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A NOTE ON A MULTIVALUED ITERATIVE EQUATION

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Abstract. In this note, we consider a second order multivalued iterative equation, and the result on decreasing solutions is given.

1. Introduction

Let X be a topological space and for integer $n \geq 0$ the n-th iterate of a mapping f is defined by $f^n = f \circ f^{n-1}$ and $f^0 = \mathbf{id}$, where $\circ$ denotes the composition of mappings and **id** denotes the identity mapping. As an important class of functional equations [**1, 2**], the polynomial-like iterative equation is a linear combination of iterates, which is of the general form

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \ldots + \lambda_n f^n(x) = F(x), \qquad x \in X, \tag{1}$$

where X is a Banach space or its closed subset, where F is a given mapping, f is an unknown mapping, and λ_i $(i = 1, \ldots, n)$ are real constants. Equation (1) has been studied extensively on the existence, uniqueness and stability of its solutions [**1, 2**], it was also considered in the class of multifunctions [**3**].

Let $I = [a, b]$ be a given interval and $cc(I)$ denote the family of all nonempty convex compact subsets of I. This family endowed with the Hausdorff distance is defined by

$$h(A, B) = \max\,\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\},$$

where $d(a, B) = \inf\{|a - b| : b \in B\}$.

A multifunction $F : I \to cc(I)$ is *decreasing* (resp. *strictly decreasing*) if $\max F(x) \leq \min F(y)$ (resp. $\max F(x) < \min F(y)$) for every $x, y \in I$ with $x > y$.

Let $\Gamma(I)$ be the family of all multifunctions $F : I \to cc(I)$ and $\Phi(I)$ be defined by

$$\Phi(I) = \{F \in \Gamma(I) : \text{is USC, increasing, } F(a) = \{a\}, F(b) = \{b\}\},$$

and endowed with the metric

$$D(F_1, F_2) = \sup\{h(F_1(x), F_2(x)) : x \in I\}, \qquad \forall F_1, F_2 \in \Phi(I).$$

In [**3**], the authors investigated the second order multivalued iterative equation

$$\lambda_1 F(x) + \lambda_2 F^2(x) = G(x), \tag{2}$$

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in an interval $I = [a, b]$, and the following results were obtained:

Lemma 1. ([**3**, Lemma 1.]) *The metric space* $(\Phi(I), D)$ *is complete.*

Lemma 2. ([**3**, Lemma 2.]) *If* $F, G \in \Phi$ *and* $F(x) \subset G(x)$ *for all* $x \in I$*, then* $F = G$.

Lemma 3. ([**3**, Theorem 1.]) *Let* $G \in \Phi(I)$*,* $\lambda_1 > \lambda_2 \geq 0$ *and* $\lambda_1 + \lambda_2 = 1$*. Then the equation* (2) *has a unique solution* $F \in \Phi(I)$*.*

In this paper, as in [**3**], we are still interested in a multivalued solution of the equation (2), and the decreasing solutions are given.

2. Main Result

Let $I = [-a, a]$ be a given interval and $\Psi(I)$ be defined by

$$\Psi(I) = \{F : I \to cc(I), \text{ is USC, decreasing, } F(-a) = \{a\},\ F(a) = \{-a\}\}.$$

We endow $\Psi(I)$ with the metric

$$D(F_1, F_2) = \sup\{h(F_1(x), F(x_2)) : x \in I\}, \qquad \forall F_1, F_2 \in \Psi(I)$$

Obviously, by Lemma 1 the metric space $(\Psi(I), D)$ is complete.

Lemma 4. *If* $G, F \in \Psi(I)$ *and* $F(x) \subset G(x)$ *for all* $x \in I$*, then* $F = G$*.*

This Lemma follows from Lemma 2. More concretely, we should consider the monotonicity of decreasing.

Theorem 1. *Let* $G \in \Psi(I)$*,* $\lambda_1 > 1/2$*,* $\lambda_2 < 0$ *and* $\lambda_1 - \lambda_2 = 1$*. Then the equation* (2) *has a unique solution* $F \in \Psi(I)$*.*

Proof. Define the mapping $L : \Psi(I) \to \Gamma(I)$

$$LF(x) = \lambda_1 x + \lambda_2 F(x), \qquad \forall x \in I,$$

where $F \in \Psi(I)$. Obviously, LF is USC and $LF(-a) = -\lambda_1 a + \lambda_2 a = \{-a\}$, $LF(a) = \lambda_1 a - \lambda_2 a = \{a\}$. Moreover, for any $x_2 > x_1$ in I, we have $\max F(x_2) - \min F(x_1) \leq 0$ since F is decreasing. Therefore,

$$\begin{aligned}\min LF(x_2) - \max LF(x_1) &= \lambda_1(x_2 - x_1) + \lambda_2(\min F(x_2) - \max F(x_1)) \\ &\geq \lambda_1(x_2 - x_1) > 0\end{aligned}$$

for $\lambda_1 > 0$ and $\lambda_2 < 0$, which implies that LF is strictly increasing and the multifunction $(LF)^{-1}$ defined by $(LF)^{-1}(y) = \{x \in I : y \in LF(x)\}$ is single-valued and continuous.

Define the mapping $\Upsilon : \Psi(I) \to \Gamma(I)$ as

$$\Upsilon F(x) = (LF)^{-1}(G(x)), \qquad \forall F \in \Psi(I), \quad \forall x \in I,$$

Hence, ΥF is also USC and $\Upsilon F(-a) = (LF)^{-1}(G(-a)) = \{a\}$, $\Upsilon F(a) = (LF)^{-1}(G(a)) = \{-a\}$. Moreover, ΥF is decreasing since $(LF)^{-1}$ is increasing and G is decreasing.

Finally, by (c.f. [**3**, pp. 431–432]), we have

$$\begin{aligned} D(\Upsilon F_1, \Upsilon F_2) &= \sup_{x\in I} h((LF_1)^{-1}(G(x)), (LF_2)^{-1}(G(x))) \\ &\le \frac{1}{\lambda_1} \sup_{x\in I} h(LF_1(x), LF_2(x)), \end{aligned}$$

for every $x \in I$ and $F_1, F_2 \in \Psi(I)$. Hence, we obtain that

$$\begin{aligned} D(\Upsilon F_1, \Upsilon F_2) &\le \frac{1}{\lambda_1} \sup_{x\in I} h(LF_1(x), LF_2(x)) \\ &\le \frac{1}{\lambda_1} \sup_{x\in I} h(\lambda_1 x + \lambda_2 F_1(x), \lambda_1 x + \lambda_2 F_2(x)) \\ &= \frac{|\lambda_2|}{\lambda_1} \sup_{x\in I} h(F_1(x), F_2(x)) \\ &\le \frac{|\lambda_2|}{\lambda_1} D(F_1, F_2) \\ &= \left(\frac{1}{\lambda_1} - 1\right) D(F_1, F_2) \\ &< D(F_1, F_2) \end{aligned}$$

which implies that Υ is a contraction. Therefore, by the Banach fixed point principle, Υ has a unique fixed point F in $\Psi(I)$, i.e.

$$(LF)^{-1}(G(x)) = F(x), \qquad \forall x \in I.$$

Consequently, by Lemma 4, we have

$$\lambda_1 F(x) + \lambda_2 F^2(x) = G(x), \qquad \forall x \in I.$$

The proof is completed. □

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