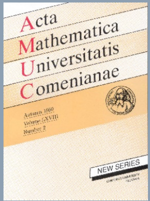

ON THE OSTROWSKI TYPE INTEGRAL INEQUALITY

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Abstract. In this note, we establish an inequality of Ostrowski-type involving functions of two independent variables newly by using certain integral inequalities.

1. Introduction

In [3], Ujević proved the following double integral inequality:

Theorem 1. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a,b) *and suppose that* $\gamma \leq f''(t) \leq \Gamma$ *for all* $t \in (a,b)$*. Then we have the double inequality*

$$\frac{3S-\Gamma}{24}(b-a)^2 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq \frac{3S-\gamma}{24}(b-a)^2 \tag{1.1}$$

where $S = \dfrac{f'(b)-f'(a)}{b-a}$.

In a recent paper [2], Liu et al. have proved the following two sharp inequalities of perturbed Ostrowski-type

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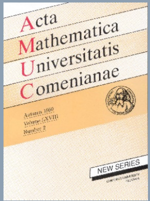

Go back
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Close
Quit

Theorem 2. *Under the assumptions of Theorem 1, we have*

$$
\begin{aligned}
&\frac{\Gamma[(x-a)^3-(b-x)^3]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^2 (S-\Gamma) \\
&\leq \frac{1}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\
&\leq \frac{\gamma[(x-a)^3-(b-x)^3]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^2 (S-\gamma),
\end{aligned} \tag{1.2}
$$

for all $x \in [a,b]$,

where $S = \dfrac{f'(b)-f'(a)}{b-a}$. *If* γ, Γ *are given by*

$$\gamma = \min_{t\in[a,b]} f''(t), \qquad \Gamma = \max_{t\in[a,b]} f''(t)$$

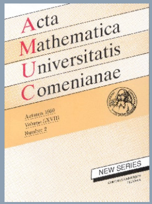
then the inequality given by (2) is sharp in the usual sense.

In [1], Cheng has proved the following integral inequality

Theorem 3. *Let* $I \subset \mathbb{R}$ *be an open interval,* $a, b \in I$, $a < b$. $f : I \to \mathbb{R}$ *is a differentiable function such that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \leq f'(x) \leq \Gamma$, $x \in [a,b]$. *Then we have*

$$
\begin{aligned}
&\left|\frac{1}{2}f(x) - \frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \\
&\leq \frac{(x-a)^2+(b-x)^2}{8(b-a)}(\Gamma-\gamma),
\end{aligned} \tag{1.3}
$$

for all $x \in [a,b]$.



Go back
Full Screen
Close
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The main purpose of this paper is to establish new inequality similar to the inequalities (1)–(3) involving functions of two independent variables.

2. Main Result

Theorem 4. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be an absolutely continuous fuction such that the partial derivative of order 2 exists and supposes that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma$ *for all* $(t,s) \in [a,b] \times [c,d]$. *Then, we have*

$$\begin{aligned}
&\Bigg| \frac{1}{4} f(x,y) + \frac{1}{4} H(x,y) - \frac{1}{2(b-a)} \int_a^b f(t,y)\mathrm{d}t - \frac{1}{2(d-c)} \int_c^d f(x,s)\mathrm{d}s \\
&\quad - \frac{1}{2(b-a)(d-c)} \int_a^b [(y-c)f(t,c) + (d-y)f(t,d)]\mathrm{d}t \\
&\quad - \frac{1}{2(b-a)(d-c)} \int_c^d [(x-a)f(a,s) + (b-x)f(b,s)]\mathrm{d}s \\
&\quad + \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\mathrm{d}s\,\mathrm{d}t \Bigg| \\
&\leq \frac{[(x-a)^2 + (b-x)^2][(y-c)^2 + (d-y)^2]}{32(b-a)(d-c)} (\Gamma - \gamma),
\end{aligned} \tag{2.1}$$

for all $(x, y) \in [a, b] \times [c, d]$ *where*

$$H(x,y) = \frac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]+(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)} + \frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a} + \frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}.$$

Proof. We define the functions: $p : [a, b] \times [a, b] \to \mathbb{R}$, $q : [c, d] \times [c, d] \to \mathbb{R}$ given by

$$p(x,t) = \begin{cases} t - \dfrac{a+x}{2}, & t \in [a,x] \\ t - \dfrac{b+x}{2}, & t \in (x,b] \end{cases}$$

and

$$q(y,s) = \begin{cases} s - \dfrac{c+y}{2}, & s \in [c,y] \\ s - \dfrac{d+y}{2}, & s \in (y,d]. \end{cases}$$

By definitions of $p(x,t)$ and $q(y,s)$, we have

$$
\begin{aligned}
\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\mathrm{d}t &= \int_a^x\int_c^y\left(t-\frac{a+x}{2}\right)\left(s-\frac{c+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t\\
&+\int_a^x\int_y^d\left(t-\frac{a+x}{2}\right)\left(s-\frac{d+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t\\
&+\int_x^b\int_c^y\left(t-\frac{b+x}{2}\right)\left(s-\frac{c+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t\\
&+\int_x^b\int_y^d\left(t-\frac{b+x}{2}\right)\left(s-\frac{d+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
\tag{2.2}
$$

Integrating by parts twice, we can state:

$$
\begin{aligned}
&\int_a^x\int_c^y\left(t-\frac{a+x}{2}\right)\left(s-\frac{c+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t\\
&=\frac{(x-a)(y-c)}{4}[f(x,y)+f(a,y)+f(x,c)+f(a,c)]\\
&\quad-\frac{y-c}{2}\int_a^x[f(t,y)+f(t,c)]\mathrm{d}t-\frac{x-a}{2}\int_c^y[f(x,s)+f(a,s)]\mathrm{d}s+\int_a^x\int_c^y f(t,s)\mathrm{d}s\mathrm{d}t.
\end{aligned}
\tag{2.3}
$$

$$\int_a^x \int_y^d \left(t - \frac{a+x}{2}\right)\left(s - \frac{d+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t$$

$$(2.4) \quad = \frac{(x-a)(d-y)}{4}[f(x,y)+f(x,d)+f(a,y)+f(a,d)]$$

$$- \frac{d-y}{2}\int_a^x [f(t,d)+f(t,y)]\mathrm{d}t - \frac{x-a}{2}\int_y^d [f(x,s)+f(a,s)]\mathrm{d}s + \int_a^x \int_y^d f(t,s)\mathrm{d}s\,\mathrm{d}t.$$

$$\int_x^b \int_c^y \left(t - \frac{b+x}{2}\right)\left(s - \frac{c+y}{2}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t$$

$$(2.5) \quad = \frac{(b-x)(y-c)}{4}[f(x,y)+f(b,y)+f(x,c)+f(b,c)]$$

$$- \frac{y-c}{2}\int_x^b [f(t,c)+f(t,y)]\mathrm{d}t - \frac{b-x}{2}\int_c^y [f(x,s)+f(b,s)]\mathrm{d}s + \int_x^b \int_c^y f(t,s)\mathrm{d}s\,\mathrm{d}t.$$

$$\int_x^b \int_y^d (t - \frac{b+x}{2})(s - \frac{d+y}{2})\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\,\mathrm{d}t$$

$$(2.6) \quad = \frac{(b-x)(d-y)}{4}[f(x,y)+f(x,d)+f(b,y)+f(b,d)]$$

$$- \frac{d-y}{2}\int_x^b [f(t,d)+f(t,y)]\mathrm{d}t - \frac{b-x}{2}\int_y^d [f(x,s)+f(b,s)]\mathrm{d}s + \int_x^b \int_y^d f(t,s)\mathrm{d}s\,\mathrm{d}t.$$

Adding (6)–(9) and rewriting, we easily deduce

$$\begin{aligned}
\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\,\mathrm{d}t = \frac{1}{4}\{&(b-a)(d-c)f(x,y)\\
&+[(x-a)f(a,y)+(b-x)f(b,y)](d-c)\\
&+[(y-c)f(x,c)+(d-y)f(x,d)](b-a)\\
&+[(y-c)f(a,c)+(d-y)f(a,d)](x-a)\\
&+[(y-c)f(b,c)+(d-y)f(b,d)](b-x)\}\\
-\frac{d-c}{2}\int_a^b f(t,y)\mathrm{d}t-\frac{b-a}{2}\int_c^d f(x,s)\mathrm{d}s&\\
-\frac{1}{2}\int_a^b[(y-c)f(t,c)+(d-y)f(t,d)]\mathrm{d}t&\\
-\frac{1}{2}\int_c^d[(x-a)f(a,s)+(b-x)f(b,s)]\mathrm{d}s+\int_a^b\int_c^d f(t,s)\mathrm{d}s\,\mathrm{d}t.&
\end{aligned} \tag{2.7}$$

We also have

$$\int_a^b\int_c^d p(x,t)q(y,s)\mathrm{d}s\,\mathrm{d}t = 0. \tag{2.8}$$

Let $M = \frac{\Gamma+\gamma}{2}$. From (10) and (11), it follows that

$$\int_a^b \int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right] \mathrm{d}s\,\mathrm{d}t = \int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s} \mathrm{d}s\,\mathrm{d}t. \tag{2.9}$$

On the other hand, we get

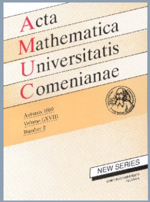
$$\left|\int_a^b \int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right] \mathrm{d}s\mathrm{d}t\right| \leq \max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right| \int_a^b \int_c^d |p(x,t)q(y,s)|\,\mathrm{d}s\,\mathrm{d}t. \tag{2.10}$$

We also have

$$\max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right| \leq \frac{\Gamma-\gamma}{2} \tag{2.11}$$

and

$$\int_a^b \int_c^d |p(x,t)q(y,s)|\,\mathrm{d}s\,\mathrm{d}t = \frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{16}. \tag{2.12}$$



From (13) to (15), we easily get

$$\left|\int_a^b \int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right] \mathrm{d}s\,\mathrm{d}t\right| \le \frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{32}(\Gamma-\gamma). \tag{2.13}$$

From (12) and (16), we see that (4) holds. □

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