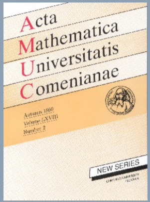

VALUATIONS ON THE RING OF ARITHMETICAL FUNCTIONS

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Abstract. In this paper we study a class of nontrivial independent absolute values on the ring A of arithmetical functions over the field $\mathbb{C}$ of complex numbers. We show that A is complete with respect to the metric structure obtained from each of these absolute values. We also consider an Artin-Whaples type theorem in this context.

1. Introduction

Let A denote the set of complex valued arithmetical functions $f : \mathbb{N} \to \mathbb{C}$, where $\mathbb{N}$ is the set of positive integers. For $f, g \in A$ their Dirichlet convolution is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$$

for $n \in \mathbb{N}$. A is a ring with the usual addition of functions and Dirichlet convolution. It is known that A is a unique factorization domain. This was proved by Cashwell and Everett [5]. Schwab and Silberberg [7] constructed an extension of A which is a discrete valuation ring, and in [8], they showed that A is a quasi-noetherian ring. Yokom [9] investigated the prime factorization of arithmetical functions in a certain subring of the regular convolution ring. He also determined a discrete valuation subring of the unitary ring of arithmetical functions. Some questions on the

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structure of the ring of arithmetical functions in several variables have been recently investigated by Alkan and the authors in [1], [2], [3]. Our aim in the present paper is to construct an infinite class of valuations on A which are independent of each other. To keep the exposition short and simple, we will restrict to the case of arithmetical functions of one variable, with values in $\mathbb{C}$. We construct these valuations as follows. Let P be the set of prime numbers. Fix a weight function $w : P \to \mathbb{R}$ such that for all $p \in P$, $w(p) \geq 0$. Given $n \in \mathbb{N}$ with prime factorization $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, we define $\Omega_w(n) = \alpha_1 w(p_1) + \dots + \alpha_k w(p_k)$. Also for $f \in A$, let $\operatorname{supp}(f)$ denote the support of f, so $\operatorname{supp}(f) = \{n \in \mathbb{N} | f(n) \neq 0\}$, and define

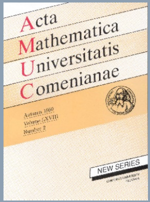
$$V_w(f) = \inf_{n \in \operatorname{supp}(f)} \Omega_w(n),$$

with the convention $\min(\emptyset) = \infty$. Then V_w is a valuation on A. Next, we extend V_w to a valuation, also denoted by V_w, on the field of fractions $\mathbb{K} = \left\{ \frac{f}{g} | f, g \in A, g \neq 0 \right\}$ of A by letting $V_w\left(\frac{f}{g}\right) = V_w(f) - V_w(g)$. We also fix a number $\rho \in (0, 1)$ and define an absolute value $|.|_w : \mathbb{K} \to \mathbb{R}$ by

$$|x|_w = \rho^{V_w(x)} \text{ if } x \neq 0, \qquad \text{and} \qquad |x|_w = 0 \text{ if } x = 0.$$

In Section 2 we show that V_w is indeed a valuation, and so $|.|_w$ is a non-archimedian absolute value. In Section 3 we show that A is complete with respect to the metric structure obtained from the absolute value $|.|_w$.

Lastly, we take a finite number $w_1, \dots, w_s$ of weight functions on P for which the absolute values $|.|_{w_1}, \dots, |.|_{w_s}$ are independent, and consider the completions $\mathbb{K}_{w_1}, \dots, \mathbb{K}_{w_s}$ of $\mathbb{K}$ with respect to $|.|_{w_1}, \dots, |.|_{w_s}$. Define the function $\psi : \mathbb{K} \to \mathbb{K}_{w_1} \times \dots \times \mathbb{K}_{w_s}$ by $x \to \psi(x) = (x, \dots, x)$. By the Artin-Whaples Theorem [4], we know that the topological closure of $\psi(\mathbb{K})$ in $\mathbb{K}_{w_1} \times \dots \times \mathbb{K}_{w_s}$ coincides with $\mathbb{K}_{w_1} \times \dots \times \mathbb{K}_{w_s}$. Since we are more interested in the ring A than in its field of



fractions $\mathbb{K}$, a natural question to ask is what the topological closure of the image $\psi(A)$ of A under ψ is in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. We show that this topological closure is $\psi(A)$ itself.

2. Absolute Values

Theorem 1.

(i) *For any $f, g \in A$, we have*

$$V_w(f+g) \geq \min(\{V_w(f), V_w(g)\}).$$

(ii) *For any $f, g \in A$, we have*

$$V_w(f * g) = V_w(f) + V_w(g).$$

Proof. (i) Let $f, g \in A$. Since $\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$, we get that for any $n \in \operatorname{supp}(f+g)$, either $n \in \operatorname{supp}(f)$, or $n \in \operatorname{supp}(g)$. Thus we have that $\Omega_w(n) \geq V_w(f)$, or $\Omega_w(n) \geq V_w(g)$ for any $n \in \operatorname{supp}(f+g)$. So, it follows immediately that

$$V_w(f+g) \geq \min(\{V_w(f), V_w(g)\}).$$

(ii) Again let $f, g \in A$. Let $n \in \operatorname{supp}(f)$, and $m \in \operatorname{supp}(g)$. Suppose that k, and l satisfy the equations $\Omega_w(n) = k$, and $\Omega_w(m) = l$ respectively. Also assume that $V_w(f) = k$ and $V_w(g) = l$. Now,

$$V_w(f) + V_w(g) = k + l.$$

Let a be a positive integer such that $a \in \operatorname{supp}(f * g)$. Then,

$$0 \neq (f * g)(a) = \sum_{d|a} f(d) g\left(\frac{a}{d}\right).$$

Therefore $f(d) \neq 0$, and $g\left(\frac{a}{d}\right) \neq 0$ for some $d|a$. It follows that for any $a \in \text{supp}(f * g)$,

$$\begin{aligned} V_w(f) + V_w(g) &\leq \Omega_w(d) + \Omega_w\left(\frac{a}{d}\right) \\ &= \Omega_w(a). \end{aligned}$$

So, $V_w(f) + V_w(g) \leq V_w(f * g)$.

To show the reverse inequality, we first define the following two sets.

$$\mathfrak{C}_f = \{a \in \mathbb{N} : f(a) \neq 0 \qquad \text{and} \qquad \Omega_w(a) = k\}$$

and

$$\mathfrak{C}_g = \{b \in \mathbb{N} : g(b) \neq 0 \qquad \text{and} \qquad \Omega_w(b) = l\}.$$

Let n be the smallest element of $\mathfrak{C}_f$. Also let m be the smallest element of $\mathfrak{C}_g$. Denote $u = nm$. We have that

$$V_w(f) + V_w(g) = \Omega_w(n) + \Omega_w(m) = \Omega_w(u).$$

So if we show that $(f * g)(u) \neq 0$, then we will be done. To show that $(f * g)(u) \neq 0$, we consider the identity

$$(f * g)(u) = \sum_{de=nm} f(d)g(e)$$

and show that all terms in this sum vanish except for the term $f(n)g(m)$ which is nonzero. Suppose that $f(d)g(e)$ is a nonzero term of the sum. Then note that none of the inequalities $\Omega_w(d) < k$ and $\Omega_w(e) < l$ can hold since otherwise the term $f(d)g(e)$ is zero. Also observe that if $\Omega_w(d) > k$, then $\Omega_w(e) < l$ and the latter inequality cannot hold as we have seen above. Similarly if $\Omega_w(e) > l$, then $\Omega_w(d) < k$ and again the latter inequality cannot hold. We conclude that $\Omega_w(d) = k$ and $\Omega_w(e) = l$. It follows that $d \in \mathfrak{C}_f$, and $e \in \mathfrak{C}_g$. Since we have $d \leq n$ and $e \leq m$, it is clear from the definition of n and n that $d = n$, and $e = m$. Hence, if $f(d)g(e)$ is a nonzero term of the sum, then it follows that $d = n$, and $e = m$. Thus (ii) holds, and this completes the proof of the theorem. $\square$

It follows from the above theorem and [6, Proposition 3.1.10] that $|.|_w$ is a non-archimedian absolute value on $\mathbb{K}$.

3. Completeness and Topological Closure

Define a distance d_w on $\mathbb{K}$ by putting for $x, y \in \mathbb{K}$, $d_w(x, y) = |x - y|_w$, and consider also the restriction of this distance to A.

Theorem 2. *The metric space (A, d_w) with respect to the distance d_w defined above is complete.*

Proof. Let $(f_n)_{n\geq 0}$ be a Cauchy sequence in A. Then for each $\varepsilon > 0$, there exists $N = N_\epsilon \in \mathbb{N}$ such that $|f_m - f_n|_w < \epsilon$ for all $m, n \geq N_\varepsilon$. For each $k \in \mathbb{N}$, taking $\varepsilon = \rho^k$, there exists $N_k \in \mathbb{N}$ such that $|f_m - f_n|_w < \rho^k$ for all $m, n \geq N_k$. Equivalently, $V_w(f_m - f_n) > k$ for all $m, n \geq N_k$, i.e., we have that for all $m, n \geq N_k$,

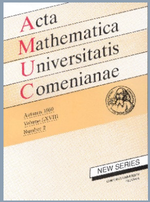
$$f_m(l) = f_n(l)$$

whenever $\Omega_w(l) \leq k$, for all $l \in \mathbb{N}$. We choose for each $k \in \mathbb{N}$, the smallest natural number N_k with the above property such that

$$N_1 < N_2 < \ldots < N_k < N_{k+1} < \ldots.$$

Let us define $f : \mathbb{N} \to \mathbb{C}$ as follows. Given $l \in \mathbb{N}$, let k be the smallest positive integer such that $k > \Omega_w(l)$. We set $f(l) = f_{N_k}(l)$. Then f is the limit of the sequence $(f_n)_{n\geq 0}$. This completes the proof of Theorem 2. □

Remark 1. *Let w, w' be weight functions on P. If the absolute values $|.|_w, |.|_{w'}$, correspondingly the valuations $V_w, V_{w'}$, arising from w and w' respectively are dependent, then there exists a constant C such that $w(p) = Cw'(p)$ for all primes p.*



Proof. Let w, w' be weight functions on P. If $V_w, V_{w'}$ are dependent on $\mathbb{K}$, then there exists a constant C such that $V_w(x) = CV_{w'}(x)$ for all $x \in \mathbb{K}$. For each prime number p, define $\delta_p \in A$ by

$$\delta_p(n) = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{else} \end{cases}$$

for all $n \in \mathbb{N}$. Then $V_w(\delta_p) = w(p)$, and $V_{w'}(\delta_p) = w'(p)$. Hence $w(p) = Cw'(p)$ for all primes p as claimed. □

We have seen that each weight function $w : P \to \mathbb{R}$ on the set P of prime numbers gives rise, via the function $|.|_w$, to an absolute value on $\mathbb{K}$. We denote by $\mathbb{K}_w$ the completion of $\mathbb{K}$ with respect to the absolute value $|.|_w$.

Let $s > 0$ be an integer. Let $w_1, \ldots, w_s$ be weight functions on P. Suppose that the absolute values $|.|_{w_1}, \ldots, |.|_{w_s}$, corresponding to the valuations $V_{w_1}, \ldots, V_{w_s}$ on $\mathbb{K}$, are independent. Consider the product topology on $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. Define the function $\psi : \mathbb{K} \to \mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ by $x \to \psi(x) = (x, \ldots, x)$. Then the topological closure of $\psi(\mathbb{K})$ in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ coincides with $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. We would like to identify the topological closure of the image $\psi(A)$ of A under ψ in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. For a subset F of $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ we denote by $\overline{F}$ the topological closure of F in $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$. Then $\overline{\psi(A)} \subseteq A \times \cdots \times A$ since A is complete with respect to each absolute value $|.|_{w_i}$ $(i = 1, \ldots, s)$.

Theorem 3. *The topological closure of* $\psi(A)$ *in* $\mathbb{K}_{w_1} \times \cdots \times \mathbb{K}_{w_s}$ *is* $\psi(A)$ *itself.*

Proof. We have already seen that $\overline{\psi(A)} \subseteq A \times \cdots \times A$. Let $f_1, \ldots, f_s \in A \times \cdots \times A$, and assume that $(f_1, \ldots, f_s) \in \overline{\psi(A)}$. We want to show that $(f_1, \ldots, f_s) \in \psi(A)$. By the above assumption we know that there exists a sequence $(h_n)_{n \in \mathbb{N}}$ in A such that $\psi(h_n)$ converges to $(f_1, \ldots, f_s)$. So for each $i \in \{1, \ldots, s\}$, we have that $|h_n - f_i|_{w_i} \to 0$ as $n \to \infty$, correspondingly, $V_{w_i}(h_n - f_i) \to \infty$ as $n \to \infty$.

Fix $m = p_1^{\alpha_1} \cdots P_u^{\alpha_u}$. Also fix $j \in \{1, \ldots, s\}$. Since $V_{w_i}(h_n - f_i) \to \infty$ as $n \to \infty$, there exists $N_j \in \mathbb{N}$ such that

$$V_{w_j}(h_n - f_j) > \alpha_1 w_j(p_1) + \alpha_2 w_j(p_2) + \cdots + \alpha_u w_j(p_u) \qquad \text{for all } n > N_j.$$

for all $n > N_j$. Let N be the maximum of $N_1, \ldots, N_s$. Then, for all $j \in \{1, \ldots, s\}$, and all $n > N$, we have that

$$V_{w_j}(h_n - f_j) > \alpha_1 w_j(p_1) + \alpha_2 w_j(p_2) + \cdots + \alpha_u w_j(p_u).$$

Thus, $m \notin \operatorname{supp}(h_n - f_j)$ and therefore, $h_n(m) = f_j(m)$ for all $j \in \{1, \ldots, s\}$, and all $n > N$. Hence, $f_1(m) = f_2(m) \cdots = f_s(m)$. Since m is arbitrary, it follows that the arithmetical functions $f_1, f_2, \ldots, f_s$ are identical, and hence $(f_1, \ldots, f_s) \in \psi(A)$. This completes the proof of the theorem. □

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