Acta Math. Univ. Comenianae
Vol. LXXVIII, 2(2009), pp. 287–302

EXISTENCE THEOREMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. This paper concernes with the study of existence theorems for a general class of functional differential equations of the form

$$u'(t) = f(t, u \circ \gamma(t, \cdot)).$$

The obtained results generalize the retarded functional differential equations [**5, 6, 8**] and cover singular functional differential equations [**1, 2, 4, 7, 9, 12**].

1. Introduction

Let $(E, |\cdot|_E)$ be a Banach space. For a fixed $r > 0$, we define $\mathcal{C} = C([-r, 0]; E)$ to be the Banach space of continuous E-valued functions on $J := [-r, 0]$ with the usual supremum norm $\|\varphi\| = \sup_{\theta\in[-r,0]} |\varphi(\theta)|_E$.

For a continuous function $u : \mathbb{R} \to E$ and any $t \in \mathbb{R}$, we denote by u_t the element of $\mathcal{C}$, defined by

$$u_t(\theta) = u(t + \theta), \qquad \theta \in J.$$

For each $(\sigma, a) \in \mathbb{R} \times \mathbb{R}^*_+$, we consider

$$\Gamma_{\sigma,a} = \{\, \gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a] \text{ continuous functions such that for all } \theta \in [-r, 0],\ s \in [0, a], \quad \gamma(\sigma, \theta) = \sigma + \theta \text{ and } \gamma(\sigma + s, 0) = \sigma + s \,\}.$$

It is clear that if $u \in C([\sigma - r, \sigma + a]; E)$ and $\gamma \in \Gamma_{\sigma,a}$, then $u \circ \gamma(t, \cdot) \in \mathcal{C}$ and $t \mapsto u \circ \gamma(t, \cdot)$ is a continuous function for $t \in [\sigma, \sigma + a]$, where $u \circ \gamma(t, \cdot)(\theta) := u(\gamma(t, \theta))$ for all $\theta \in J$, in particular, if $\gamma(t, \theta) = t + \theta$, then $u \circ \gamma(t, \cdot) = u_t \in \mathcal{C}$ and $t \mapsto u_t$ is continuous for $t \in [\sigma, \sigma + a]$.

Now we introduce a general class of functional differential equations

$$u'(t) = f(t, u \circ \gamma(t, \cdot)) \tag{1.1}$$

where f is a continuous function from $[\sigma, \sigma + a] \times \mathcal{C}$ into E.

If $\gamma(t, \theta) = t + \theta$, then the equation $R(f, \gamma)$ coincides with the classical retarded functional differential equation $u'(t) = f(t, u_t)$ (see, for example [**5, 6, 8**]).

Received July 18, 2008; revised December 4, 2008.
2000 *Mathematics Subject Classification.* Primary 34K05; 34K30.
Key words and phrases. class of functional differential equations; singular functional differential equations.

If $\gamma(t,\theta) = \rho(\rho^{-1}(t)+\theta)$ where $\rho : [\sigma - r, \sigma + b] \to [\sigma - r, \sigma + a]$, $(b > 0)$ is defined by

$$\rho(\tau) = \begin{cases} \sigma + \displaystyle\int_\sigma^\tau \frac{\mathrm{d}s}{\psi(s)} & \text{if} \quad \tau \in [\sigma, \sigma + b] \\ \tau & \text{if} \quad \tau \in [\sigma - r, \sigma], \end{cases}$$

$\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$, then the equation $R(f,\gamma)$ coincides with the following initial value problem for the singular functional differential equation (see [**7**]):

$$\begin{cases} \psi(\tau)x'(\tau) = g(\tau, x_\tau), & \tau \in (\sigma, \sigma + b] \\ x_\sigma = \varphi, \end{cases}$$

where $g : [\sigma, \sigma + b] \times \mathcal{C} \to E$ is completely continuous and $f(t,\phi) := g(\rho^{-1}(t), \phi)$.

Also, in the Section 5, we shall study the general form

$$\begin{cases} \psi(\tau)x^{(n)}(\tau) = g(\tau, x_\tau, x'_\tau, \ldots, x_\tau^{(n-1)}), & \tau \in (\sigma, \sigma + b], \quad (b > 0) \\ x_\sigma = \varphi, \end{cases}$$

or the second order delay equation of the form

$$\begin{cases} \psi(\tau)x''(\tau) = g(\tau, x(\tau), x(\tau - r_1), x'(\tau), x'(\tau - r_2)), & \tau \in (\sigma, \sigma + b], \quad (b > 0) \\ x_\sigma = \varphi, \quad x'_\sigma = \varphi' \quad \text{on } [-r, 0], \end{cases}$$

where $r = \max(r_1, r_2)$, (see [**12**]).

2. Preliminaries

Let D be a subset of $\mathbb{R} \times \mathcal{C}$ and let f be a continuous function from D into E. In the sequel, we give $(\sigma, a) \in \mathbb{R} \times \mathbb{R}_+^*$ and $\gamma \in \Gamma_{\sigma,a}$. We say that the relation

$$u'(t) = f(t, u \circ \gamma(t, \cdot)), \qquad ((t, u \circ \gamma(t, \cdot)) \in D),$$

is a functional differential equation on D and will denote this equation by $R(f,\gamma)$.

Definition 2.1. A function u is said to be a solution of the equation $R(f,\gamma)$, if there exists a real A such that $0 < A \leq a$ and $u \in C([\sigma - r, \sigma + A); E)$, $(t, u \circ \gamma(t, \cdot)) \in D$ and u satisfies the equation $R(f,\gamma)$ for $t \in [\sigma, \sigma + A)$. Then, we say that u is a solution of $R(f,\gamma)$ on $[\sigma, \sigma + A)$

For $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$, we say $u := u(\sigma, \varphi)$ is a solution of equation $R(f,\gamma)$ through (σ, φ), if there is A such that $0 < A \leq a$ and $u(\sigma, \varphi)$ is a solution of $R(f,\gamma)$ on $[\sigma - r, \sigma + A)$ and $u_\sigma(\sigma, \varphi) = \varphi$.

Let $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$, we consider the function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t - \sigma) & \text{if} \quad t \in [\sigma - r, \sigma] \\ \varphi(0) & \text{if} \quad t \geq \sigma. \end{cases}$$

We have $\widetilde{\varphi} \in C([\sigma - r, +\infty); E)$, $\widetilde{\varphi}_\sigma = \varphi$ and $\widetilde{\varphi}(t + \sigma) = \varphi(0)$ for $t \geq 0$.

It is easy to see that the following result is immediate.

Lemma 2.1. *Suppose that $f \in C(D;E)$, $\varphi \in \mathcal{C}$ and $0 < A \leq a$. Then, there are equivalent statements:*

i) *u is solution of $R(f,\gamma)$ on $[\sigma - r, \sigma + A)$ through (σ, φ),*

ii) *$u \in C([\sigma - r, \sigma + A); E)$, $(t, u \circ \gamma(t,\cdot)) \in D$ for all $t \in [\sigma, \sigma + A)$ and*

$$\begin{cases} u_\sigma = \varphi \\ u(t) = \varphi(0) + \displaystyle\int_\sigma^t f(s, u \circ \gamma(s,\cdot))\mathrm{d}s, \qquad t \in [\sigma, \sigma + A); \end{cases}$$

iii) *there exists $y \in C([-r, A); E)$ such that $(\sigma + t, y_t + \widetilde{\varphi} \circ \gamma(\sigma + t,\cdot)) \in D$ for all $t \in [0, A)$ and*

$$\begin{cases} y_0 = 0 \\ y(t) = \displaystyle\int_0^t f(\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s,\cdot))\mathrm{d}s, \qquad t \in [0, A). \end{cases}$$

For any real $\alpha, \beta > 0$, define

$$I_\alpha = [0,\alpha], \qquad \widehat{I_\alpha} = (0,\alpha], \qquad B_\beta = \{\psi \in \mathcal{C} : \|\psi\| \leq \beta\},$$
$$A(\alpha,\beta) = \{y \in C([-r,\alpha]; E) : y_0 = 0 \ \text{ and } \ y_t \in B_\beta, \ \ t \in I_\alpha\}$$

and

$$C^0(D,E) = \{f \in C(D,E) : f \text{ is bounded on } D\}.$$

We have $A(\alpha,\beta)$ is a closed bounded convex subset of $C([-r,\alpha];E)$ and $C^0(D,E)$ is a Banach space with the norm $\|f\|_0 = \sup_{(t,\varphi)\in D} |f(t,\varphi)|_E$.

Lemma 2.2. *Suppose that $\Omega \subset \mathbb{R} \times \mathcal{C}$ is open, $W \subset \Omega$ is compact and $f^0 \in C(\Omega; E)$. Then, there exists a neighborhood $V \subset \Omega$ of W such that $f^0 \in C^0(V;E)$, there exists a neighborhood $U \subset C^0(V;E)$ of f^0 and three positive constants M, $\alpha \leq a$ and β such that*

$$|f(\sigma,\varphi)|_E < M \qquad \text{for all } (\sigma,\varphi) \in V \text{ and } f \in U,$$

$(\sigma^0 + t, y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t,\cdot)) \in V$ for any $(\sigma^0,\varphi^0) \in W$, $t \in I_\alpha$, $y \in A(\alpha,\beta)$ and $\gamma \in \Gamma_{\sigma^0,a}$.

Proof. Since $f^0(W)$ is a compact subset of the Banach space E, it is bounded, and therefore exists $M > 0$ such that

$$\left|f^0(\sigma^0,\varphi^0)\right|_E < \frac{M}{3}$$

for all $(\sigma^0,\varphi^0) \in W$. However f^0 is continuous at (σ^0,φ^0), and therefore for $0 < \varepsilon < \frac{M}{3}$, there exists $(\alpha',\beta') \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ (with $\alpha' \leq a$) such that for $(t,\psi) \in \left(\sigma^0 - \alpha', \sigma^0 + \alpha'\right) \times B(\varphi^0,\beta') \subset \Omega$ we have

$$\left|f^0(t,\psi)\right|_E \leq \left|f^0(t,\psi) - f^0(\sigma^0,\varphi^0)\right|_E + \left|f^0(\sigma^0,\varphi^0)\right|_E < \frac{2M}{3}.$$

Consider β such that $0 < \beta < \beta'$ and $\gamma \in \Gamma_{\sigma',a}$. Since the function $s \in [\sigma^0, \sigma^0 + a] \mapsto \widetilde{\varphi^0} \circ \gamma(s, \cdot)$ is continuous at σ^0, then there exists α such that $0 < \alpha < \alpha'$ and

$$\left\|\widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot) - \widetilde{\varphi^0} \circ \gamma(\sigma^0, \cdot)\right\| = \left\|\widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0\right\| < \beta' - \beta, \qquad t \in I_\alpha.$$

Define $V = \bigcup_{(\sigma^0,\varphi^0)\in W} \left(\sigma^0 - \alpha', \sigma^0 + \alpha'\right) \times B(\varphi^0, \beta')$. Then $W \subset V \subset \Omega$, V is a neighborhood of W and $f^0 \in C^0(V; E)$. Moreover $(\sigma^0 + t, y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) \in V$ for all $t \in I_\alpha$, $y \in A(\alpha, \beta)$, $\gamma \in \Gamma_{\sigma^0,a}$. Indeed $\sigma^0 + t \in \left(\sigma^0 - \alpha', \sigma^0 + \alpha'\right)$ and $y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot) \in B(\varphi^0, \beta')$ because

$$\left\|y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0\right\| \le \|y_t\| + \left\|\widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0\right\| \le \beta'.$$

Define $U = \left\{f \in C^0(V; E) : \left\|f - f^0\right\|_0 < \frac{M}{3}\right\}$. Then U is a neighborhood of f^0, $U \subset C^0(V; E)$ and for all $(\sigma, \varphi) \in V$, $f \in U$

$$|f(\sigma, \varphi)|_E \le \left|f(\sigma, \varphi) - f^0(\sigma, \varphi)\right|_E + \left|f^0(\sigma, \varphi)\right|_E < M.$$

□

The next lemma will be used to apply fixed point theorems for existence of solutions of the equation $R(f, \gamma)$.

Lemma 2.3. *Suppose that $\Omega \subset \mathbb{R} \times \mathcal{C}$ is open, $W = \{(\sigma, \varphi)\} \subset \Omega$ and $f^0 \in C(\Omega; E)$ are given, the neighborhoods V, U and the constants M, α and β are the ones obtained from Lemma 2.2. Define an operator $T : U \times A(\alpha, \beta) \to C([-r, \alpha]; E)$ by*

$$T(f, y)(t) = \begin{cases} 0 & \text{if } t \in [-r, 0] \\ \displaystyle\int_0^t f((\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot))\mathrm{d}s & \text{if } t \in I_\alpha. \end{cases}$$

If $M\alpha \le \beta$, then $T : U \times A(\alpha, \beta) \to A(\alpha, \varphi)$ and T is continuous on $U \times A(\alpha, \beta)$.

Proof. It is clear that T maps $U \times A(\alpha, \beta)$ into $C([-r, \alpha]; E)$ and by Lemma 2.2, for all $t, t' \in I_\alpha$

$$|T(f, y)(t) - T(f, y)(t')|_E \le M\,|t - t'| \qquad \text{and} \qquad |T(f, y)(t)|_E \le M\alpha.$$

It is easy to see that for all $t, t' \in [-r, \alpha]$

$$|T(f, y)(t) - T(f, y)(t')|_E \le M\,|t - t'| \qquad \text{and} \qquad |T(f, y)(t)|_E \le M\alpha.$$

Hence the family $\Im = \{T(f, y) : (f, y) \in U \times A(\alpha, \beta)\}$ is bounded and uniformly equicontinuous. Also, we have $(T(f, y))_0 = 0$ and $(T(f, y))_t \in B_\beta$ if $M\alpha \le \beta$, thus T maps $U \times A(\alpha, \beta)$ into $A(\alpha, \beta)$.

It remains to show that T is continuous on $U \times A(\alpha, \beta)$.

Let (f^n, y^n) be a sequence in $U \times A(\alpha, \beta)$ that converges to a member (f, y) of $U \times A(\alpha, \beta)$.

It is clear that for each $s \in I_\alpha$

$$\|y_s^n - y_s\| = \sup_{\theta\in[-r,0]} |y^n(s+\theta) - y(s+\theta)|_E \le \|y^n - y\|_1$$
$$:= \sup_{t\in[-r,\alpha]} |y^n(t) - y(t)|_E .$$

We have $(\sigma+s, y_s^n + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)$, $(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot) \in V$ because $s \in I_\alpha$ and $y^n, y \in A(\alpha,\beta)$. Since (f^n) converges uniformly to f in V, then the sequence $(f^n(\sigma+s, y_s^n + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)))$ converges to $f(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))$ in E, but f^n and f are bounded on V (see Lemma 2.2) and by the Lebesgue dominated convergence theorem, we obtain

$$\int_0^t f^n(\sigma+s, y_s^n + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s \longrightarrow \int_0^t f(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s$$

in E. Hence for all $t \in I_\alpha$, $T(f^n, y^n)(t)$ converges to $T(f,y)(t)$ in E and then $(T(f^n, y^n)(t))$ converges to $T(f,y)(t)$ in E for all $t \in [-r,\alpha]$. This implies that the set $\{T(f^n, y^n)(t) : t \in [-r,\alpha]\}$ is relatively compact in E, but the family $\{T(f^n, y^n) : n \in \mathbb{N}\}$ is bounded and uniformly equicontinuous and therefore by the Ascoli theorem [**3, 11**], the family$\{T(f^n, y^n) : n \in \mathbb{N}\}$ is relatively compact in $C([-r,\alpha]; E)$. We shall show that $T(f^n, y^n)$ converges to $T(f,y)$ in $C([-r,\alpha]; E)$. Suppose, for the sake of contradiction, that there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$

$$\exists n > N : \ \|T(f^n, y^n) - T(f,y)\|_1 \ge \varepsilon.$$

Then for

$$N = n_0, \quad \exists n_1 > n_0 : \qquad \|T(f^{n_1}, y^{n_1}) - T(f,y)\|_1 \ge \varepsilon$$

and for $k > 1$ and

$$N = n_{k-1}, \quad \exists n_k > n_{k-1} : \quad \|T(f^{n_k}, y^{n_k}) - T(f,y)\|_1 \ge \varepsilon.$$

If necessary, passing, to a subsequence, we can assume that $(T(f^{n_k}, y^{n_k}))$ converges to $z \in A(\alpha,\beta)$ such that $\|z - T(f,y)\|_1 \ge \varepsilon$. Since $(T(f^{n_k}, y^{n_k}))$ converges to z in $C([-r,\alpha]; E\ )$, then $(T(f^{n_k}, y^{n_k}))(t)$ converges to $z(t)$ in E for each $t \in [-r,\alpha]$, but this sequence converges to $T(f,y)(t)$ in E, which is a contradiction. Therefore T is continuous on $U \times A(\alpha,\beta)$. □

3. Local existence of solutions

In this section we shall show existence theorem of solutions to $R(f,\gamma)$ by using the results obtained in the section two.

Definition 3.1. Suppose that Ω is an open set in $\mathbb{R}\times\mathcal{C}$. A function $f \in C(\Omega; E)$ is said to have the condition (l) if, for all $(\sigma,\varphi) \in \Omega$, there exists a neighborhood $V' \subset \Omega$ of (σ,φ) and a positive constant k such that for all bounded $I \times S_1 \subset V'$ with bounded $f(I \times S_1)$, then $\chi(f(I, S_1)) \le k\chi_0(S_1)$ where χ (resp. χ_0) is the measure of noncompactness [**3, 11**] on E (resp. $\mathcal{C}$).

Theorem 3.1. *Suppose that Ω is an open set in $\mathbb{R}\times\mathcal{C}$ and $f\in C(\Omega;E)$. If f is compact or satisfying the condition (l) (resp. $f(t,\cdot)$ is locally Lipschitz), then for all $(\sigma,\varphi)\in\Omega$ and $\gamma\in\Gamma_{\sigma,a}$ there exists a positive constant $\alpha\leq a$ and a solution (resp. a unique solution) of the equation $R(f,\gamma)$ on $[\sigma-r,\sigma+\alpha]$ through (α,φ).*

Proof. By notations of Lemmas 2.2 and 2.3 with $W=\{(\sigma,\varphi)\}$, the operator $T_1=T(f,\cdot)$ maps $A(\alpha,\beta)$ into $A(\alpha,\beta)$ if $\alpha\leq\frac{\beta}{M}$ and T_1 is continuous.

<u>*First case*</u>. If f is compact we shall show that T_1 is compact.

Let B be a bounded subset of $A(\alpha,\beta)$ and (z^n) a sequence of T_1B, then there exists a sequence (y^n) of B such that $z^n=T_1y^n$.

The set $\{f(\sigma+s,y_s^n+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):n\in\mathbb{N},\ s\in I_\alpha\}$ is relatively compact because f is completely continuous. By Mazur theorem [**3, 11**] its closed convex hull is compact. But, for all $t\in\widehat{I_\alpha}$, we have (see [**11**, p. 25])

$$\frac{1}{t}\int_0^t f(\sigma+s,y_s^n+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s\in\overline{\mathrm{Co}}\{f(\sigma+s,y_s^n+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):n\in\mathbb{N},\ s\in I_\alpha\}.$$

Then,

$$\{T_1y^n(t):n\in\mathbb{N},s\in\widehat{I_\alpha}\}\subset\alpha\overline{\mathrm{Co}}\{f(\sigma+s,y_s^n+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):n\in\mathbb{N},s\in I_\alpha\}$$

which is compact, hence $\{T_1y^n(t):n\in\mathbb{N},s\in\widehat{I_\alpha}\}$ is relatively compact. However $\{T_1y^n(t):n\in\mathbb{N}\}$ is bounded and uniformly equicontinuous, then by the Ascoli theorem, $\{T_1y^n(t):n\in\mathbb{N}\}$ is relatively compact, thus (z^n) has a subsequence that converges in $C([-r,\alpha];E)$. By Schauder fixed-point theorem and Lemma 2.3, $R(f,\gamma)$ has a solution on $[\sigma-r,\sigma+\alpha]$ through (σ,φ).

<u>*Second case*</u>. If f satisfies the condition (l).

Let $V=(\sigma-\alpha',\sigma+\alpha')\times B(\varphi,\beta')$ be the neighborhood obtained in the Lemma 2.2 and by the condition (l), there exist $V'=(\sigma-\alpha'',\sigma+\alpha'')\times B(\varphi,\beta'')$ and $k>0$ such that if $f(I\times S_1)$ is bounded for all bounded $I\times S_1\subset V'$, then $\chi(f(I\times S_1))\leq k\chi_0(S_1)$. Take $\alpha_2=\min(\alpha',\alpha'')$, $\beta_2=\min(\beta',\beta'')$ and $V_1=V\cap V'$. Let $0<\beta_1<\beta_2$, then there exists α_1 such that $0<\alpha_1<\alpha_2$ (see proof of Lemma 2.2) and for all $s\in I_{\alpha_1}$

$$\|\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)-\widetilde{\varphi}\circ\gamma(\sigma,\cdot)\|=\|\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)-\varphi\|<\beta_2-\beta_1.$$

Then for every $s\in I_{\alpha_1}$ and every $y\in A(\alpha_1,\beta_1)$, $(\sigma+s,y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\in V_1$. Thus, if $\alpha_1\leq\frac{\beta_1}{M}$, then T_1 maps $A(\alpha_1,\beta_1)$ into itself and T_1 is continuous. Moreover $A(\alpha_1,\beta_1)$ is closed, bounded convex subset of $C([-r,\alpha_1];E)$. In order to apply Darboux fixed point theorem [**3, 11**], we shall show that there exists $\delta\in[0,1)$ such that $\chi_0^1(T_1S)\leq\delta\chi_0^1(S)$ for all $S\subset A(\alpha_1,\beta_1)$ where χ_0^1 is the measure of noncompactness on $C([-r,\alpha_1];E)$.

Let $S \subset A(\alpha_1, \beta_1)$, then for each $t \in \widehat{I_{\alpha_1}}$

$$\begin{aligned}\chi(T_1S(t)) &= \chi\left(\left\{\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s :\ y\in S\right\}\right)\\ &= \chi\left(\left\{t\frac{1}{t}\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s :\ y\in S\right\}\right)\\ &\le \alpha_1\chi(\overline{\mathrm{Co}}\{f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) :\ y\in S,\ s\in[0,t]\})\\ &\le \alpha_1\chi(\{f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) :\ y\in S,\ s\in[0,t]\}).\end{aligned}$$

By definition of V_1, we have for each $t \in \widehat{I_{\alpha_1}}$

$$\{(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) :\ y\in S,\ s\in[0,t]\} \subset V_1.$$

Take

$$I = \{\sigma+s :\ s\in[0,t]\} \subset [\sigma, \sigma+\alpha_1] \qquad \text{and}$$

$$S_1 = \{y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,.) :\ y\in S,\ s\in[0,t]\}.$$

Then, $I\times S_1$ and $f(I\times S_1)$ are bounded (because f is bounded on V see Lemma 2.2). Hence, for each $t \in \widehat{I_{\alpha_1}}$, $\chi(T_1S(t)) \le k\alpha_1\chi_0(S_1)$ and

$$\begin{aligned}\chi_0(S_1) &= \chi_0(\{y_s :\ y\in S,\ s\in[0,t]\}) + \chi_0(\{\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot) :\ s\in[0,t]\})\\ &\le \chi_0(\{y_s : y\in S,\ s\in[0,t]\}).\end{aligned}$$

Since $\chi_0(\{\widetilde{\varphi}\circ\gamma(\sigma+s,.) : s\in[0,t]\}) = 0$, the set $\{\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot) : s\in[0,t]\}$ is relatively compact.

Thus, for each $t \in \widehat{I_{\alpha_1}}$

$$\chi(T_1S(t)) \le k\alpha_1\chi_0(S_s) \le k\alpha_1\chi_0^1(S) \qquad (\text{see } [\mathbf{13}])$$

where $S_s = \{y_s : s\in[0,t],\ y\in S\}$.

But the family $\left\{t\mapsto\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\ \mathrm{d}s\right\}$ is uniformly bounded and equicontinuous, then by Ambrosetti theorem [**3, 11**], we obtain

$$\chi_0^1(T_1S) = \sup_{t\in[-r,\alpha_1]}\chi(T_1S(t)) \le k\alpha_1\chi_0^1(S).$$

Take $\alpha_1 < \min\left\{\frac{1}{k}, \frac{\beta_1}{M}\right\}$ and $\delta = k\alpha_1 \in [0,1)$.

Third case. If $f(t,\cdot)$ is locally Lipschitz.

Let $V = (\sigma-\alpha', \sigma+\alpha')\times B(\varphi,\beta')$ be the neighborhood obtained in the Lemma 2.2 and since $f(t,\cdot)$ is locally Lipschitz then there exists $V' = (\sigma-\alpha'', \sigma+\alpha'')\times B(\varphi,\beta'')$ such that $f(t,\cdot)$ is Lipschitz on V'. T_1 maps $A(\alpha_1,\beta_1)$ into $A(\alpha_1,\beta_1)$ if $\alpha_1 \le \frac{\beta_1}{M}$ (see the second case of the existence of α_1, β_1) and T_1 is a

contraction strict if $\alpha_1 < \min\left\{\frac{\beta_1}{M}, \frac{1}{k}\right\}$. Indeed, for all $y, z \in A(\alpha_1, \beta_1)$

$$\begin{aligned}\|T_1 y - T_1 z\|_1 &= \sup_{t\in[-r,\alpha_1]} |T_1 y(t) - T_1 z(t)|_E \\ &\leq \sup_{t\in\widehat{I_{\alpha_1}}} \int_0^t |f(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) \\ &\qquad - f(\sigma+s, z_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))|_E \mathrm{d}s \\ &\leq k \sup_{t\in\widehat{I_{\alpha_1}}} \int_0^t \|y_s - z_s\| \,\mathrm{d}s,\end{aligned}$$

but for all $s \in (0,t] \subset \widehat{I_{\alpha_1}}$

$$\|y_s - z_s\| \leq \sup_{\theta\in[-r,0]\cap[-s,0]} |y(s+\theta) - z(s+\theta)|_E \leq \|y - z\|_1 .$$

Finally

$$\|T_1 y - T_1 z\|_1 \leq k\alpha_1 \|y - z\|_1 \qquad \text{for all } y, z \in A(\alpha_1, \beta_1).$$

Thus, the equation $R(f,\gamma)$ has a unique solution on $[\sigma - r, \sigma + \alpha_1]$ through (σ, φ). □

Remark. If $f = f_1 + f_2$ where f_1 is completely continuous and f_2 is locally Lipschitz, then the condition (l) is verified.

4. Global existence solutions

Definition 4.1. Let u (resp. v) be a solution of $R(f,\gamma)$ on $J_u = [\sigma - r, \sigma + A)$ (resp. $J_v = [\sigma - r, \sigma + B)$) where $0 < A, B \leq a$. The solution v is said to be a continuation of u if $J_v \supset J_u$ and $v = u$ on J_u.

The solution u is said to be noncontinuable if it has no proper continuation.

The following result of the existence of noncontinuable solutions follows from Zorn lemma [**14**].

Proposition 4.1. *If u is a solution of the equation $R(f,\gamma)$ on J_u, then there exists a noncontinuable solution $\widehat{u}$ of $R(f,\gamma)$ on $J_{\widehat{u}}$ such that $\widehat{u}$ is a continuation of u.*

Theorem 3.1 gives a criterion of local existence for solutions to $R(f,\gamma)$, then we use the previous proposition to study the continuation of solutions to the equation $R(f,\gamma)$.

Theorem 4.1. *Suppose that Ω is an open subset of $\mathbb{R}\times\mathcal{C}$ and $f \in C(\Omega; E)$. If f is compact or verifies the condition (l) (resp. $f(t,\cdot)$ is locally Lipschitz), then for all $(\sigma,\varphi) \in \Omega$ and $\gamma \in \Gamma_{\sigma,a}$, there exists a noncontinuable solution (resp. a unique solution) of $R(f,\gamma)$ on $[\sigma - r, \sigma + \alpha)$ $(0 < \alpha \leq a)$ through (σ,φ).*

Proof. It remains to show unicity of a noncontinuable solution if $f(t,\cdot)$ is locally Lipschitz. Suppose that there exist two noncontinuable solutions $u : [\sigma - r,$

$\sigma+\alpha_u) \to E$ and $v : [\sigma-r, \sigma+\alpha_v) \to E$ of $R(f,\gamma)$ through (σ,φ), then $u_\sigma = v_\sigma = \varphi$. If $\alpha_u < \alpha_v$, then v is a continuation of u, which is a contradiction, so $\alpha_u = \alpha_v$.

By Lemma 2.2, we associated with u (resp. v), y (resp. z) and we will see $y = z$ on $J = (0, \alpha)$. Suppose that in J there exists $t' > 0$ such that $y(t') \neq z(t')$. Define $t_0 = \inf\{t \in J : y(t) \neq z(t)\}$, by continuity of y and z, then $y(t_0) = z(t_0)$ and $y_{t_0} = z_{t_0}$. Set $\varphi^0 = y_{t_0} + \widetilde{\varphi} \circ \gamma(\sigma + t_0, \cdot) = z_{t_0} + \widetilde{\varphi} \circ \gamma(\sigma + t_0, \cdot)$, then $(\sigma + t_0, \varphi^0) \in \Omega$ and there exist a neighborhood $V \subset \Omega$ of (σ, φ) and a positive constant k such that f is k-Lipschitz on V. Let $\alpha' > 0$ such that for all $t \in J$, $0 < t - t_0 \leq \alpha'$, then $(\sigma + t, y_t + \widetilde{\varphi} \circ \gamma(\sigma + t, \cdot))$, $(\sigma + t, z_t + \widetilde{\varphi} \circ \gamma(\sigma + t, \cdot)) \in V$. However

$$\begin{aligned} y(t) - z(t) &= y(t) - y(t_0) + z(t_0) - z(t) \\ &= \int_{t_0}^{t} [f(\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot)) - f(\sigma + s, z_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot))] \mathrm{d}s, \end{aligned}$$

and thus

$$|y(t) - z(t)|_E \leq \int_{t_0}^{t} k \, \|y_s - z_s\| \, \mathrm{d}s.$$

It follows from Gronwall inequality [**5**] that $y(t) = z(t)$ for all $t_0 \leq t \leq t_0 + \alpha'$, which is a contradiction to the definition of t_0. □

Now, let $\sigma \in \mathbb{R}$ and Γ_σ be a set of continuous functions $\gamma : [\sigma, +\infty) \times [-r, 0] \to [\sigma - r, +\infty)$ such that $\gamma(\sigma, \theta) = \sigma + \theta, \gamma(\sigma + t, 0) = \sigma + t$ and $\gamma([\sigma, \sigma + t] \times [-r, 0]) = [\sigma - r, \sigma + t]$ for all $\theta \in [-r, 0]$, $t \geq 0$.

The following theorem gives a global solutions of the equation $R(f,\gamma)$ on $[\sigma - r, +\infty)$.

Theorem 4.2. *Let $f : \mathbb{R} \times \mathcal{C} \to E$ be a continuous function. Suppose that f is compact or verifies the condition (l) (resp. $f(t,\cdot)$ is locally Lipschitz). Suppose further that there exists a continuous function $m : \mathbb{R} \to \mathbb{R}^+$ such that*

$$|f(t,\psi)|_E \leq m(t)h(\|\psi\|), \qquad (t,\psi) \in \mathbb{R} \times \mathcal{C},$$

where h is continuous nondecreasing on $\mathbb{R}^+$, positive on $\mathbb{R}_*^+$ and $\int_0^{+\infty} \frac{\mathrm{d}s}{h(s)} = +\infty$. Then, for all $(\sigma,\varphi) \in \mathbb{R} \times \mathcal{C}$ and $\gamma \in \Gamma_\sigma$, there exists a function (resp. a unique solution) $u \in C([\sigma - r, +\infty); E)$ which verifies the Cauchy problem:

$$\text{(4.1)} \qquad \begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \geq 0 \\ u_\sigma = \varphi. \end{cases}$$

Proof. By Theorem 4.1, there exists a noncontinuable solution (resp. a unique solution) u of problem (4.1) on $[\sigma - r, \beta)$ where $\beta > \sigma$. We shall show $\beta = +\infty$. Suppose that $\beta < +\infty$. By Lemma 2.1, we have for $t \in [\sigma, \beta)$

$$\text{(*)} \qquad \begin{aligned} |u(t)|_E &\leq |\varphi(0)|_E + \int_\sigma^t |f(s, u \circ \gamma(s, \cdot))|_E \, \mathrm{d}s \\ &\leq \|\varphi\| + \int_\sigma^t m(s)h(\|u \circ \gamma(s, \cdot)\|)\mathrm{d}s. \end{aligned}$$

Consider the function v given by

$$v(t) = \sup\left\{|u(s)|_E : \sigma - r \le s \le t\right\}, \qquad t \in [\sigma, \beta).$$

It is clear that

$$(**) \qquad v(t) \le \|\varphi\| + \int_\sigma^t m(s)h(\|u \circ \gamma(s,\cdot)\|)\mathrm{d}s \le \|\varphi\| + \int_\sigma^t m(s)h(v(s))\mathrm{d}s.$$

Denoting by $w(t)$ the right-hand side of the above inequality $(**)$, we obtain $w(\sigma) = \|\varphi\|$ and

$$v(t) \le w(t), \qquad w'(t) = m(t)h(v(t)) \le m(t)h(w(t)), \qquad t \in [\sigma, \beta).$$

Integrating over $[\sigma, t]$, we obtain

$$\int_\sigma^t \frac{w'(s)}{h(w(s))}\mathrm{d}s = \int_{w(\sigma)}^{w(t)} \frac{\mathrm{d}s}{h(s)} \le \int_\sigma^t m(s)\mathrm{d}s < +\infty.$$

This inequality implies that there is a positive constant c such that for all $t \in [\sigma, \beta)$, $w(t) \le c$, then $v(t) \le c$. This majoration implies $|u'(t)|_E$ is bounded, hence u is uniformly continuous on $[\sigma - r, \beta)$, then there exists a unique continuous function $\overline{u} : [\sigma - r, \beta] \to E$ defined by

$$\overline{u}(t) = \begin{cases} u(t) & \text{if } \ t < \beta \\ \lim\limits_{s \to \beta} u(s) & \text{if } \ t = \beta. \end{cases}$$

Since $\gamma(s, \theta) \in [\sigma - r, s]$, then $\overline{u} \circ \gamma(s, \cdot) = u \circ \gamma(s, \cdot)$ and

$$\begin{aligned} \overline{u}(\beta) &= \lim_{s \to \beta} u(s) = \varphi(0) + \lim_{s \to \beta} \int_\sigma^s f(s', u \circ \gamma(s', \cdot))\mathrm{d}s' \\ &= \varphi(0) + \lim_{s \to \beta} \int_\sigma^s f(s', \overline{u} \circ \gamma(s', \cdot))\mathrm{d}s' \\ &= \varphi(0) + \int_\sigma^\beta f(s', \overline{u} \circ \gamma(s', \cdot))\mathrm{d}s'. \end{aligned}$$

This implies $\overline{u}$ is a solution to (4.1) on $[\sigma - r, \beta]$, which is a contradiction, and thus $\beta = +\infty$. □

Remark. By the fixed-point theorem for a strict contraction, we obtain the following result easily.

Theorem 4.3. *Let Γ be the set of continuous functions $\gamma : \mathbb{R} \times [-r, 0] \to \mathbb{R}$ such that*

$$\gamma([\sigma, \sigma + T] \times [-r, 0]) \subset [\sigma - r, \sigma + T] \qquad \textit{for all } \ (\sigma, T) \in \mathbb{R} \times \mathbb{R}_+^*$$

and $f : \mathbb{R} \times \mathcal{C} \to E$ be a continuous function such that $f(t, \cdot)$ is Lipschitz. Then, for all $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$ and $\gamma \in \Gamma$, there exists a unique function $u \in C([\sigma - r, +\infty); E) \cap C^1([\sigma, +\infty); E)$ which verifies the Cauchy problem:

$$(4.2) \qquad \begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \ge 0 \\ u_\sigma = \varphi. \end{cases}$$

5. Applications

Retarded functional differential equations

Let $(\sigma, a) \in \mathbb{R} \times \mathbb{R}_+^*$. Consider $\gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a]$ defined by $\gamma(t, \theta) = t + \theta$. Then $\gamma \in \Gamma_{\sigma,a}$ and the equation $R(f, \gamma)$ coincides with the classical retarded functional differential equations $u'(t) = f(t, u_t)$ (see, for example [**5, 6, 8**]).

Singular functional differential equations

Singular functional differential equations have been studied by many authors, for instance, Baxely [**1**], Bobisud and O'Regan [**2**], Gatica and al [**4**], Huaxing and Tadeusz [**7**], Labovskii [**9**] and O'Regan [**12**].

Our purpose in this section is to apply the previous results to give some theorems of existence for singular functional differential equations.

Theorem 5.1. *Consider the initial value problem for singular functional differential equations*

$$\begin{cases} \psi(\tau)x'(\tau) = g(\tau, x_\tau), \qquad \tau \in (\sigma, \sigma + b], \ (b > 0) \\ x_\sigma = \varphi \end{cases} \tag{5.1}$$

where $g : [\sigma, \sigma + b] \times \mathcal{C} \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b)$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$. Then, (5.1) *has at least one noncontinuable solution.*

Proof. Let $\rho : [\sigma - r, \sigma + b] \to [\sigma - r, \sigma + a]$ defined by

$$\rho(\tau) = \begin{cases} \sigma + \displaystyle\int_\sigma^\tau \frac{\mathrm{d}s}{\psi(s)} & \text{if } \ \tau \in [\sigma, \sigma + b] \\ \tau & \text{if } \ \tau \in [\sigma - r, \sigma], \end{cases}$$

then ρ is bijective and continuous.

For all $\tau \in [\sigma, \sigma + b]$ and $\theta \in [-r, 0]$, put

$$u(\rho(\tau + \theta)) = x(\tau + \theta).$$

Then, for all $\tau \in (\sigma, \sigma + b]$,

$$x'(\tau) = u'(\rho(\tau))\rho'(\tau) = u'(\rho(\tau))\frac{1}{\psi(\tau)}.$$

Hence $u'(\rho(\tau)) = \psi(\tau)x'(\tau) = g(\tau, x_\tau)$, and thus

$$u'(t) = g(\rho^{-1}(t), x_{\rho^{-1}(t)}), \qquad t \in (\sigma, \sigma + a], \qquad \text{and}$$

$$x_{\rho^{-1}(t)}(\theta) = x(\rho^{-1}(t) + \theta) = u(\rho(\rho^{-1}(t) + \theta)).$$

Consider the function $\gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a]$ defined by $\gamma(t, \theta) = \rho(\rho^{-1}(t) + \theta)$. It is clear that $\gamma \in \Gamma_{\sigma,a}$, $x_{\rho^{-1}(t)} = u \circ \gamma(t, \cdot)$ and

$$u_\sigma(\theta) = u(\sigma + \theta) = u(\gamma(\sigma, \theta)) = u(\rho(\rho^{-1}(\sigma) + \theta)) = x_\sigma(\theta) = \varphi(\theta).$$

□

Finally, the initial value problem (5.1) is equivalent to following problem

$$\begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \varphi, \end{cases} \tag{5.2}$$

where $f(t, \phi) = g(\rho^{-1}(t), \phi)$, f is also completely continuous and by Theorem 4.1, the problem (5.2) has at least one noncontinuable solution u on $[\sigma - r, \sigma + \alpha)$ with $\alpha \leq a$. It easy to see that the problem (5.1) has also a noncontinuable solution x on $[\sigma - r, \sigma + \beta)$ where α, β are fasten by the relation $\alpha = \int_\sigma^{\sigma+\beta} \frac{\mathrm{d}s}{\psi(s)}$.

An important criteria given by the following theorem assure the existence of global solutions of (5.1).

Theorem 5.2. *Assume the conditions of Theorem 5.1 are satisfied. Suppose further that*

(1) *for all* $(\tau, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$

$$|g(\tau, \phi)|_E \leq m(\tau) h(\|\phi\|),$$

where $m : [\sigma, \sigma + b] \to \mathbb{R}^+$ *and* $h : \mathbb{R}^+ \to \mathbb{R}_*^+$ *are continuous,* h *nondecreasing on* $\mathbb{R}^+$ *and* $\int_\sigma^{\sigma+b} \frac{m(s)}{\psi(s)}\, \mathrm{d}s < \int_{\|\varphi\|}^{+\infty} \frac{\mathrm{d}s}{h(s)}$, *or*

(2) *for all* $(\tau, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$

$$|g(\tau, \phi)|_E \leq h(\tau, |\phi(0)|_E),$$

where $h : [\sigma, \sigma + b] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and the set of solutions to the singular ordinary differential equation*

$$\begin{cases} \psi(\tau) y'(\tau) = h(\tau, y(\tau)) \\ y(\lambda) = \mu \end{cases}$$

is bounded on $C([\lambda, \sigma + b]; \mathbb{R})$, *or*

(3) (ξ) *there are three functions* $V \in C([\sigma, \sigma + b] \times \mathcal{C}; \mathbb{R}^+)$, $a_1, a_2 \in C(\mathbb{R}^+; \mathbb{R}^+)$ *with* $\lim\limits_{s \to +\infty} a_1(s) = +\infty$ *and* $a_1(\|\phi\|) \leq V(t, \phi) \leq a_2(\|\phi\|)$ *for all* $(t, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$

(ξ') *for any* $0 < \beta \leq b$ *and for any solution* x *of* (5.1) *on* $[\sigma - r, \sigma + \beta)$, *we have for all* $t \in (\sigma, \sigma + \beta)$

$$\begin{aligned} D^+ V(t, x_t(\sigma, \varphi)) &:= \limsup_{k \to 0^+} \frac{1}{k} [V(t + k, x_{t+k}(t, \varphi)) - V(t, x_t(\sigma, \varphi))] \\ &\leq [\psi(t)]^{-1} h(t, V(t, x_t(\sigma, \varphi)) \end{aligned}$$

where $h : [\sigma, \sigma + b] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and the set of solutions of the singular ordinary differential equation*

$$\begin{cases} \psi(t) y'(t) = h(t, y(t)) \\ y(\lambda) = \mu \end{cases}$$

is bounded on $C([\lambda, \sigma + b]; \mathbb{R})$.

Then (5.1) *has at least one global solution on* $[\sigma - r, \sigma + b]$.

Proof.

– Suppose that the condition (1) is verified.

Let x (resp. u) be a solution of (5.1) (resp. (5.2)) on $[\sigma - r, \sigma + \beta_1)$ (resp. on $[\sigma - r, \sigma + \alpha_1)$) with $\alpha_1 = \int_\sigma^{\sigma+\beta_1} \frac{\mathrm{d}s}{\psi(s)}$. By using the same argument seen in the Theorem 4.2, we obtain

$$\lim_{\tau \to \sigma+\beta_1} x(\tau) = \lim_{t \to \sigma+\alpha_1} u(t) \text{ exist.}$$

Take

$$x^1(\tau) = \begin{cases} x(\tau) & \text{if } \tau \in [\sigma - r, \sigma + \beta_1) \\ \lim\limits_{\tau' \to \sigma+\beta_1} x(\tau') & \text{if } \tau = \sigma + \beta_1. \end{cases}$$

Then x^1 is a solution to (5.1) on $[\sigma - r, \sigma + \beta_1]$. If $\beta_1 < b$, consider the problem

$$\begin{cases} \psi(\tau)x'(\tau) = g(\tau, x_\tau), & \tau \in [\sigma + \beta_1, \sigma + b] \\ x_{\sigma+\beta_1} = x^1_{\sigma+\beta_1}, \end{cases}$$

then this problem has a solution x^2 on $[\sigma + \beta_1 - r, \sigma + \beta_1 + \beta_2]$. Define

$$z(t) = \begin{cases} x^1(t) & \text{if} \quad t \in [\sigma - r, \sigma + \beta_1] \\ x^2(t) & \text{if} \quad t \in [\sigma + \beta_1, \sigma + \beta_1 + \beta_2]. \end{cases}$$

Then z is a solution of (5.1) on $[\sigma - r, \sigma + \beta_1 + \beta_2]$. Repeating this method, we can get a global solution of (5.1) on $[\sigma - r, \sigma + b]$.

– Suppose that the condition (2) is verified.

Let x be a solution of (5.1) on $[\sigma - r, \sigma + \beta_1)$. Take $m(\tau) = |x(\tau)|_E$, $\tau \in [\sigma, \sigma + \beta)$, then $m(\sigma) \leq \|\varphi\| = y(\sigma)$ (with $\mu = \|\varphi\|$) and for all $\tau \in (\sigma, \sigma + \beta)$

$$\psi(\tau)D^+m(\tau) := \psi(\tau)\limsup_{k \to 0^+} \frac{m(\tau + k) - m(\tau)}{k} \leq \psi(\tau)\,|x'(\tau)|_E \leq h(\tau, m(\tau)).$$

Consequently, by [**10**] we obtain $m(\tau) \leq y_{\max}(\tau)$ (where $y_{\max}$ is a maximal solution of singular the ordinary differential equation). Let $M = \sup\{y_{\max}(\tau) : \tau \in [\sigma, \sigma + b]\}$. Note that $|x(\tau)|_E \leq M$, $\tau \in [\sigma, \sigma + \beta_1)$. We shall prove that $\lim\limits_{\tau \to \sigma+\beta_1} x(\tau)$ exists. For $\sigma < \tau < \tau' < \sigma + \beta$ we have

$$\begin{aligned} |x(\tau) - x(\tau')|_E &\leq \int_\tau^{\tau'} [\psi(s)]^{-1}\,|g(s, x_s)|_E\,\mathrm{d}s \\ &\leq \int_\tau^{\tau'} [\psi(s)]^{-1} h(s, |x(s)|_E)\mathrm{d}s \leq M' \int_\tau^{\tau'} [\psi(s)]^{-1}\mathrm{d}s, \end{aligned}$$

where $M' = \max\{h(s,t) \ : s \in [\sigma, \sigma + b],\ t \leq M\}$.

For any $\varepsilon > 0$, we can find $\eta > 0$ such that

$$\left| \int_\tau^{\tau'} \frac{\mathrm{d}s}{\psi(s)} \right| < \frac{\varepsilon}{M'}$$

whenever $|\tau - \tau'| < \eta$, now for any $\tau < \tau'$ such that $|\tau - (\sigma + \beta)| < \frac{\eta}{2}$ and $|\tau' - (\sigma + \beta)| < \frac{\eta}{2}$, then

$$|x(\tau) - x(\tau')|_E \leq M' \int_{\tau}^{\tau'} [\psi(s)]^{-1} \mathrm{d}s < \varepsilon.$$

The rest of the proof is identical to the condition (1).

– Suppose that the condition (3) is verified.
Let $m(t) = V(t, x_t)$. By (ξ) we obtain

$$m(\sigma) \leq a_2(\|x_\sigma\|) = a_2(\|\varphi\|) := y(\sigma)$$

and by (ξ') we have

$$\psi(t) D^+ m(t) \leq h(t, m(t)), \qquad t \in (\sigma, \sigma + \beta).$$

Then, $m(t) \leq y_{\max}(t)$ where $y_{\max}$ is a maximal solution of the singular ordinary differential equation. Hence

$$a_1(\|x_t\|) \leq V(t, \phi) = m(t) \leq y_{\max}(t) \leq M.$$

But $\lim\limits_{s \to +\infty} a_1(s) = +\infty$, there exists $M' > 0$ such that $M < a_1(M')$, so $\|x_t\| \leq M'$. Let $M_1 = \sup \{|g(t, \varphi)|_E : t \in (\sigma, \sigma + \beta),\ \|\varphi\| < M'\}$, then the rest of the proof is similar to the condition (2). □

Theorem 5.3. *Consider the initial value problem for singular functional differential equations*

$$\text{(5.3)} \qquad \begin{cases} \psi(\tau) x^{(n)}(\tau) = g(\tau, x_\tau, x'_\tau, \ldots, x_\tau^{(n-1)}), \quad \tau \in (\sigma, \sigma + b],\ (b > 0) \\ x_\sigma = \varphi \in C^{(n-1)}([-r, 0]; E), \end{cases}$$

where $g : [\sigma, \sigma + b] \times \mathcal{C}^n \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma + b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$. Then, (5.3) *has at least one noncontinuable solution.*

Proof. Let ρ and γ be the functions defined in Theorem 5.1. For all $\tau \in [\sigma, \sigma + b]$ and $\theta \in [-r, 0]$, put

$$u(\rho(\tau + \theta)) = \begin{pmatrix} u^1(\rho(\tau + \theta)) \\ u^2(\rho(\tau + \theta)) \\ \vdots \\ u^n(\rho(\tau + \theta)) \end{pmatrix} = \begin{pmatrix} x(\tau + \theta) \\ x'(\tau + \theta) \\ \vdots \\ x^{(n-1)}(\tau + \theta) \end{pmatrix}.$$

Using the same technique as in the proof of Theorem 5.1, the problem (5.3) becomes equivalent to the following problem

$$\text{(5.4)} \qquad \begin{cases} u'(t) = F_1(t, u \circ \gamma(t, \cdot)) + F_2(t, u \circ \gamma(t, \cdot)), \quad t \in (\sigma, \sigma + a] \\ u_\sigma = (\varphi, \varphi', \ldots, \varphi^{(n-1)}), \end{cases}$$

where $F_1, F_2 : [\sigma, \sigma + a] \times C([-r, 0]; E^n) \to E^n$ are defined by

$$F_1(t,\phi) = \begin{pmatrix} \psi(\rho^{-1}(t))\phi_2(0) \\ \vdots \\ \psi(\rho^{-1}(t))\phi_n(0) \\ 0 \end{pmatrix}, \qquad F_2(t,\phi) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(\rho^{-1}(t), \phi_1, \phi_2, \ldots, \phi_n) \end{pmatrix}$$

with $\phi_1, \phi_2, \ldots, \phi_n$ which are the components of $\phi \in C([-r, 0]; E^n)$. It is easy to see that $F_1(t, \cdot)$ is Lipschitz and F_2 is completely continuous, then $F = F_1 + F_2$ verifies the condition (l) and by Theorem 4.1, the problem (5.4) has at least one noncontinuable solution and therefore, the problem (5.3) has also at least a noncontinuable solution. □

Theorem 5.4. *Consider the initial value problem for singular functional equations*

$$(5.5) \quad \begin{cases} \psi(\tau)x'(\tau) = g(\tau, x(\tau - r_1), \ldots, x(\tau - r_m)), & \tau \in]\sigma, \sigma + b], \ (b > 0) \\ x_\sigma = \varphi & \text{on } [-r, 0] \text{ with} \\ & r = \max\limits_{1 \le i \le m} (r_i), \ r_i \ge 0 \end{cases}$$

where $g : [\sigma, \sigma + b] \times E^2 \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$.
Then, (5.5) *has at least one noncontinuable solution.*

Proof. Using the same argument as in the proof of Theorem 4.1, we can see that the problem (5.5) is equivalent to the following problem

$$(5.6) \quad \begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \varphi, \end{cases}$$

where $f(t, \phi) = g(\rho^{-1}(t), \phi(-r_1), \ldots, \phi(-r_m))$. □

Theorem 5.5. *Consider the initial value problem for singular functional equations*

$$(5.7) \quad \begin{cases} \psi(\tau)x''(\tau) = g(\tau, x(\tau), x(\tau - r_1), x'(\tau), x'(\tau - r_2)), & \tau \in (\sigma, \sigma + b] \\ x_\sigma = \varphi, \ x'_\sigma = \varphi', & \text{on } [-r, 0] \text{ with} \\ & r = \max(r_1, r_2) \end{cases}$$

where $g : [\sigma, \sigma + b] \times E^2 \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$.

Then, (5.7) *has at least one noncontinuable solution.*

Proof. The problem (5.7) is equivalent to the following problem

$$(5.8) \quad \begin{cases} u'(t) = F_1(t, u \circ \gamma(t, \cdot)) + F_2(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \Phi & \text{with } \Phi = (\varphi, \varphi'), \end{cases}$$

where $F_1, F_2 : [\sigma, \sigma + a] \times C([-r, 0]; E^2) \to E^2$ are defined by

$$F_1(t, \phi) = \begin{pmatrix} \psi(\rho^{-1}(t))\phi_2(0) \\ 0 \end{pmatrix}$$

and

$$F_2(t, \phi) = \begin{pmatrix} 0 \\ g(\rho^{-1}(t), \phi_1(0), \phi_1(-r_1)\phi_2(0), \phi_2(-r_2)) \end{pmatrix}$$

with $\phi : \theta \in [-r, 0] \to \phi(\theta) = (\phi_1(\theta), \phi_2(\theta))$, $\phi_1, \phi_2 \in \mathcal{C}$. □

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