Acta Math. Univ. Comenianae
Vol. LXXX, 2 (2011), pp. 251–253

A CLOSED-FORM EXPRESSION FOR SECOND-ORDER RECURRENCES

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Abstract. This paper introduces a closed-form expression for the second-order recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, in which c_1 and c_2 are fixed constants and the value of two arbitrary terms a_{n-p} and a_{n-q} are known where p and q are positive integers and $p > q$. This expression is

$$a_n = \frac{S(p+1)}{S(p-q+1)} a_{n-q} - (-c_2)^{(p-q)} \frac{S(q+1)}{S(p-q+1)} a_{n-p}$$

where

$$S(n) = \sum_{i=0}^{[n/2]} \binom{n-i}{i} c_1^{n-2i} c_2^i.$$

1. Introduction

The Fibonacci and Lucas numbers are famous for possessing fabulous properties. For example, the ratio of Fibonacci numbers converges to the golden ratio, and the sum and differences of Fibonacci numbers are Fibonacci numbers. Nevertheless, many of these properties can be stated and proved for a much more general class of sequences, namely, second-order recurrences [**1, 2**], defined by $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, in which c_1 and c_2 are fixed constants. Lucas in his foundation paper [**3**] studied these sequences.

One of the difficulties in using recurrence relations is the need to compute all intermediate values up to the required term. In general, to overcome this for a d-order linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}$$

we can state $a_n = \sum_{i=1}^{d} k_i r_i^n$, where r_i are the roots of the characteristic polynomial

$$P(t) = t^d - c_1 t^{d-1} - c_2 t^{d-2} - \ldots - c_d = \prod_{i=1}^{d} (t - r_i)$$

Received November 16, 2010; revised March 15, 2011.
2010 *Mathematics Subject Classification.* Primary 11B37, 11B39.
Key words and phrases. Second-order recurrence; characteristic polynomial; closed-form solution.
The authors gratefully acknowledge the constructive comments of the referee which considerably improved the paper.

and the constants k_i are evaluated by looking at the initial values [**4**]. For the Fibonacci numbers, this solution reduces to the well-known Binet formula.

We recall the polynomial identity

$$\sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} (xy)^i (x+y)^{n-2i} = x^n + x^{n-1}y + \ldots + y^n = \frac{x^{n+1} - y^{n+1}}{x-y}$$

observed in [**7**]. Sury stated that since $\sum_{i\geq 0} \binom{n-i}{i}$ satisfies the same recursion as the Fibonacci sequence and starts with F_2 and F_3, it follows by induction that $\sum_{i\geq 0} \binom{n-i}{i} = F_{n+1}$, and he used this polynomial to obtain the Binet formula [**6**]. He also used this identity to derive trigonometric expressions for the Fibonacci and Lucas numbers [**5**].

In this paper, the second-order recurrences $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ are studied and this polynomial identity is applied to the general solution $a_n = k_1 r_1 + k_2 r_2$ to derive an explicit expression for a_n in terms of two arbitrary known terms a_{n-p} and a_{n-q}.

2. Closed-form expression for second-order recurrences

Let c_1 and c_2 be fixed constants. Consider the linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

For this recursion, the characteristic polynomial is $p(t) = t^2 - c_1 t - c_2$. Let α and β be its roots; hence $a_n = k_1 \alpha^n + k_2 \beta^n$ for certain constants k_1 and k_2. From the equalities

$$a_{n-p} = k_1 \alpha^{n-p} + k_2 \beta^{n-p}$$

$$a_{n-q} = k_1 \alpha^{n-q} + k_2 \beta^{n-q}$$

we can solve for k_1 and k_2 and obtain finally that

$$a_n = \frac{\alpha^p - \beta^p}{\alpha^{p-q} - \beta^{p-q}} a_{n-q} - \frac{(\alpha\beta)^{p-q}(\alpha^q - \beta^q)}{\alpha^{p-q} - \beta^{p-q}} a_{n-p}.$$

In order to simplify this expression further, we rearrange it to form the aforementioned polynomial identity. We also apply the equality $\alpha\beta = -c_2$; thus

$$a_n = \frac{\dfrac{\alpha^p - \beta^p}{\alpha - \beta}}{\dfrac{\alpha^{p-q} - \beta^{p-q}}{\alpha - \beta}} a_{n-q} - \frac{(-c_2)^{p-q} \dfrac{\alpha^q - \beta^q}{\alpha - \beta}}{\dfrac{\alpha^{p-q} - \beta^{p-q}}{\alpha - \beta}} a_{n-p}.$$

As $\alpha + \beta = c_1$ and $\alpha\beta = -c_2$, the polynomial identity results in

$$\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \sum_{i=0}^{[n/2]} \binom{n-i}{i} c_1^{n-2i} c_2^i.$$

Thus if we denote $S(n) = \sum_{i=0}^{[n/2]} \binom{n-i}{i} c_1^{n-2i} c_2^i$, we have

$$a_n = \frac{S(p+1)}{S(p-q+1)} a_{n-q} - (-c_2)^{(p-q)} \frac{S(q+1)}{S(p-q+1)} a_{n-p}.$$

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