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THE CONTINUOUS DUAL OF THE SEQUENCE SPACE $l_p(\Delta^n)$, $(1 \leq p \leq \infty, n \in \mathbb{N})$

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Abstract. The space $l_p(\Delta^m)$ consisting of all sequences whose m^{th} order differences are p-absolutely summable was recently studied by Altay [*On the space of p-summable difference sequences of order m,* $(1 \leq p < \infty)$, Stud. Sci. Math. Hungar. **43(4)** (2006), 387–402]. Following Altay [2], we have found the continuous dual of the spaces $l_1(\Delta^n)$ and $l_P(\Delta^n)$. We have also determined the norm of the operator Δ^n acting from l_1 to itself and from l_∞ to itself, and proved that Δ^n is a bounded linear operator on the space $l_p(\Delta^n)$.

1. Preliminaries, Definitions and Notations

Let ω denote the space of all complex-valued sequences, i.e. $\omega = \mathbb{C}^{\mathbb{N}}$ where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Any vector subspace of ω which contains ϕ, the set of all finitely non-zero sequences, is called a sequence space. The continuous dual of a sequence space λ which is denoted by λ^* is the set of all bounded linear functionals on λ. Suppose Δ be the difference operator with matrix representation

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and suppose $x = (x_k)_{k=0}^{\infty} \in \omega$, then $\Delta x = (x_k - x_{k-1})_{k=0}^{\infty}$ and $\Delta^n x = \Delta(\Delta^{n-1}x)$ for all $n \geq 2$ where any x with negative index is zero. For every $n \in \mathbb{N} \setminus \{0\}$,

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Δ^n has a triangle matrix representation, so it is invertible and

$$\Delta^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\binom{n}{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \binom{n}{2} & -\binom{n}{1} & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\binom{n}{3} & \binom{n}{2} & -\binom{n}{1} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ (-1)^n\binom{n}{n} & (-1)^{n-1}\binom{n}{n-1} & \vdots & \vdots & -\binom{n}{1} & 1 & 0 & 0 & \cdots \\ 0 & (-1)^n\binom{n}{n} & (-1)^{n-1}\binom{n}{n-1} & \vdots & \vdots & -\binom{n}{1} & 1 & 0 & \cdots \\ 0 & 0 & (-1)^n\binom{n}{n} & (-1)^{n-1}\binom{n}{n-1} & \vdots & \vdots & -\binom{n}{1} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\Delta^{-n} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \binom{n}{1} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \binom{n+1}{2} & \binom{n}{1} & 1 & 0 & 0 & 0 & \cdots \\ \binom{n+2}{3} & \binom{n+1}{2} & \binom{n}{1} & 1 & 0 & 0 & \cdots \\ \binom{n+3}{4} & \binom{n+2}{3} & \binom{n+1}{2} & \binom{n}{1} & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$, there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0, \tag{1}$$

then (b_n) is called a Schauder basis for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

2. The space $l_p(\Delta^n)$

Now we introduce an apparently new sequence space and denote it by $l_p(\Delta^n)$ like Kizmaz who defined and studied $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$.

$$l_p(\Delta^n) = \{x \in \omega : \Delta^n x \in l_p\} \tag{2}$$

$$\|x\|_{l_p(\Delta^n)} = \|\Delta^n x\|_{l_p} \tag{3}$$

Trivially $l_p(\Delta) = bv_p$.

Theorem 2.1. *$l_p(\Delta^n)$ is a Banach space.*

Proof. Since it is a routine verification to show that $l_p(\Delta^n)$ is a normed space with the norm defined by (3) and coordinate-wise addition and scalar multiplication we omit the details. To prove the theorem, we show that every Chauchy sequence in $l_p(\Delta^n)$ has a limit. Suppose $(x^{(m)})_{m=0}^{\infty}$ is a Chauchy sequence in $l_p(\Delta^n)$. So

(4)

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall r, s \geq N)(\|\Delta^n x^{(r)} - \Delta^n x^{(s)}\|_{l_p} = \|x^{(r)} - x^{(s)}\|_{l_p(\Delta^n)} < \varepsilon)$$

So the sequence $(\Delta^n x^{(m)})_{m=0}^{\infty}$ in l_p is Chauchy and since l_p is Banach, there exists $x \in l_p$ such that

$$\|\Delta^n x^{(m)} - x\|_{l_p} \to 0 \qquad \text{as} \;\; m \to \infty \tag{5}$$

But $x = (\Delta^n)(\Delta^n)^{-1}x$, so $\|\Delta^n x^{(m)} - \Delta^n(\Delta^n)^{-1}x\|_{l_p} = \|x^{(m)} - (\Delta^n)^{-1}x\|_{l_p(\Delta^n)} \to 0$ as $m \to \infty$. Now, since $(\Delta^n)^{-1}x \in l_p(\Delta^n)$ we are done. □

Theorem 2.2. *$l_p(\Delta^n)$ is isometrically isomorphic to l_p.*

Proof. Let

$$T : l_p(\Delta^n) \to l_p \tag{6}$$

defined by $T(x) = \Delta^n x$. Since T is bijective and norm preserving, we are done. □

Theorem 2.3. *Except the case $p = 2$, the space $l_p(\Delta^n)$ is not an inner product space and hence not a Hilbert space for $1 \le p < \infty$.*

Proof. First we show that $l_2(\Delta^n)$ is a Hilbert space. It suffices to show that $l_2(\Delta^n)$ has an inner product. Since

$$\|x\|_{l_2(\Delta^n)} = \|\Delta^n x\|_{l_2} =< \Delta^n x, \Delta^n x >^{\frac{1}{2}}, \tag{7}$$

$l_2(\Delta^n)$ is a Hilbert space. Now, we show that if $p \ne 2$, then $l_p(\Delta^n)$ is not Hilbert. Let

$$u = (\Delta^{n-1})^{-1}(1, 2, 2, 2, \cdots)$$
$$e = (\Delta^{n-1})^{-1}(1, 0, 0, 0, \cdots).$$

Then $\|u\|_{l_p(\Delta^n)} = \|e\|_{l_p(\Delta^n)} = 2^{\frac{1}{p}}$ and $\|u + e\|_{l_p(\Delta^n)} = \|u - e\|_{l_p(\Delta^n)} = 2$. So the parallelogram equality does not satisfy. Hence the space $l_p(\Delta^n)$ with $p \ne 2$ is not a Hilbert space. □

Theorem 2.4. *If $1 \le p < q < \infty$, then $l_p(\Delta^n) \subseteq l_q(\Delta^n) \subseteq l_\infty(\Delta^n)$.*

Proof. We only point out that if $1 \le p < q < \infty$, then $l_p \subseteq l_q \subseteq l_\infty$. □

Theorem 2.5. $l_p \subseteq l_p(\Delta) \subseteq l_p(\Delta^2) \subseteq l_p(\Delta^3) \subseteq \cdots$

Proof. Since $l_p \subseteq bv_p$, it is trivial that $l_p \subseteq l_p(\Delta)$. Now, if $x \in l_p(\Delta^n)$, then $\Delta^n x \in l_p \subseteq l_p(\Delta)$. So

$$\Delta^n x \in l_p(\Delta) \;\; \Rightarrow \;\; \Delta(\Delta^n x) \in l_p \;\; \Rightarrow \;\; \Delta^{n+1} x \in l_p \;\; \Rightarrow \;\; x \in l_p(\Delta^{n+1}).$$

□

Theorem 2.6. $\|x\|_{l_p(\Delta^n)} \le 2^n \|x\|_{l_p}$

Proof. Since $\|x\|_{l_p(\Delta)} = \|x\|_{bv_p} \le 2\|x\|_{l_p}$, $\|x\|_{l_p(\Delta^2)} \le 2\|x\|_{l_p(\Delta)} \le 2 \cdot 2 \cdot \|x\|_{l_p} = 2^2\|x\|_{l_p}$. Now by induction, we are done. □

3. Schauder basis for space $l_p(\Delta^n)$

Suppose e^k is a sequence whose only nonzero term is 1 in the $(k+1)^{\text{th}}$ place. The sequence $(\Delta^{-n}e^k)_{k=0}^{\infty}$ is a sequence of elements of $l_p(\Delta^n)$ since for all $k \in \mathbb{N}$, $e^k \in l_p$. We assert that this sequence is a Schauder basis for $l_p(\Delta^n)$. Suppose $x \in l_p(\Delta^n)$, $x^{[m]} = \sum_{k=0}^{m} (\Delta^n x)_k (\Delta^{-n} e^k) = \sum_{k=0}^{m} \Delta^{-n}((\Delta^n x)_k e^k)$. Then since $x \in l_p(\Delta^n)$, we have $\Delta^n x \in l_p$ such that

$$\left(\sum_{i=0}^{\infty} |(\Delta^n x)_i|^p \right)^{\frac{1}{p}} = s < \infty \tag{8}$$

$$\Rightarrow (\forall \varepsilon > 0)(\exists m_0 \in \mathbb{N}) \left(\sum_{i=m}^{\infty} |(\Delta^n x)_i|^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{2} \qquad \text{for all } m \geq m_0 \tag{9}$$

$$\Rightarrow \|x - x^{[m]}\|_{l_p(\Delta^n)} = \|\Delta^n x - \Delta^n x^{[m]}\|_{l_p} \tag{10}$$

$$= \| \sum_{k=0}^{\infty} (\Delta^n x)_k e^k - \sum_{k=0}^{m} (\Delta^n x)_k e^k \|_{l_p}$$

$$= \| \sum_{k=m+1}^{\infty} (\Delta^n x)_k e^k \|_{l_p} = \left(\sum_{k=m+1}^{\infty} |\Delta^n x|_k^p \right)^{\frac{1}{p}} \tag{11}$$

$$\leq \left(\sum_{k=m_0}^{\infty} |\Delta^n x|_k^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{2}$$

So $x = \sum_{k=0}^{\infty} (\Delta^n x)_k (\Delta^{-n} e^k) = \sum_{k=0}^{\infty} \Delta^{-n}((\Delta^n x)_k e^k)$. Now, we show the uniqueness of this representation. Suppose $x = \sum_{k=0}^{\infty} \mu_k (\Delta^{-n} e^k) = \sum_{k=0}^{\infty} \Delta^{-n}(\mu_k e^k)$, so $\Delta^n x = \sum_{k=0}^{\infty} \mu_k e^k$. On the other hand $\Delta^n x = \sum_{k=0}^{\infty} (\Delta^n x)_k e^k$. Hence $\mu_k = (\Delta^n x)_k$, for all $k \in \mathbb{N}$. So this representation is unique.

4. Continuous dual of $l_p(\Delta^n)$

Sequence space bv_p is $l_p(\Delta)$ so $l_p(\Delta^n)$ is an extension of this space. In [**1**] the continuous dual of bv_p was studied. The idea was wrong. We showed a counter example and then corrected it in [**4**]. Now, we introduce the continuous dual of $l_p(\Delta^n)$.

Suppose $1 \leq q < \infty$ and let

$$d_q^n = \left\{ a \in \omega : \|a\|_{d_q^n} = \left\| D^{(n)} a \right\|_{l_q} = \left(\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} D_{kj}^{(n)} a_j \right|^q \right)^{\frac{1}{q}} < \infty \right\} \tag{12}$$

$$d_\infty^n = \left\{ a \in \omega : \|a\|_{d_\infty^n} = \left\| D^{(n)} a \right\|_{l_\infty} = \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} D_{kj}^{(n)} a_j \right| < \infty \right\}, \tag{13}$$

where

$$(14)\qquad D^{(n)} = \begin{bmatrix} 1 & \binom{n}{1} & \binom{n+1}{2} & \binom{n+2}{3} & \binom{n+3}{4} & \binom{n+4}{5} & \cdots \\ 0 & 1 & \binom{n}{1} & \binom{n+1}{2} & \binom{n+2}{3} & \binom{n+3}{4} & \cdots \\ 0 & 0 & 1 & \binom{n}{1} & \binom{n+1}{2} & \binom{n+2}{3} & \cdots \\ 0 & 0 & 0 & 1 & \binom{n}{1} & \binom{n+1}{2} & \cdots \\ 0 & 0 & 0 & 0 & 1 & \binom{n}{1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

since $D^{(n)}$ is triangle, then ${D^{(n)}}^{-1}$ exists. Trivially d_q^n and d_∞^n are normed spaces with respect to coordinate-wise addition and scalar multiplication. d_q^n and d_∞^n are Banach spaces since if $(x^{(m)})_{m=0}^\infty$ is a Chauchy sequence in d_q^n, then

$$(15)\ (\forall\varepsilon > 0)(\exists N \in \mathbb{N})(\forall r, s > N)\|D^{(n)}(x^{(r)} - x^{(s)})\|_{l_q} = \|x^{(r)} - x^{(s)}\|_{d_q^n} < \varepsilon$$

so the sequence $(D^{(n)}(x^{(m)}))_{m=0}^\infty$ is Chauchy in l_q and since l_q is Banach, there exists y in l_q such that $\|D^{(n)}x^{(m)} - y\|_{l_q} \to 0$ as $m \to \infty$. But $y = D^{(n)}{D^{(n)}}^{-1}y$, so $\|D^{(n)}x^{(m)} - D^{(n)}{D^{(n)}}^{-1}y\|_{l_q} = \|x^{(m)} - {D^{(n)}}^{-1}y\|_{d_q^n} \to 0$ as $m \to \infty$. On the other hand ${D^{(n)}}^{-1}y \in d_q^n$. So d_q^n is Banach. In a similar way d_∞^n is Banach.

Theorem 4.1. *$l_1(\Delta^n)^*$ is isometrically isomorphic to d_∞^n.*

Proof. Let

$$(16)\qquad T : l_1(\Delta^n)^* \to d_\infty^n$$

defined by $Tf = (f(e^0), f(e^1), f(e^2), f(e^3), \cdots)$. Trivially T is linear and since $x = \sum_{k=0}^\infty (\Delta^n x)_k(\Delta^{-n}e^k)$ we have $f(x) = \sum_{k=0}^\infty (\Delta^n x)_k f(\Delta^{-n}e^k)$. But

$$(17)\qquad \begin{aligned} \Delta^{-n}e^k &= (\underbrace{0, 0, \cdots, 0}_{k \text{ term}}, 1, \tbinom{n}{1}, \tbinom{n+1}{2}, \tbinom{n+2}{3}, \tbinom{n+3}{4}, \cdots) \\ &= e^k + \tbinom{n}{1}e^{k+1} + \tbinom{n+1}{2}e^{k+2} + \tbinom{n+2}{3}e^{k+3} + \cdots \end{aligned}$$

so

$$(18)\qquad f(x) = \sum_{k=0}^\infty \left[(\Delta^n x)_k \cdot (f(e^k) + \tbinom{n}{1}f(e^{k+1}) + \tbinom{n+1}{2}f(e^{k+2}) + \cdots)\right]$$

If $f_j = f(e^j)$, then with respect to (14), we have

$$\begin{aligned} f(x) &= \sum_{k=0}^\infty \left[(\Delta^n x)_k \cdot (D_{kk}^{(n)}f_k + D_{k(k+1)}^{(n)}f_{k+1} + D_{k(k+2)}^{(n)}f_{k+2} + \cdots)\right] \\ &= \sum_{k=0}^\infty \left[(\Delta^n x)_k \cdot \sum_{j=k}^\infty D_{kj}^{(n)} f_j\right] \end{aligned}$$

So $|f(x)| \le \sum_{k=0}^\infty |\Delta^n x|_k \cdot \sup_{k\in\mathbb{N}} |\sum_{j=k}^\infty D_{kj}^{(n)} f_j| = \|(f_0, f_1, f_2, \cdots)\|_{d_\infty^n} \cdot \|x\|_{l_1(\Delta^n)}$. So $\|f\| \le \|(f_0, f_1, f_2, \cdots)\|_{d_\infty^n}$. So T is surjective. T is injective since $T(f) = 0$

implies $f = 0$. Finally T is norm preserving since

$$|f(x)| \leq \sum_{k=0}^{\infty} |\Delta^n x|_k \cdot \sup_{k\in\mathbb{N}} \big| \sum_{j=k}^{\infty} D_{kj}^{(n)} f_j \big| = \|x\|_{l_1(\Delta^n)} \cdot \|Tf\|_{d_\infty^n} \tag{19}$$

So

$$\|f\| \leq \|Tf\|_{d_\infty^n} \tag{20}$$

On the other hand,

$$\big| \sum_{j=k}^{\infty} D_{kj}^{(n)} f_j \big| = |f(\Delta^{-n} e^k)| \leq \|f\| \cdot \|\Delta^{-n} e^k\|_{l_1(\Delta^n)} = \|f\| \quad \text{for all } k \in \mathbb{N} \tag{21}$$

So

$$\|Tf\|_{d_\infty^n} = \sup_{k\in\mathbb{N}} \big| \sum_{j=k}^{\infty} D_{kj}^{(n)} f_j \big| \leq \|f\| \tag{22}$$

From (20) and (22), we have

$$\|Tf\|_{d_\infty^n} = \|f\|.$$

So T is norm preserving and it completes the proof. □

Theorem 4.2. *If* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then* $l_p(\Delta^n)^*$ *is isometrically isomorphic to* d_q^n.

Proof. Let

$$T : l_p(\Delta^n)^* \to d_q^n \tag{23}$$

defined by $Tf = (f(e^0), f(e^1), f(e^2), f(e^3), \cdots)$. Trivially T is linear and (18) implies that

$$\begin{aligned} |f(x)| = \left| \sum_{k=0}^{\infty} [(\Delta^n x)_k \cdot \sum_{j=k}^{\infty} D_{kj}^{(n)} f_j] \right| &\leq \left[\sum_{k=0}^{\infty} |\Delta^n x|_k^p \right]^{\frac{1}{p}} \left[\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} D_{kj}^{(n)} f_j \right|^q \right]^{\frac{1}{q}} \\ &= \|x\|_{l_p(\Delta^n)} \cdot \|(f_0, f_1, f_2, \cdots)\|_{d_q^n}. \end{aligned}$$

The above computations show that T is surjective. Moreover T is injective since $Tf = 0$ implies $f = 0$. T is norm preserving since $|f(x)| \leq \|x\|_{l_p(\Delta^n)} \cdot \|(f_0, f_1, f_2, \cdots)\|_{d_q^n} = \|x\|_{l_p(\Delta^n)} \cdot \|Tf\|_{d_q^n}$. So

$$\|f\| \leq \|Tf\|_{d_q^n}. \tag{24}$$

On the other hand, let $x^{(m)} = (x_k^{(m)})$ where

$$(\Delta^n x^{(m)})_k = \begin{cases} \big| \sum_{j=k}^{\infty} D_{kj}^{(n)} f_j \big|^{q-1} \operatorname{sgn}\big(\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j \big) & 0 \leq k \leq m \\ 0 & k > m \end{cases} \tag{25}$$

Then $x^{(m)} \in l_p(\Delta^n)$ since $\Delta^n x^{(m)} \in l_p$. So

$$\begin{aligned} f(x^{(m)}) &= f\Big(\sum_{k=0}^{\infty}(\Delta^n x^{(m)})_k \cdot (\Delta^{-n}e^k)\Big) = f\Big(\sum_{k=0}^{m}(\Delta^n x^{(m)})_k \cdot (\Delta^{-n}e^k)\Big) \\ &= \sum_{k=0}^{m}(\Delta^n x^{(m)})_k f(\Delta^{-n}e^k) = \sum_{k=0}^{m}(\Delta^n x^{(m)})_k \sum_{j=k}^{\infty} D_{kj}^{(n)} f_j \\ &= \sum_{k=0}^{m}\Big|\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big|^{q-1} \operatorname{sgn}\Big(\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big)\Big(\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big) \\ &= \sum_{k=0}^{m}\Big|\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big|^{q} \le \|f\| \cdot \|x^{(m)}\|_{l_p(\Delta^n)}. \end{aligned}$$

So

$$\begin{aligned} \|x^{(m)}\|_{l_p(\Delta^n)} = \|\Delta^n x^{(m)}\|_{l_p} &= \Big(\sum_{k=0}^{\infty}|\Delta^n x^{(m)}|_k^p\Big)^{\frac{1}{p}} = \Big(\sum_{k=0}^{m}|\Delta^n x^{(m)}|_k^p\Big)^{\frac{1}{p}} \\ &= \Big(\sum_{k=0}^{m}\Big|\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big|^{p(q-1)}\Big|\operatorname{sgn}\Big(\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big)\Big|^{p}\Big)^{\frac{1}{p}} \\ &= \Big(\sum_{k=0}^{m}\Big|\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big|^{q}\Big)^{\frac{1}{p}} \end{aligned}$$

So

$$\Big(\sum_{k=0}^{m}\Big|\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big|^{q}\Big)^{1} \le \|f\| \cdot \Big(\sum_{k=0}^{m}\Big|\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big|^{q}\Big)^{\frac{1}{p}}.$$

So

$$\|f\| \ge \Big(\sum_{k=0}^{m}\Big|\sum_{j=k}^{\infty} D_{kj}^{(n)} f_j\Big|^{q}\Big)^{\frac{1}{q}} = \|Tf\| d_q^n. \tag{26}$$

From (24) and (26), we have

$$\|Tf\|_{d_q^n} = \|f\|.$$

So T is norm preserving and this completes the proof. □

5. Continuity of Δ^n on some sequence spaces

Lemma 5.1. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ if and only if the supremum of l_1 norms of the columns of A is bounded. In fact,* $\|A\|_{(l_1,l_1)} = \sup_n \sum_{k=0}^{\infty} |a_{nk}|$.

Corollary 5.2. $\|\Delta^n\|_{(l_1,l_1)} = 2^n$.

Lemma 5.3. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_\infty)$ if and only if the supremum of l_1 norms of the rows of A is bounded. In fact,* $\|A\|_{(l_\infty,l_\infty)} = \sup_k \sum_{n=0}^{\infty} |a_{nk}|$.

Corollary 5.4. $\|\Delta^n\|_{(l_\infty,l_\infty)} = 2^n$.

Lemma 5.5. *Let $1 < p < \infty$ and let $A \in (l_\infty, l_\infty) \bigcap (l_1, l_1)$. Then $A \in (l_p, l_p)$.*

Corollary 5.6. *For every integer n and $1 < p < \infty$ holds $\Delta^n \in B(l_p)$.*

Proof. With respect to the matrix representation of Δ^n and Lemma 5.1 and 5.3 $\Delta^n \in (l_\infty, l_\infty) \bigcap (l_1, l_1)$ and so by Lemma 5.5, $\Delta^n \in (l_p, l_p)$. □

Theorem 5.7. $\Delta^n \in B(l_p(\Delta^n))$.

Proof. Suppose $\Delta^n : l_p \to l_p$ and $x \in l_p$. Then by Corollary 5.6, there exists $M_n^p \in \mathbb{N}$ such that $\|\Delta^n x\|_{l_p} \le M_n^p \|x\|_{l_p}$. So if $\Delta^n : l_p(\Delta^n) \to l_p(\Delta^n)$ and $x \in l_p(\Delta^n)$, then $\|\Delta^n x\|_{l_p(\Delta^n)} = \|\Delta^n(\Delta^n x)\|_{l_p} \le M_n^p \cdot \|\Delta^n x\|_{l_p} = M_n^p \cdot \|x\|_{l_p(\Delta^n)}$. So $\|\Delta^n\|_{(l_p(\Delta^n), l_p(\Delta^n))} \le M_n^p$ and it completes the proof. □

In [**1**, Theorem 3.2] claims that the norm of operator Delta is 2 i.e. Δ is a bounded operator on $l_p(\Delta)$ which confirms Theorem 5.7 in case $n = 1$.

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