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REGULARITY AND A PRIORI ESTIMATES OF SOLUTIONS FOR SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. We improve recent results of Li [**11**] on L^∞-regularity and a priori estimates for non-negative very weak solutions of elliptic systems in bounded domains. The proof is based on an alternate-bootstrap procedure in the scale of weighted Lebesgue spaces.

1. Introduction

The aim of this paper is to extend some recent results of Li [**11**] on L^∞-regularity and a priori estimates for solutions of elliptic systems of the form

$$\left.\begin{array}{rcl} \left.\begin{array}{rcl} -\Delta u &=& f(\cdot\,,u,v) \\ -\Delta v &=& g(\cdot\,,u,v) \end{array}\right\} & \text{in } \Omega, \\ \left.\begin{array}{rcl} u &=& 0 \\ v &=& 0 \end{array}\right\} & \text{on } \partial\Omega. \end{array}\right\} \tag{1}$$

Throughout this paper we will assume that Ω is a bounded domain in $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$ and f, g are non-negative Carathéodory functions. We are mainly interested in very weak solutions (u, v) of problem (1).

Let us first consider the corresponding scalar problem

$$\left.\begin{array}{rclr} -\Delta u &=& f(x\,,u) & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2}$$

where f satisfies the growth assumption

$$0 \le f(x,u) \le C(1+|u|^p), \quad p>0. \tag{3}$$

Let us denote by $L^k_\delta(\Omega)$ the weighted Lebesgue space $L^k(\Omega, \delta(x)dx)$ where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. We call u an L^1_δ-solution (or a very weak solution) of (2) if $u \in L^1(\Omega)$, $f(\cdot, u(\cdot)) \in L^1_\delta(\Omega)$ and

$$\int_\Omega (u(x)\Delta\varphi(x) - f(x\,,u)\varphi(x))\mathrm{d}x = 0 \quad \text{for all } \varphi \in C^2(\overline{\Omega}),\ \varphi|_{\partial\Omega} = 0. \tag{4}$$

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It is known, see [**4**] and [**17**], that all very weak solutions of (2) belong to $L^\infty(\Omega)$ provided $p < p_c$, where p_c is defined by

$$p_c := \begin{cases} \infty, & \text{if } N < 2, \\ \dfrac{N+1}{N-1}, & \text{if } N \geq 2. \end{cases} \tag{5}$$

On the other hand, unbounded very weak solutions of (2) were constructed for $p \geq p_c$ in [**7**], [**19**], see also [**2**], [**3**]. If Ω is not smooth, then the critical exponent p_c depends also on Ω, see [**9**], [**13**]. The critical exponent will also change if we replace the homogeneous Dirichlet boundary condition with homogeneous Neumann or Newton boundary condition, see [**17**]. The critical exponent for scalar problems with nonlinear boundary conditions in smooth domains was established in [**16**].

In the case of systems, very weak solutions of (1) are defined analogously to the scalar case, see [**11**, Definition 2.1] for details. The boundedness of very weak solutions of systems and their a priori estimates were studied in [**5**], [**10**], [**11**], [**12**], [**17**] and [**19**]. Let us mention some related results from [**11**], [**17**] and [**19**].

In 2004, P. Quittner and Ph. Souplet [**17**] showed that any non-negative L^1_δ-solution (u, v) of system (1) belongs to $L^\infty(\Omega)$ and has the a priori bound

$$\|u\|_\infty + \|v\|_\infty \leq C(\Omega, p, q, \gamma, \sigma, N, C_1, M) \tag{6}$$

provided

$$\|u\|_{L^1_\delta} + \|v\|_{L^1_\delta} \leq M,$$

$$\begin{aligned} 0 \leq & f(x, u, v) \leq C_1(1 + |v|^p + |u|^\gamma), \\ 0 \leq & g(x, u, v) \leq C_1(1 + |u|^q + |v|^\sigma), \end{aligned} \tag{7}$$

where

$$\max\{p+1, q+1\} > \frac{pq-1}{p_c - 1}, \tag{8}$$

$$1 \leq \gamma, \sigma < p_c \tag{9}$$

and $p, q > 0$. Their proof was based on a bootstrap argument using L^p_δ-regularity of the Dirichlet Laplacian, see [**8**] and Lemma 2.1 below. They also found sufficient conditions on f, g guaranteeing the estimate (7).

In 2005, Ph. Souplet [**19**] showed that the exponent p_c appearing in (8), (9) is optimal. Assuming

$$\max\{p+1, q+1\} < \frac{pq-1}{p_c - 1}, \tag{10}$$

he constructed functions $a, b \in L^\infty(\Omega)$, $a, b \geq 0$ such that the problem

$$\left.\begin{aligned} \left.\begin{aligned} -\Delta u &= av^p \\ -\Delta v &= bu^q \end{aligned}\right\} & \text{ in } \Omega, \\ \left.\begin{aligned} u &= 0 \\ v &= 0 \end{aligned}\right\} & \text{ on } \partial\Omega, \end{aligned}\right\} \tag{11}$$

admits a positive very weak solution $(u, v) \notin L^\infty(\Omega) \times L^\infty(\Omega)$.

Recently, Y.-X. Li [**11**] improved the results in [**17**]. He proved that (7) implies (6) under more general assumptions on f, g:

$$\left.\begin{array}{rcl} 0 \le f(x,u,v) & \le & C_1(1+|u|^r|v|^p+|u|^\gamma), \\ 0 \le g(x,u,v) & \le & C_1(1+|u|^q|v|^s+|v|^\sigma), \end{array}\right\} \tag{12}$$

where

$$r, s, \min\{p+r, q+s\} \in [0, p_c)\,, \tag{13}$$

$$\max\{p+1-s, q+1-r\} > \frac{pq-(1-r)(1-s)}{p_c-1}, \tag{14}$$

$p, q > 0$ and (9) is true. Notice that if $r = s = 0$, then the assumptions (13), (14) are equivalent to (8) (since (8) guarantees that $\min\{p, q\} < p_c$). Similarly to [**19**], Li also constructed an example showing that his results are optimal in some sense. In this paper, we obtain the following improvement of his results.

Theorem 1.1. *Let $f, g : \Omega \times \mathbb{R}^2 \to [0, \infty)$ satisfy*

$$\left.\begin{array}{rcl} f(x,u,v) & \le & C_1(1+|u|^{r_1}|v|^{p_1}+|u|^{r_2}|v|^{p_2}+|u|^\gamma), \\ g(x,u,v) & \le & C_1(1+|u|^{q_1}|v|^{s_1}+|u|^{q_2}|v|^{s_2}+|v|^\sigma), \end{array}\right\} \tag{15}$$

where $p_i, q_i, r_i, s_i \ge 0$ for $i = 1,\ 2$, $\max\{p_1, p_2\}, \max\{q_1, q_2\} > 0$ and (9) is true. Assume also that

$$\begin{array}{l} \min\{\max\{p_1+r_1,\ p_2+r_2\}, \qquad \max\{q_1+s_1,\ q_2+s_2\}\} < p_c, \\ r_i, s_i < p_c, \quad i = 1, 2, \end{array} \tag{16}$$

$$\begin{array}{l} \max\{p_i+1-s_j,\ q_j+1-r_i\} > \dfrac{p_iq_j-(1-r_i)(1-s_j)}{p_c-1}, \\ i, j = 1, 2, \end{array} \tag{17}$$

and (u, v) is a non-negative very weak solution of (1) satisfying

$$\|u\|_{L^1_\delta} + \|v\|_{L^1_\delta} \le M. \tag{18}$$

Then (u, v) belongs to $L^\infty(\Omega) \times L^\infty(\Omega)$ and

$$\|u\|_{L^\infty} + \|v\|_{L^\infty} \le C(\Omega, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, \gamma, \sigma, N, C_1, M). \tag{19}$$

Remark 1.2. Actually, if we replace growth assumption (15) by

$$\left.\begin{array}{rcl} f(x,u,v) & \le & C_1(1+(1+|u|)^{r_1}(1+|v|)^{p_1} \\ & & +(1+|u|)^{r_2}(1+|v|)^{p_2}+|u|^\gamma), \\ g(x,u,v) & \le & C_1(1+(1+|u|)^{q_1}(1+|v|)^{s_1} \\ & & +(1+|u|)^{q_2}(1+|v|)^{s_2}+|v|^\sigma), \end{array}\right\} \tag{20}$$

the results in Theorem 1.1 remain valid.

Remark 1.3. If we set $p_2 = q_2 = r_2 = s_2 = 0$, Theorem 1.1 recovers Li's result [**11**] since (16), (17) are equivalent to (13), (14) in this case. In Section 3 below, we show that all assumptions of Theorem 1.1 are satisfied for $N = 3$ and

$$\left.\begin{aligned} f(x,u,v) &= u^{1-\varepsilon}v + v^{\frac{5}{4}-\varepsilon} \\ g(x,u,v) &= u^4 v \end{aligned}\right\} \tag{21}$$

where $\varepsilon \in \left(0, \frac{1}{7}\right)$, but f, g do not satisfy Li's assumptions (9), (12), (13) and (14).

Remark 1.4. Similarly to Li [**11**], the same argument as in the proof of Theorem 1.1 can be used in order to get L^∞ regularity of H^1_0- or L^1-solutions of (1) (see [**11**, Definition 2.1] for precise definitions of such solutions). In the case of H^1_0-solutions, p_c has to be replaced by the Sobolev exponent p_S

$$p_S := \begin{cases} \infty, & \text{if } N < 3, \\ \dfrac{N+2}{N-2}, & \text{if } N \geq 3 \end{cases} \tag{22}$$

and in the case of L^1-solutions p_c has to be replaced by the singular exponent p_{sg} defined by

$$p_{sg} := \begin{cases} \infty, & \text{if } N < 3, \\ \dfrac{N}{N-2}, & \text{if } N \geq 3. \end{cases} \tag{23}$$

Notice that in the case of H^1_0-solutions, the L^∞ a priori bound (19) requires the estimate

$$\|u\|_{H^1_0} + \|v\|_{H^1_0} \leq M$$

instead of (18) and obtaining this estimate (unlike estimate (18) in the case of L^1_δ-solutions) is far from easy, see [**17**], [**18**] and the references therein, for example. L^1-solutions are in particular important in the case of Neumann or Newton boundary conditions where the bootstrap argument works as well and, in addition, one can easily find conditions on f, g guaranteeing the necessary initial bound

$$\|u\|_{L^1} + \|v\|_{L^1} \leq M,$$

see [**17**].

A significant difference between H^1_0-solutions and L^1- (or L^1_δ-) solutions can be observed in the critical case: While H^1_0-solutions of the scalar problem (2) are regular in the critical case $p = p_S$, see [**6**] or [**18**, Corollary 3.4], singular L^1- or L^1_δ-solutions of (2) exist if $p = p_{sg}$ or $p = p_c$ respectively, see [**1**], [**14**], [**15**] and [**7**].

This paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. In Section 3, we construct an example of system (1) which satisfies the assumptions of Theorem 1.1 but not assumptions in [**11**].

2. Proof of Theorem 1.1

In order to give a complete proof of Theorem 1.1, we will need the following regularity results for very weak solutions of the scalar problem

$$\left.\begin{aligned} -\Delta u &= \phi \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{24}$$

see [**17**] and [**8**].

Lemma 2.1. *Let* $1 \le m \le k \le \infty$ *satisfy*

$$\frac{1}{m} - \frac{1}{k} < \frac{1}{p_c'},$$

where p_c' *satisfies* $\frac{1}{p_c} + \frac{1}{p_c'} = 1$. *Let* $u \in L^1_\delta(\Omega)$ *be the unique* L^1_δ*-solution of* (24). *If* $\phi \in L^m_\delta(\Omega)$, *then* $u \in L^k_\delta(\Omega)$ *and* u *satisfies the estimate*

$$\|u\|_{L^k_\delta} \le C(\Omega, m, k)\|\phi\|_{L^m_\delta}.$$

Now, we can give the *proof* of Theorem 1.1:

Proof. Without loss of generality we can assume

$$p_2 + r_2 \le p_1 + r_1, \quad q_2 + s_2 \le q_1 + s_1 \tag{25}$$

and

$$p_1 + r_1 \le q_1 + s_1, \tag{26}$$

which together with (16) imply

$$p_1 + r_1 < p_c. \tag{27}$$

Moreover, we can assume $p_1 \ne p_c - 1$, $p_2 \ne p_c - 1$, otherwise we can increase the values of exponents p_1 and/or p_2 (and q_1 if necessary) in such a way that (16), (17), (25) and (26) remain true.

We will denote by C a constant, which may vary from line to line, but is independent of (u, v). For simplicity, we denote by $|\cdot|_k$ the norm $\|\cdot\|_{L^k_\delta}$. Let $\varphi_1 > 0$ be the first eigenfunction of the negative Dirichlet Laplacian (normalized in L^∞, for example). Notice that there exist $c_1, c_2 > 0$ such that

$$c_1\delta \le \varphi_1 \le c_2\delta. \tag{28}$$

Testing both equations of (1) with φ_1, using Green's Theorem, (28) and the nonnegativity of f, g, u, v yield

$$|f|_1 \le C|u|_1 \quad \text{and} \quad |g|_1 \le C|v|_1.$$

Then, application of Lemma 2.1 and (18) imply

$$|u|_k + |v|_k \le C, \qquad \text{for all } k \in [1, p_c).$$

We distinguish several cases:

Case 1: $r_2 \le r_1$ and $p_2 \ge p_1$.

1a. If $p_2 < p_c - 1$, using bootstrap on the first equation of (1), we will obtain $|u|_\infty \le C$.

(i) First assume $r_1 < 1$. (9), (25) and (27) imply that there exists k such that

$$\max\{\gamma, p_1 + r_1\} < k < p_c, \quad \frac{p_2}{k} < \frac{1}{p_c'}. \tag{29}$$

For such a fixed k, we can find ε small enough to satisfy

$$\left.\begin{aligned} &\frac{\gamma}{k+m\varepsilon} - \frac{1}{k+(m+1)\varepsilon} < \frac{1}{p_c'}, \\ &\qquad\qquad \text{for any } m \in \mathbb{N}_0 = \{0,1,2,\dots\}, \\ &\frac{r_i}{k+m\varepsilon} + \frac{p_i}{k} - \frac{1}{k+(m+1)\varepsilon} < \frac{1}{p_c'}, \\ &\qquad\qquad \text{for } i = 1,2 \text{ and any } m \in \mathbb{N}_0. \end{aligned}\right\} \tag{30}$$

For $m \in \mathbb{N}_0$, set
$$\frac{1}{\rho_m} = \frac{r_1}{k+m\varepsilon} + \frac{p_1}{k}, \qquad \frac{1}{\nu_m} = \frac{r_2}{k+m\varepsilon} + \frac{p_2}{k}, \qquad \frac{1}{\varrho_m} = \frac{\gamma}{k+m\varepsilon}.$$

Using (25) and (29), we obtain that $\rho_m,\ \nu_m,\ \varrho_m > 1$. Denote $m_0 = \min\{m : \min\{\rho_m, \nu_m, \varrho_m\} > p_c'\}$. We claim that after m_0-th bootstrap on the first equation, we arrive at the desired result.

Assume the estimate $|u|_{k+m\varepsilon} \le C$ holds for some $m \in [0, m_0] \cap \mathbb{N}_0$ (which is true for $m = 0$). Then (30) implies

$$\frac{1}{\min\{\rho_m, \nu_m, \varrho_m\}} - \frac{1}{k+(m+1)\varepsilon} < \frac{1}{p_c'},$$

hence Lemma 2.1 together with (15) and the Hölder inequality imply

$$\begin{aligned} |u|_{k+(m+1)\varepsilon} &\le C|f|_{\min\{\rho_m, \varrho_m, \nu_m\}} \\ &\le C(\| u|^{r_1}|v|^{p_1}|_{\rho_m} + \| u|^{r_2}|v|^{p_2}|_{\nu_m} + \| u|^{\gamma}|_{\varrho_m} + 1) \\ &\le C(|u|^{r_1}_{k+m\varepsilon}|v|^{p_1}_k + |u|^{r_2}_{k+m\varepsilon}|v|^{p_2}_k + |u|^{\gamma}_{k+m\varepsilon} + 1) \\ &\le C \end{aligned}$$

So $|u|_{k+(m_0+1)\varepsilon} \le C$ and another application of Lemma 2.1 yields $|u|_\infty \le C$.

(ii) If $r_1 \ge 1$, (9), (25) and (27) imply that there exist k and η,

$$\max\{\gamma, p_1 + r_1\} < k < p_c, \quad \frac{p_2}{k} < \frac{1}{p_c'}, \quad k \text{ close enough to } p_c,$$
$$1 < \eta, \quad \eta \text{ close enough to } 1,$$

such that

$$\left.\begin{aligned} \frac{\gamma}{\eta^m k} - \frac{1}{\eta^{m+1}k} &< \frac{1}{p_c'}, \\ \frac{r_i}{\eta^m k} + \frac{p_i}{k} - \frac{1}{\eta^{m+1}k} &< \frac{1}{p_c'}, \qquad i=1,2, \end{aligned}\right\} \tag{31}$$

for any $m \in \mathbb{N}_0$. Similarly to the case **1a(i)**, we obtain $|u|_\infty \leq C$.

Now, we can carry on the bootstrap on the second equation of (1). From (9), (16), there exist l close enough to p_c and $\eta > 1$ such that

$$\alpha := \max\{\sigma, s_1, s_2\} < l < p_c \quad \text{ and } \quad \frac{\alpha}{l} - \frac{1}{\eta l} < \frac{1}{p_c'}.$$

Applying Lemma 2.1 we conclude after finitely many steps

$$|v|_\infty \leq C.$$

1b. In case $p_c - 1 < p_1 \leq p_2$, let us denote by k_1^* and k_2^* the solutions of

$$\frac{r_i}{k_i^*} + \frac{p_i}{p_c} - \frac{1}{k_i^*} = \frac{1}{p_c'}, \qquad i = 1,2. \tag{32}$$

We claim that $|u|_{k'} \leq C$, $k' \in [1, k^*)$ where $k^* = \min\{k_1^*, k_2^*\}$. Inequality $p_c - 1 < p_1 \leq p_2$ and (25), (27) imply $r_2 \leq r_1 < 1$. Remark that

$$k^* > p_c \tag{33}$$

since $p_i + r_i < p_c$ for $i = 1, 2$ due to (25) and (27). As in [**11**], let us denote $k_\varepsilon := k^* - \varepsilon$ for any $0 < \varepsilon \ll 1$ and $k_\varepsilon^{\tau^m} := k_\varepsilon - \tau^m(k_\varepsilon - k)$ for $m \in \mathbb{N}_0$. Thanks to (9), (25), (32) and (27), we can find $k = k(\varepsilon)$ and $\tau = \tau(\varepsilon)$ such that

$$\begin{aligned} \max\{\gamma, p_1 + r_1\} < k < p_c, &\qquad k \text{ close enough to } p_c, \\ r_2 \leq r_1 < \tau < 1, &\qquad \tau \text{ close enough to } 1, \\ r_2 k_\varepsilon^\tau \leq r_1 k_\varepsilon^\tau < \tau k, & \end{aligned}$$

and

$$\left.\begin{aligned} \frac{\gamma}{k} - \frac{1}{k_\varepsilon^\tau} &< \frac{1}{p_c'}, \\ \frac{r_i}{k_\varepsilon} + \frac{p_i}{k} - \frac{1}{k_\varepsilon} &< \frac{1}{p_c'}, \quad i = 1,2. \end{aligned}\right\} \tag{34}$$

Using $r_2 k_\varepsilon^\tau \leq r_1 k_\varepsilon^\tau < \tau k$ and $\gamma \geq 1$ we get

$$\left.\begin{aligned} \frac{\gamma}{k_\varepsilon^{\tau^m}} - \frac{1}{k_\varepsilon^{\tau^{(m+1)}}} &\leq \frac{\gamma}{k} - \frac{1}{k_\varepsilon^\tau}, \\ \frac{r_i}{k_\varepsilon^{\tau^m}} - \frac{1}{k_\varepsilon^{\tau^{(m+1)}}} &< \frac{r_i}{k_\varepsilon} - \frac{1}{k_\varepsilon}, \quad i = 1,2, \end{aligned}\right\} \tag{35}$$

for all $m \in \mathbb{N}_0$. Now setting

$$\frac{1}{\rho_m} = \frac{r_1}{k_\varepsilon^{\tau^m}} + \frac{p_1}{k},$$
$$\frac{1}{\nu_m} = \frac{r_2}{k_\varepsilon^{\tau^m}} + \frac{p_2}{k},$$
$$\frac{1}{\varrho_m} = \frac{\gamma}{k_\varepsilon^{\tau^m}},$$

and using similar bootstrap argument to the case **1a** lead to

$$|u|_{k_\varepsilon^{\tau(m+1)}} \le C, \quad m \in \mathbb{N}_0.$$

As $k_\varepsilon^{\tau^m}$ tends to k_ε with m going to infinity, we obtain

$$|u|_{k'} \le C, \quad k' \in [1, k^*).$$

To continue the bootstrap on the second equation of (1), we first show that

$$\frac{q_i}{k^*} + \frac{s_i}{p_c} < 1, \quad i = 1, 2. \tag{36}$$

Inequality (36) is true for $i = 1$ thanks to (17) and (26). Let $j \in \{1, 2\}$ be such that $k^* = k_j^*$. If $i = 2$, then (36) follows from (17) if $p_j + r_j \le q_2 + s_2$ and from inequality

$$(q_2 + 1 - r_j)(p_c - p_j - r_j) > 0$$

otherwise.
From the definition of k^*, it is easy to see that

$$\frac{r_i}{k^*} + \frac{p_i}{p_c} - \frac{1}{k^*} \le \frac{1}{p_c'}. \tag{37}$$

Thanks to (9), (16), (25), (27), (33), (36) and (37) we can choose l, k_1 and η satisfying

$$\left.\begin{aligned} \max\{p_1 + r_1, \sigma, s_1, s_2\} < l < p_c, &\qquad l \text{ close enough to } p_c, \\ p_c < k_1 < k^*, &\qquad k_1 \text{ close enough to } k^*, \\ 1 < \eta, &\qquad \eta \text{ close enough to } 1, \end{aligned}\right\} \tag{38}$$

such that

$$\frac{q_i}{k_1} + \frac{s_i}{l} < 1, \quad i = 1, 2,$$
$$\frac{\sigma}{l} - \frac{1}{\eta l} < \frac{1}{p_c'},$$
$$\frac{\gamma}{k_1} - \frac{1}{\eta k_1} < \frac{1}{p_c'},$$
$$\frac{q_i}{k_1} + \frac{s_i}{l} - \frac{1}{\eta l} < \frac{1}{p_c'}, \quad i = 1, 2,$$
$$\frac{r_i}{k_1} + \frac{p_i}{\eta l} - \frac{1}{\eta k_1} < \frac{1}{p_c'}, \quad i = 1, 2.$$

Multiplying the LHS of the inequalities above by $1/\eta^m$, we get

$$
\begin{aligned}
\frac{q_i}{\eta^m k_1} + \frac{s_i}{\eta^m l} &< 1, \quad i = 1,2,\\
\frac{\sigma}{\eta^m l} - \frac{1}{\eta^{m+1} l} &< \frac{1}{p_c'},\\
\frac{\gamma}{\eta^m k_1} - \frac{1}{\eta^{m+1} k_1} &< \frac{1}{p_c'},\\
\frac{q_i}{\eta^m k_1} + \frac{s_i}{\eta^m l} - \frac{1}{\eta^{m+1} l} &< \frac{1}{p_c'}, \quad i = 1,2,\\
\frac{r_i}{\eta^m k_1} + \frac{p_i}{\eta^{m+1} l} - \frac{1}{\eta^{m+1} k_1} &< \frac{1}{p_c'}, \quad i = 1,2,
\end{aligned} \tag{39}
$$

for all $m \in \mathbb{N}_0$. Set

$$
\frac{1}{\mu_m} = \frac{q_1}{\eta^m k_1} + \frac{s_1}{\eta^m l}, \qquad \frac{1}{\varsigma_m} = \frac{q_2}{\eta^m k_1} + \frac{s_2}{\eta^m l}, \qquad \frac{1}{\sigma_m} = \frac{\sigma}{\eta^m l},
$$

$$
\frac{1}{\rho_m} = \frac{r_1}{\eta^m k_1} + \frac{p_1}{\eta^{m+1} l}, \qquad \frac{1}{\nu_m} = \frac{r_2}{\eta^m k_1} + \frac{p_2}{\eta^{m+1} l}, \qquad \frac{1}{\varrho_m} = \frac{\gamma}{\eta^m k_1}.
$$

It is easy to see that $\mu_m, \varsigma_m, \sigma_m, \rho_m, \nu_m, \varrho_m > 1$ thanks to (9), (25), (27), (38) and (39). Assume the estimate $|u|_{\eta^m k_1} + |v|_{\eta^m l} \le C$ holds for some $m \in \mathbb{N}_0$ (which is true for $m = 0$). Then the inequalities above imply

$$
\begin{aligned}
\frac{1}{\min\{\mu_m, \varsigma_m, \sigma_m\}} - \frac{1}{\eta^{m+1} l} &< \frac{1}{p_c'},\\
\frac{1}{\min\{\rho_m, \nu_m, \varrho_m\}} - \frac{1}{\eta^{m+1} k_1} &< \frac{1}{p_c'}.
\end{aligned}
$$

Hence Lemma 2.1 together with (15) and the Hölder inequality imply

$$
\begin{aligned}
|v|_{\eta^{m+1} l} &\le C|g|_{\min\{\mu_m, \varsigma_m, \sigma_m\}}\\
&\le C(\||u|^{q_1}|v|^{s_1}|_{\mu_m} + \||u|^{q_2}|v|^{s_2}|_{\varsigma_m} + \||v|^{\sigma}|_{\sigma_m} + 1)\\
&\le C(|u|^{q_1}_{\eta^m k_1}|v|^{s_1}_{\eta^m l} + |u|^{q_2}_{\eta^m k_1}|v|^{s_2}_{\eta^m l} + |v|^{\sigma}_{\eta^m l} + 1)\\
&\le C
\end{aligned}
$$

$$
\begin{aligned}
|u|_{\eta^{m+1} k_1} &\le C|f|_{\min\{\rho_m, \varrho_m, \nu_m\}}\\
&\le C(\||u|^{r_1}|v|^{p_1}|_{\rho_m} + \||u|^{r_2}|v|^{p_2}|_{\nu_m} + \||u|^{\gamma}|_{\varrho_m} + 1)\\
&\le C(|u|^{r_1}_{\eta^m k_1}|v|^{p_1}_{\eta^{m+1} l} + |u|^{r_2}_{\eta^m k_1}|v|^{p_2}_{\eta^{m+1} l} + |u|^{\gamma}_{\eta^m k_1} + 1)\\
&\le C.
\end{aligned}
$$

Denote $m_0 := \min\{m \in \mathbb{N}_0 : \max\{\min\{\rho_m, \varrho_m, \nu_m\}, \min\{\mu_m, \varsigma_m, \sigma_m\}\} > p_c'\}$. As in [**11**, Case III in the proof of Theorem 2.4] after m_0-th alternate bootstrap on system (1), we arrive at the desired result $|v|_\infty \le C$. So we also have $|u|_\infty \le C$ thanks to (9), (16) and Lemma 2.1.

1c. In case $p_1 < p_c - 1 < p_2$, we have $r_2 < 1$ from (25) and (27). Let us denote

$$k^* := k_2^* = \frac{p_c(1-r_2)}{p_2 - (p_c - 1)},$$

we claim that

$$|u|_{k'} \le C \quad k' \in [1, k^*).$$

(i) If $r_1 < 1$, similarly to case **1b**, due to (9), (25) and (27), there exist k and τ such that

$$\begin{array}{lll} \max\{\gamma, p_1 + r_1\} < k < p_c, & \dfrac{p_1}{k} < \dfrac{1}{p_c'}, & k \text{ close enough to } p_c, \\ r_2 \le r_1 < \tau < 1, & \tau \text{ close enough to } 1, & r_2 k_\varepsilon^\tau \le r_1 k_\varepsilon^\tau < \tau k, \end{array}$$

where

$$k_\varepsilon = k^* - \varepsilon$$

and (34), (35) are satisfied. By the same bootstrap on the first equation as in case **1b**, we obtain

$$|u|_{k'} \le C, \qquad k' \in [1, k^*).$$

(ii) If $r_1 \ge 1$, due to (9), (25) and (27), there exist k and η such that

$$\begin{array}{lll} \max\{\gamma, p_1 + r_1\} < k < p_c, & \dfrac{p_1}{k} < \dfrac{1}{p_c'}, & k \text{ close enough to } p_c, \\ 1 < \eta, & \eta r_2 < 1, & \eta \text{ close enough to } 1, \end{array}$$

and inequalities

$$\left.\begin{aligned} \frac{\gamma}{\eta^m k} - \frac{1}{\eta^{m+1}k} &< \frac{1}{p_c'}, \\ \frac{r_i}{\eta^m k} + \frac{p_i}{k} - \frac{1}{\eta^{m+1}k} &< \frac{1}{p_c'}, \quad i = 1, 2, \end{aligned}\right\} \tag{40}$$

are satisfied for all $m \in \mathbb{N}_0$ such that

$$k' := \eta^{m+1} k < \frac{p_c k (1 - \eta r_2)}{p_2 p_c - k(p_c - 1)}.$$

As the expression on the right-hand side of the last inequality goes to $\frac{(1-\eta r_2)k^*}{1-r_2}$ when k approaches p_c, by the bootstrap on the first equation of (1) we obtain

$$|u|_{k'} \le C \quad k' \in [1, k^*),$$

because we can make $\frac{(1-\eta r_2)k^*}{1-r_2}$ arbitrarily close to k^* by the choice of η.

Now, we can carry on the alternate bootstrap procedure just like in the case **1b** to obtain

$$|u|_\infty + |v|_\infty \le C.$$

Case 2: $r_2 \geq r_1$ and $p_2 < p_1$

Application of the Young inequality implies

$$|u|^{r_2}|v|^{p_2} \leq C(|u|^{r_1}|v|^{p_1} + |u|^{\frac{r_2p_1-r_1p_2}{p_1-p_2}}).$$

Then (16) and (25) imply

$$0 < \frac{r_2p_1 - r_1p_2}{p_1 - p_2} < p_c,$$

so we can simply set new γ by

$$\gamma := \max\left\{\gamma, \frac{r_2p_1 - r_1p_2}{p_1 - p_2}\right\}.$$

From in [**11**, Lemmas 2.5, 2.6], we get

$$\left.\begin{aligned} &|u|_\infty \leq C, && && \text{if } p_1 < p_c - 1,\\ &|u|_{k_1} \leq C, && \text{for all } k_1 \in [1, k^*), && \text{if } p_1 > p_c - 1, \end{aligned}\right\} \tag{41}$$

where k^* is the solution of (32) with $i = 1$. Using the bootstrap on the second equation similarly to [**11**] leads to $|v|_\infty \leq C$ thanks to (16) and (17). In particular:
2a. If $p_1 < p_c - 1$ using (9), (16), similarly to the case **1a**, we obtain $|v|_\infty \leq C$.

2b. If $p_1 > p_c - 1$, we first show that

$$\frac{q_i}{k^*} + \frac{s_i}{p_c} < 1, \quad i = 1, 2. \tag{42}$$

This inequality holds if $i = 1$ thanks to (17) and (26). If $i = 2$, then (42) is true if $p_1 + r_1 \leq q_2 + s_2$ due to (17), otherwise it can be derived from the inequality

$$(q_2 + 1 - r_1)(p_c - p_1 - r_1) > 0.$$

We can choose l, k_1 and η satisfying

$$\begin{aligned} \max\{p_1 + r_1, \sigma, s_1, s_2\} < l < p_c, &\qquad l \text{ close enough to } p_c,\\ p_c < k_1 < k^*, &\qquad k_1 \text{ close enough to } k^*,\\ 1 < \eta\ , &\qquad \eta \text{ close enough to } 1, \end{aligned}$$

such that

$$\begin{aligned} \frac{q_i}{k_1} + \frac{s_i}{l} &< 1, \quad i = 1, 2,\\ \frac{\sigma}{l} - \frac{1}{\eta l} &< \frac{1}{p_c'},\\ \frac{\gamma}{k_1} - \frac{1}{\eta k_1} &< \frac{1}{p_c'},\\ \frac{q_i}{k_1} + \frac{s_i}{l} - \frac{1}{\eta l} &< \frac{1}{p_c'}, \quad i = 1, 2,\\ \frac{r_1}{k_1} + \frac{p_1}{\eta l} - \frac{1}{\eta k_1} &< \frac{1}{p_c'}. \end{aligned}$$

We can carry on the alternate bootstrap procedure to obtain $|v|_\infty \le C$, then we can use the bootstrap on the first equation again to obtain $|u|_\infty \le C$ thanks to (9) and (16).

Case 3: $r_2 < r_1$ and $p_2 < p_1$

We recall Remark 1.2. As $(1+|u|)^{r_2}(1+|v|)^{p_2} \le (1+|u|)^{r_1}(1+|v|)^{p_1}$, we can replace r_2 and p_2 by r_1 and p_1, respectively. □

3. Example

As we have already mentioned in Remark 1.3, we consider system (1) with $N = 3$ and

$$\left.\begin{aligned} f(x,u,v) &= u^{1-\varepsilon}v + v^{\frac{5}{4}-\varepsilon}, \\ g(x,u,v) &= u^4 v, \end{aligned}\right\} \tag{43}$$

where

$$\varepsilon \in \left(0, \frac{1}{7}\right).$$

Notice that $p_c = 2$. It is easy to see that any non-negative very weak solution (u, v) of (43) belongs to $L^\infty(\Omega) \times L^\infty(\Omega)$ thanks to Theorem 1.1 with $p_1 = 1-\varepsilon$, $4r_1 = 14$, $p_2 = \frac{5}{4} - \varepsilon$, $r_2 = 0$, $\gamma = 1$, $q_1 = 4$, $s_1 = 1$, $q_2 = s_2 = 0$, $\sigma = 1$. Next, we will show that f, g do not satisfy Li's assumptions (9), (12), (13) and (14). Assume for contradiction

$$u^{1-\varepsilon}v + v^{\frac{5}{4}-\varepsilon} \le C(u^r v^p + u^2 + 1) \tag{44}$$

$$u^4 v \le C(u^q v^s + v^2 + 1) \tag{45}$$

where p, r, s and q satisfy (13) and (14). If we take $v = 1$ in (45) and send u to infinity, we obtain $q \ge 4$. Hence (13) guarantees $p + r < 2$. Setting $v = u^{4-\delta}$ with $0 < \delta \ll 1$ in (45) yields

$$8 - \delta \le q + (4-\delta)s,$$

which (taking $\delta \to 0$) leads to

$$2 - \frac{q}{4} \le s. \tag{46}$$

Since $p + r < 2 < q + s$, (14) implies $q + 1 - r > pq - (1-r)(1-s)$. This is equivalent to

$$p < 1 + \frac{(1-r)(2-s)}{q}. \tag{47}$$

Now, setting $u = 1$ in (44) and sending v to infinity lead to

$$\frac{5}{4} - \varepsilon \le p. \tag{48}$$

Thus $r < 1$ due to $p + r < 2$. This with (46), (47) imply

$$p < \frac{5}{4} - \frac{r}{4}. \tag{49}$$

Inequalities (48), (49) lead to $r < 4\varepsilon$. Now we choose $\alpha \in (1+\varepsilon, 4-20\varepsilon)$. This choice of α implies

$$\begin{aligned} 2 <& \quad 1-\varepsilon+\alpha, \\ r+\alpha p <& \quad 1-\varepsilon+\alpha. \end{aligned}$$

Now, taking $v = u^\alpha$ in inequality (44) and sending u to infinity yield a contradiction.

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