrm{d}\mu(y) \\ &= I_1(z) + I_2(z) + I_3(z).\end{aligned}$$

It is clear that

$$I_1(z) \leq \frac{1}{K(1+\rho(0,1))} \int\limits_{B(0,1)} K(\rho(z,y))\mathrm{d}\mu(y) < \infty.$$

If $1 \leq \rho(z,y) < m\rho(0,z)$, then

$$1+\rho(0,y) \leq 1 + c(\rho(0,z)+\rho(z,y)) < 1 + c(1+m)\rho(0,z) < d\rho(0,z),$$

where $d = m + c(1+m)$. Hence

$$I_2(z) \leq \frac{1}{K(d\rho(0,z))} \int\limits_{B(0,m\rho(0,z))\setminus B(0,1)} K(\rho(z,y))\mathrm{d}\mu(y) < \infty.$$

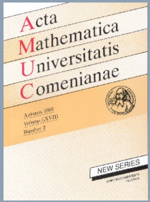
Consider $I_3(z)$. If $1 < m\rho(0,z) \leq \rho(z,y)$, then there exists $C_1 \geq 1$ such that

$$\begin{aligned}\frac{K(\rho(z,y))}{K(1+\rho(0,y))} &\leq \frac{K(\rho(z,y))}{K(1+c(\rho(0,z)+\rho(z,y)))} \leq \frac{K(\rho(z,y))}{K(1+c\left(1+\frac{1}{m}\right)\rho(z,y))} \\ &\leq \frac{K(\rho(z,y))}{K((1+c\left(1+\frac{1}{m}\right))\rho(z,y))} \leq C_1\end{aligned}$$

Then $I_3(z) \leq C_1 \int_X \mathrm{d}\mu(y)$. By Lemma 2, we have $I_3(z) < \infty$. Therefore

$$\int\limits_X \frac{K(\rho(z,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) < \infty.$$

The necessary part of the theorem has been proved.



Now let $\int_X \frac{K(\rho(x,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) < \infty$ for any $x \in X$. To prove that integral (1) exists on X and is $O(K(\rho(0,x)))$ as $\rho(0,x) \to \infty$, take $a \in (0, c^{-1})$. Then

$$\begin{aligned} K_\mu(x) &= \int\limits_{X \setminus B(x, a\rho(0,x))} K(\rho(x,y))\mathrm{d}\mu(y) + \int\limits_{B(x,a\rho(0,x))} K(\rho(x,y))\mathrm{d}\mu(y) \\ &= J_1(x) + J_2(x). \end{aligned}$$

It is clear that

$$\int\limits_X \mathrm{d}\mu(y) \le \int\limits_X \frac{K(\rho(0,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) < \infty.$$

Then

$$J_1(x) \le K(a\rho(0,x)) \int\limits_{X \setminus B(x,a\rho(0,x))} \mathrm{d}\mu(y) \le C_2 K(\rho(0,x)).$$

Consider $J_2(x)$. If $\rho(x,y) < a\rho(0,x)$, then

$$1 + \rho(0,y) > c^{-1}\rho(0,x) - \rho(x,y) > (c^{-1} - a)\rho(0,x).$$

Hence

$$J_2(x) \le K((c^{-1} - a)\rho(0,x)) \int\limits_{B(x,a\rho(0,x))} \frac{K(\rho(x,y))}{K(1+\rho(0,y))}\mathrm{d}\mu(y) = C_3 K(\rho(0,x)).$$

From the estimates of $J_1(x)$ and $J_2(x)$ the proof of the sufficiency of the condition follows. Theorem 1 is proved. $\square$

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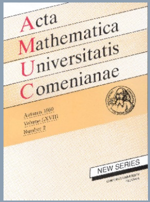


3. Limit at Infinity

For Riesz potentials, Lemmas 3 and 4 were formulated in [2] and [4].

Lemma 3. *Let X be a set with a quasi-metric ρ and a nonnegative Borel measure μ on X with* $\operatorname{supp}\mu = X$, $\operatorname{diam} X = \infty$ *and f be a nonnegative μ-locally integrable function on X. Suppose that a function $K : (0,\infty) \to [0,\infty)$ satisfies the following conditions:*

(K_1) *$K(t)$ is an almost decreasing function, i.e., there exists a constant $D > 1$ such that*

$$K(s_2) \le DK(s_1) \quad \text{for} \quad 0 < s_1 < s_2 < \infty;$$

(K_2) *there exists a constant $M \ge 1$ such that $K(r) \le MK(2r)$ for $r > 0$;*

(K_3)

$$\int\limits_{B(x,1)} K(\rho(x,y))\mathrm{d}\mu(y) < \infty.$$

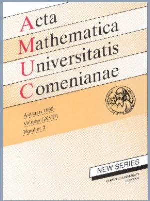
Then for the existence of

$$U_K f(x) = \int\limits_X K(\rho(x,y))f(y)\mathrm{d}\mu(y) \tag{3}$$

μ-almost everywhere on X, it is necessary and sufficient that one of the following equivalent conditions is fulfilled:

1. *there exists $x_0 \in X$ such that*

$$\int\limits_{X\setminus B(x_0,1)} K(\rho(x_0,y))f(y)\mathrm{d}\mu(y) < \infty;$$



2. *for arbitrary* $x \in X$

$$\int\limits_{X \setminus B(x,1)} K(\rho(x,y)) f(y) \mathrm{d}\mu(y) < \infty;$$

3.

$$\int\limits_{X} K(1+\rho(0,y)) f(y) \mathrm{d}\mu(y) < \infty. \tag{4}$$

Proof. First we show that from condition 1. it follows that integral (3) is finite μ-a.e. on X. For this purpose we write

$$\begin{aligned} \int\limits_{B(x_0,1)} U_K f(x) d\mu(x) &= \int\limits_{B(x_0,1)} d\mu(x) \int\limits_{B(x_0,1+c)} K(\rho(x,y)) f(y) \mathrm{d}\mu(y) \\ &\quad + \int\limits_{B(x_0,1)} d\mu(x) \int\limits_{X \setminus B(x_0,1+c)} K(\rho(x,y)) f(y) \mathrm{d}\mu(y) \\ &= J_1 + J_2. \end{aligned}$$

Consider J_1. If $y \in B(x_0, 1+c)$ and $x \in B(x_0, 1)$, then

$$\{y : \rho(x_0, y) < 1 + c\} \subset \{y : \rho(0, y) < c(1 + c + \rho(0, x_0))\};$$
$$\{x : \rho(x_0, x) < 1\} \subset \{x : \rho(x, y) < c(2 + c)\}.$$

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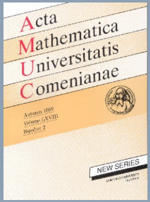

By Fubini's theorem, we have

$$J_1 = \int\limits_{B(x_0,1+c)} f(y)\mathrm{d}\mu(y) \int\limits_{B(x_0,1)} K(\rho(x,y))d\mu(x)$$

$$\leq \int\limits_{B(0,c(1+c+\rho(0,x_0)))} f(y)\mathrm{d}\mu(y) \int\limits_{B(y,c(2+c))} K(\rho(x,y))d\mu(x) < \infty.$$

Consider J_2. If $x \in B(x_0,1)$ and $y \in X \setminus B(x_0, 1+c)$, then

$$\rho(x,y) > c^{-1}\rho(x_0,y) - 1 \geq \frac{c^{-1}}{1+c}\rho(x_0,y).$$

It is clear that there exists a positive integer n such that $\frac{c^{-1}}{1+c} \geq 2^{-n}$. Then from (K_1) and (K_2) we have

$$J_2 \leq DM^n \int\limits_{B(x_0,1)} d\mu(x) \int\limits_{X\setminus B(x_0,1+c)} K(\rho(x_0,y))f(y)\mathrm{d}\mu(y)$$

$$= DM^n \mu(B(x_0,1)) \int\limits_{X\setminus B(x_0,1+c)} K(\rho(x_0,y))f(y)\mathrm{d}\mu(y).$$

From condition 1. it follows that $J_2 < \infty$. Therefore integral (3) is finite a.e. on G.

Now we show that condition 1. implies condition 2. If $\rho(x,y) \geq 1$, then

$$\rho(x_0,y) \leq c(\rho(x,y) + \rho(x,x_0)) \leq c(1+\rho(x,x_0))\rho(x,y).$$

Let n_x be a positive integer such that $c(1+\rho(x,x_0)) \leq 2^{n_x}$. Then

$$K(\rho(x,y)) \leq DK(2^{-n_x}\rho(x_0,y)) \leq DM^{n_x}K(\rho(x_0,y))$$

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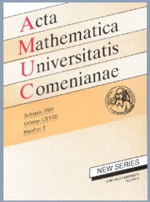

and

$$\int_{X\setminus B(x,1)} K(\rho(x,y))f(y)\mathrm{d}\mu(y) \le DK(1)\int_{B(x_0,1)} f(y)\mathrm{d}\mu(y) + \int_{(X\setminus B(x_0,1))\cap(X\setminus B(x,1))} K(\rho(x,y))f(y)\mathrm{d}\mu(y)$$
$$\le DK(1)\int_{B(x_0,1)} f(y)\mathrm{d}\mu(y) + DM^{n_x}\int_{X\setminus B(x_0,1)} K(\rho(x_0,y))f(y)\mathrm{d}\mu(y).$$

Hence condition 1. implies condition 2. Let us show that conditions 1. and 3. are equivalent. Since $\rho(x_0,y) < c(1+\rho(0,x_0))(1+\rho(0,y))$, we have

$$K(1+\rho(0,y)) \le M_1 K(\rho(x_0,y)).$$

Then

$$\int_X K(1+\rho(0,y))f(y)\mathrm{d}\mu(y) \le DK(1)\int_{B(x_0,1)} f(y)\mathrm{d}\mu(y) + \int_{X\setminus B(x_0,1)} K(1+\rho(0,y))f(y)\mathrm{d}\mu(y)$$
$$\le DK(1)\int_{B(x_0,1)} f(y)\mathrm{d}\mu(y) + M_1\int_{X\setminus B(x_0,1)} K(\rho(x_0,y))f(y)\mathrm{d}\mu(y)$$

so that condition 1. involves condition 3.

If $\rho(x_0,y)\ge 1$, then

$$1+\rho(0,y) \le \rho(x_0,y)(1+c(\rho(0,x_0)+1)).$$

Hence

$$\int_{X\setminus B(x_0,1)} K(\rho(x_0,y))f(y)\mathrm{d}\mu(y) \le M_2\int_X K(1+\rho(0,y))f(y)\mathrm{d}\mu(y).$$

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Therefore condition 1. follows from 3. The proof is completed. □

Definition 2. Let $\beta > 0$. A space $(X, \rho, \mu)_\beta$ is a set X with a quasi-metric ρ and a nonnegative Borel measure μ on X with $\operatorname{supp}\mu = X$, $\operatorname{diam} X = \infty$ such that

$$C^{-1} r^\beta \le \mu(B(x,r)) \le C r^\beta$$

for all $r > 0$ and all $x \in X$, where the constant $C \ge 1$ does not depend on x and r.

Lemma 4. *Let $K : (0,\infty) \to [0,\infty)$ be a continuous function satisfying conditions (K_1), (K_2) and*

(K_4) *there exist a constant $F > 0$ and $0 < \sigma < \beta$ such that*

$$\int\limits_{B(x,r)} K(\rho(x,y))\,\mathrm{d}\mu(y) < F r^\sigma \quad \textit{for any } r > 0.$$

Let f be a nonnegative μ-locally integrable function on X satisfying the condition

$$\int\limits_X f(y)^p w(f(y))\,\mathrm{d}\mu(y) < \infty,$$

where $p = \frac{\beta}{\sigma}$ and the following conditions are fulfilled

(w_1) *w is a positive, monotone increasing function on the interval $(0,\infty)$;*

(w_2) $$\int\limits_1^\infty w(r)^{-\frac{1}{p-1}} r^{-1}\,\mathrm{d}r < \infty;$$

(w_3) *there exists a constant $A > 0$ such that*

$$w(2r) < A w(r) \quad \textit{for any } r > 0.$$

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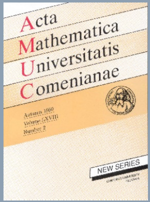


Then there exists a positive constant L such that

$$\int\limits_{\{y\in X: f(y)\geq a\}} K(\rho(x,y)) f(y)\mathrm{d}\mu(y) < L\left(\int\limits_{\{y\in X: |f(y)|\geq a\}} f(y)^p w(f(y))\mathrm{d}\mu(y)\right)^{\frac{1}{p}} \left(\int\limits_a^\infty w(t)^{-\frac{1}{p-1}} t^{-1}\mathrm{d}t\right)^{\frac{1}{p'}},$$

for any $a > 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. For $j = 1, 2, \ldots$ define

$$X_j = \left\{y \in X : 2^{j-1}a \leq f(y) < 2^j a\right\}.$$

Let $r_j = \mu(X_j)^{\frac{1}{\beta}}$. Then

$$C^{-1}\mu(X_j) \leq \mu(B(0, r_j)) \leq C\mu(X_j).$$

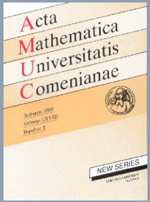
Hence

$$\begin{aligned}
\int\limits_{X_j} K(\rho(x,y))\mathrm{d}\mu(y) &\leq \int\limits_{B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) + \int\limits_{X_j\setminus B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) \\
&\leq \int\limits_{B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) + DK(r_j)\int\limits_{X_j\setminus B(x,r_j)} \mathrm{d}\mu(y) \\
&\leq \int\limits_{B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) + DCK(r_j)\mu(B(x,r_j)) \\
&\leq (1 + D^2C)\int\limits_{B(x,r_j)} K(\rho(x,y))\mathrm{d}\mu(y) \leq M_1 r_j^\sigma,
\end{aligned}$$

where $M_1 = (1 + D^2 C)F$. Therefore

$$\begin{aligned}
&\int\limits_{\{y\in X:|f(y)|\geq a\}} K(\rho(x,y))f(y)\mathrm{d}\mu(y) \\
&\quad = \sum_{j=1}^{\infty} \int\limits_{X_j} K(\rho(x,y))f(y)\mathrm{d}\mu(y) \leq \sum_{j=1}^{\infty} 2^j a \int\limits_{X_j} K(\rho(x,y))\mathrm{d}\mu(y) \\
&\quad \leq M_1 \sum_{j=1}^{\infty} 2^j a r_j^{\sigma} = 2M_1 \sum_{j=1}^{\infty} 2^{j-1} a w(2^j a)^{\frac{1}{p}} (\mu(X_j))^{\frac{1}{p}} w(2^j a)^{-\frac{1}{p}} \\
&\quad \leq 2M_1 A^{\frac{1}{p}} \sum_{j=1}^{\infty} 2^{j-1} a w(2^{j-1} a)^{\frac{1}{p}} (\mu(X_j))^{\frac{1}{p}} w(2^j a)^{-\frac{1}{p}} \\
&\quad \leq 2M_1 A^{\frac{1}{p}} \left[\sum_{j=1}^{\infty} (2^{j-1} a)^p w(2^{j-1} a)\mu(X_j)\right]^{\frac{1}{p}} \times \left[\sum_{j=1}^{\infty} w(2^j a)^{-\frac{1}{p-1}}\right]^{\frac{1}{p'}} \\
&\quad \leq 2M_1 A^{\frac{1}{p}} \left(\int\limits_{\{y\in X: f(y)\geq a\}} f(y)^p w(f(y))\mathrm{d}\mu(y)\right)^{\frac{1}{p}} \\
&\qquad\qquad \times \left(\int_a^{\infty} w^{-\frac{1}{p-1}}(t)^{-\frac{1}{p-1}} t^{-1}\mathrm{d}t\right)^{\frac{1}{p'}}.
\end{aligned}$$

Lemma 4 is proved. □



Lemma 5. *Let* (X,ρ) *be a set with a quasi-metric,* $\operatorname{diam} X = \infty$ *and* $m < c^{-1}$*. Then*

$$X \setminus B(x, m\rho(0,x)) \to X, \quad \textit{as} \quad \rho(0,x) \to \infty.$$

Proof. Assume the contrary. Suppose that there is a $y \in X$ such that for all $\delta > 0$ there exists an $x \in X$ such that $\rho(0,x) > \delta$ yields $\rho(x,y) < m\rho(0,x)$. Then by Definition 1 we have

$$\rho(0,x) \le c(\rho(x,y) + \rho(0,y)) \le c(m\rho(0,x) + \rho(0,y)).$$

Hence

$$\rho(0,x) \le \frac{c}{1-mc}\rho(0,y),$$

which is impossible under the choice $\delta > \dfrac{c}{1-mc}\rho(0,y)$. Lemma 5 is proved. □

The following theorem generalizes the corresponding theorem in [4].

Theorem 2. *Let the assumptions of Lemma 4 and condition* (4) *be fulfilled and let also* K *and* w *satisfy the conditions*

(K_5) $\lim\limits_{r\to\infty} K(r) = 0$

(w_4) $w(r^2) \le A_1 w(r)$*, for* $r \in (1,\infty)$*. Then*

$$w^*(\rho(0,x)^{-1})^{\frac{1}{p}} U_K f(x) \to 0 \quad \textit{as} \quad \rho(0,x) \to \infty,$$

where $w^*(r) = \left(\displaystyle\int_r^\infty w(t)^{-\frac{1}{p-1}} t^{-1} dt\right)^{1-p}.$

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Proof. Let $m < c^{-1}$. For $x \in X \setminus \{0\}$, we write

$$U_K f(x) = \int\limits_{X \setminus X(x, m\rho(0,x))} K(\rho(x,y)) f(y) \mathrm{d}y + \int\limits_{B(x, m\rho(0,x))} K(\rho(x,y)) f(y) \mathrm{d}y$$
$$= J_1(x) + J_2(x).$$

If $y \in X \setminus B(x, m\rho(0,x))$, then

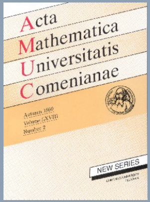
$$\rho(0,x) + \rho(0,y) \leq \rho(0,x) + c(\rho(0,x) + \rho(x,y))$$
$$\leq ((c+1)m^{-1} + 1)\rho(x,y).$$

Then one has by (K_2),

$$J_1(x) \leq \int\limits_{X \setminus B(x, m\rho(0,x))} K\Big(\frac{1}{(c+1)m^{-1} + 1}(\rho(0,x) + \rho(0,y))\Big) f(y) \mathrm{d}y$$
$$\leq C_1 \int\limits_X K(\rho(0,x) + \rho(0,y)) f(y) \mathrm{d}y.$$

By conditions (4), (K_5) and Lebesgue's dominated convergence theorem,

$$J_1(x) \to 0, \text{ as } \rho(0,x) \to \infty.$$



Consider $J_2(x)$. Let $l > \sigma$. It is clear that

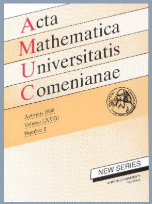
$$\begin{aligned} J_2(x) &= \int\limits_{\{y;\rho(x,y)<m\rho(0,x),f(y)<\rho(0,x)^{-l}\}} K(\rho(x,y))f(y)\mathrm{d}y \\ &\quad + \int\limits_{\{y;\rho(x,y)<m\rho(0,x),f(y)\geq\rho(0,x)^{-l}\}} K(\rho(x,y))f(y)\mathrm{d}y \\ &= J_{21}(x) + J_{22}(x). \end{aligned}$$

By (K_4), we have

$$\begin{aligned} J_{21}(x) &\leq \rho(0,x)^{-l} \int\limits_{B(x,m\rho(0,x))} K(\rho(x,y))\mathrm{d}y \\ &\leq Fm^{\sigma}\rho(0,x)^{\sigma-l} \to 0, \qquad \text{as } \rho(0,x) \to \infty. \end{aligned}$$

By Lemma 4 and the assumptions of the theorem,

$$\begin{aligned} J_{22}(x) &< L\left(\int\limits_{B(x,\rho(0,x))} f(y)^p w(f(y))\mathrm{d}\mu(y)\right)^{\frac{1}{p}} \left(\int\limits_{\rho(0,x)^{-l}}^{\infty} w(t)^{-\frac{1}{p-1}} t^{-1} dt\right)^{\frac{1}{p'}} \\ &\leq L\left(\int\limits_{B(x,\rho(0,x))} f(y)^p w(f(y))\mathrm{d}\mu(y)\right)^{\frac{1}{p}} w^*(\rho(0,x)^{-1}). \end{aligned}$$



Using Lemma 5, we have

$$w^*(\rho(0,x)^{-1})J_{22}(x) \to 0, \text{ as } \rho(0,x) \to \infty.$$

So that

$$w^*(\rho(0,x)^{-1})^{\frac{1}{p}} U_K f(x) \to 0, \text{ as } \rho(0,x) \to \infty.$$

Theorem 2 is proved. □

Remark. Typical examples of functions w satisfying conditions (w_1)-(w_4), one may take

$$w(r) = [\log(2+r)]^{\delta}, \ [\log(2+r)]^{p-1} [\log(2+\log(2+r))]^{\delta}, \ldots,$$

where $\delta > p-1 > 0$.

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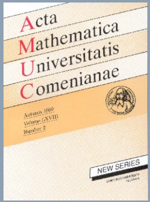

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